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SEPARATION AXIOMS IN ČECH CLOSURE ORDERED SPACES

IREM EROGLU AND ERDAL GUNER

Abstract. In this paper, we generalize closure spaces by an preorder and we give some order seperation axioms in Čech closure ordered spaces.

INTRODUCTION

Topological spaces can be generalized by many ways. Leopoldo Nachbin[6] developed a way to generalize topological spaces by an order. He defined topological ordered spaces, such that a triple $(X, \tau, \leq)$ where τ is a topology and $\leq$ is a relation of partial order on X. He investigated some properties of topological ordered spaces.

In 1968 McCartan[9] studied T_i-ordered seperation axioms $(i = 1, 2, 3, 4)$ in topological ordered spaces.

The other way to generalize topological spaces is closure operators. Eduard Čech[4] defined Čech closure spaces or dually pretopological spaces. A.S.Mashhour and M.H.Ghanim[1] investigated properties of Čech closure spaces.

The aim of this paper is to define Čech closure ordered spaces and investigate some ordered seperation axioms in this spaces. For topological spaces we refer the reader to R. Engelking[8]. For closure spaces we refer to [3],[7].

1. PRELIMINARIES

Now, we will give some basic definitions about closure spaces and orders.

Definition 1. *Let X be a set. An order (partial order) on X is a binary relation $\leq$ on X such that, for all $x, y, z \in X$*

i) $x \leq x$

ii) $x \leq y$ *and* $y \leq x$ *imply* $x = y$

iii) $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$

These conditions are referred to, respectively as reflexivity, antisymetry and transivity. A set X equipped with an order relation $\leq$ is said to be an ordered set (or

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partially ordered set).A relation $\leq$ on a set X which is reflexive and transitive but not necessarily antisymmetric is called a quasi-order or preorder **[2].**

Definition 2. *Let X and Y be ordered sets. A map φ from X to Y is said to be an order-embedding if and only if the following be satisfied $x \leq y$ in X iff $\varphi(x) \leq \varphi(y)$ in Y.*

Definition 3. *Let X designate a preordered set. A subset $A \subseteq X$ is said to be decreasing if $a \leq b$ and $b \in A$ imply $a \in A$. The smallest decreasing set containing A will be shown by $\mathrm{d}(A)$. A subset $A \subseteq X$ is said to be increasing if $a \leq b$ and $a \in A$ imply $b \in A$ and the smallest increasing set containing A will be shown by $i(A)$* **[4]**.

Definition 4. *Let us consider a topological space (X, τ) equipped with a preorder $\leq$. The triple $(X, \tau, \leq)$ is called topological ordered space* **[5]**.

Definition 5. *If X is a set and u is a single-valued relation on $P(X)$ ranging in $P(X)$, then we shall say that u is a closure operation for X provided that the following conditions are satisfied,*

c_1) $u(\emptyset) = \emptyset$
c_2) $A \subseteq u(A)$ *for each* $A \subseteq X$
c_3) $u(A \cup B) = u(A) \cup u(B)$ *for each* $A \subseteq X$ *and* $B \subseteq X$.

*A structure (X, u) where X is a set and u is a closure operation for P, will be called a closure space (***[1],[3]***).*

Definition 6. *Let (X, u) be a closure space. There is associated the interior operation int_u usually denoted by int, such that for each $A \subseteq X$,*
$int_u A = X - u(X - A)$

Definition 7. *Let X be a set, u and v are closure operators on $P(X)$. The closure operator u is said to be coarser than v, or v is said to be finer than u, if for each $A \subseteq X$, $v(A) \subseteq u(A)$.*

Definition 8. *A neighbourhood of a subset A of X is any subset U of X containing A in its interior. We will show the neighbourhood family of A by $\mathcal{N}(A)$.*

Let $x \in U$, $U \subseteq X$. U is called a neighbourhood of x if and only if $x \in int_u U$. Neighbourhoods family of a point x will be shown by $\mathcal{N}(x)$.

Definition 9. *A family $\{A_i : i \in I\}$ of subsets of a closure space (X, u) will be called closure-preserving if for each $J \subseteq I$, $\bigcup_{i \in J} u(A_i) = u(\bigcup_{i \in J} A_i)$.*

Definition 10. *The product $\prod_{\alpha \in I} (X_\alpha, u_\alpha)$ of a family $\{(X_\alpha, u_\alpha) : \alpha \in I\}$ of closure spaces is the closure space $(\prod_{\alpha \in I} X_\alpha, u)$ where $\prod_{\alpha \in I} X_\alpha$ denotes the Cartesian product of the sets $X_\alpha, \alpha \in I$ and u is the closure operator generated by the projections $\pi_\alpha : \prod_{\alpha \in I} X_\alpha \to X_\alpha$, defined by $u(A) = \prod_{\alpha \in I} u_\alpha \pi_\alpha(A)$ for each $A \subseteq \prod_{\alpha \in I} X_\alpha$.*

2. T_1-ORDERED CLOSURE ORDERED SPACES

In this section, defining T_1-ordered closure ordered space we will investigate some properties.

Definition 11. *Let (X,u) be a Čech closure space and $\leq$ be a preorder on X. Then the triple $(X,u,\leq)$ will be called closure ordered space.*

Definition 12. *Let $(X,u,\leq)$ be a closure ordered space,*

i) $(X,u,\leq)$ is upper T_1-ordered iff for each pair of elements $a \nleq b$ in X, there exists a decreasing neighbourhood U of b such that $a \notin U$.

ii) $(X,u,\leq)$ is lower T_1-ordered iff for each pair of elements $a \nleq b$ in X, there exists an increasing neighbourhood U of a such that $b \notin U$.

If both of the conditions are satisfied, then $(X,u,\leq)$ will be called T_1-ordered closure ordered space.

Example 1. *Let $X = \{a,b,c\}$, $\leq = \{(a,a),(b,b),(c,c),(a,b),(b,a)\}$ be a preorder on X and $u : P(X) \to P(X)$ is defined such that,*
$u(\{a\}) = \{a,b\}$, $u(\{b\}) = \{b\}, u(\{c\}) = \{c\}$, $u(\{a,b\}) = \{a,b\}$,
$u(\{a,c\}) = X$, $u(\{b,c\}) = \{b,c\}$, $u(X) = X$, $u(\emptyset) = \emptyset$,
Then $(X,u,\leq)$ is a T_1-ordered closure ordered space.

Theorem 1. *Let $(X,u,\leq)$ be a closure ordered space, then the following conditions are equivalent,*

i) $(X,u,\leq)$ is lower(upper) T_1-ordered space

ii) For each $a \nleq b$ in X there exists U (V) a neighbourhood of a such that $x \nleq b$ $(a \nleq x)$ for all $x \in U$ $(x \in V)$

iii) For each $x \in X$, $[\leftarrow, x]$ $([x, \rightarrow])$ is closed.

Proof. *i)* $\Rightarrow$ *ii)* Let $(X,u,\leq)$ is lower T_1-ordered space and $a \nleq b$ in X, then there exists $U \in \mathcal{N}(a)$, U is decreasing and $b \notin U$. So, $x \nleq b$ for all $x \in U$.

ii) $\Rightarrow$ *iii)* Let $x \in X$ and suppose that $[\leftarrow, x]$ is not closed, so $u([\leftarrow, x]) \neq [\leftarrow, x]$. There exists $z \in u([\leftarrow, x])$ and $z \notin [\leftarrow, x]$, so $z \nleq x$. From *ii)*, there exists $U \in \mathcal{N}(z)$ and for each $y \in U$, $y \nleq x$. But this contradicts with $z \in u([\leftarrow, x])$. Consequently, $[\leftarrow, x]$ is closed.

iii) $\Rightarrow$ *i)* Let $a, b \in X$ and $a \nleq b$. From *iii)*, $[\leftarrow, b]$ is closed and $X - [\leftarrow, b]$ is open, so $X - [\leftarrow, b] \in \mathcal{N}(a)$. We find an increasing neighbourhood of a and $b \notin X - [\leftarrow, b]$. $(X,u,\leq)$ is lower T_1-ordered space. □

It can be similarly shown for upper T_1-ordered spaces.

Proposition 1. *Let $(X,u,\leq)$ be a T_1-ordered closure ordered space. Then every closure operator weaker than u with the same preorder is T_1-ordered closure ordered space.*

Proof. Let v be a closure operator which is weaker than u and $(X, u, \leq)$ be a T_1-ordered closure ordered space. Then we can write for each $x \in X$, $v([x, \rightarrow]) \subseteq u([x, \rightarrow]) = [x, \rightarrow]$ and $v([\leftarrow, x]) \subseteq u([\leftarrow, x]) = [\leftarrow, x]$. So, $(X, v, \leq)$ is a T_1-ordered space. □

Proposition 2. *Every subspace of a T_1-ordered closure ordered space is a T_1-ordered.*

Proof. Let $(X, u, \leq)$ be a T_1-ordered closure ordered space and $(A, u_A, \leq_A)$ be a subspace of $(X, u, \leq)$. We will use Theorem1, so it will be shown that for each $a \in A$, $[a, \rightarrow]_A = [a, \rightarrow] \cap A$ and $[\leftarrow, a]_A = [\leftarrow, a] \cap A$ are closed in $(A, u_A, \leq_A)$.

$u_A([a, \rightarrow]_A) = u_A([a, \rightarrow] \cap A) = u([a, \rightarrow] \cap A) \cap A \subseteq u([a, \rightarrow]) \cap u(A) \cap A =$

$u([a, \rightarrow]) \cap A = [a, \rightarrow] \cap A = [a, \rightarrow]_A$, so $u_A([a, \rightarrow]_A) \subseteq [a, \rightarrow]_A$. Hence $[a, \rightarrow]_A \subseteq u_A([a, \rightarrow]_A)$, $u_A([a, \rightarrow]_A) = [a, \rightarrow]_A$, so $(A, u_A, \leq_A)$ upper T_1-ordered.

It can be similarly shown for $[\leftarrow, a]_A$. Consequently, $(A, u_A, \leq_A)$ $\mathrm{T}_1 - $*ordered* space. □

Proposition 3. *Let $(X, u, \leq)$ and $(Y, v, \leq')$ are closure ordered spaces and $f : (X, u, \leq) \rightarrow (Y, v, \leq')$ is a continuous and order-embedding function. If $(Y, v, \leq')$ T_1-ordered space, then $(X, u, \leq)$ is T_1-ordered space.*

Proof. Let $(Y, v, \leq')$ be a T_1-ordered space and $a, b \in X$, $a \nleq b$. Because of f is order embedding, $f(a) \nleq' f(b)$. Hence $(Y, v, \leq')$ T_1-ordered space, there exists increasing neighbourhood U of $f(a)$ and decreasing neighbourhood V of $f(b)$ such that $f(b) \notin U$, $f(a) \notin V$.

$f^{-1}(U)$ is an increasing neighbourhood of a and $f^{-1}(V)$ is a decreasing neighbourhood of b, since f is an order-embedding and continuous function. $f(b) \notin U \Rightarrow b \notin f^{-1}(U)$ and $f(a) \notin V \Rightarrow a \notin f^{-1}(V)$. Consequently, we found an increasing neighbourhood $f^{-1}(U)$ of a such that $b \notin f^{-1}(U)$ and decreasing neighbourhood $f^{-1}(V)$ of b such that $a \notin f^{-1}(V)$, so $(X, u, \leq)$ is T_1- ordered. □

Definition 13. *Let $(X, u, \leq)$ be a closure ordered space. $t^{\uparrow} = \{A \subseteq X : u(A^c) = A^c$ and A is an increasing set$\}$, $t^{\downarrow} = \{A \subseteq X : u(A^c) = A^c$ and A is a decreasing set$\}$ are topological spaces on X. They are called upper and lower topology associated with $(X, u, \leq)$.*

Proposition 4. *Let $(X, u, \leq)$ be a closure ordered space. Then the followings are true,*

i) If $(X, t^{\uparrow}, \leq)$ is lower $T_1 -$ ordered space, then $(X, u, \leq)$ is lower T_1-ordered space.

ii) If $(X, t^{\downarrow}, \leq)$ is upper $T_1 -$ ordered space, then $(X, u, \leq)$ is upper T_1-ordered space.

iii) If $(X, t^{\uparrow}, \leq)$ is lower T_1-ordered and $(X, t^{\downarrow}, \leq)$ is upper T_1-ordered space, then $(X, u, \leq)$ is T_1-ordered space.

Proof. i) Let $(X, t^{\uparrow}, \leq)$ be a lower $\mathrm{T}_1 - ordered$ space. We will show that for each $x \in X$, $[\leftarrow, x]$ is closed. If we show the closure operator of $(X, t^{\uparrow}, \leq)$ with cl, cl is coarser than the operator u. So, we can write $u([\leftarrow, x]) \subseteq cl([\leftarrow, x]) = [\leftarrow, x] \Rightarrow u([\leftarrow, x]) = [\leftarrow, x] \Rightarrow (X, u, \leq)$ is lower T_1-ordered space.

ii) Let $(X, t^{\downarrow}, \leq)$ is upper $\mathrm{T}_1 - ordered$ space. We will show that for each $x \in X$ $[x, \rightarrow]$ is closed. Because of $(X, t^{\downarrow}, \leq)$ is upper $\mathrm{T}_1 - ordered$ space, we can write $u([x, \rightarrow]) \subseteq cl([x, \rightarrow]) = [x, \rightarrow] \Rightarrow (X, u, \leq)$ is upper T_1-ordered.

iii) it can be obtained from i) and ii). □

3. T_2-ORDERED CLOSURE ORDERED SPACES

In this section, we will give the definition of T_2-ordered closure ordered spaces and we will investigate some of its properties.

Definition 14. *Let $(X, u, \leq)$ be a closure ordered space. $(X, u, \leq)$ is called T_2-ordered closure space if and only if for each $a, b \in X \ni a \nleq b$, there exist an increasing neighbourhood U of a and decreasing neighbourhood V of b such that $U \cap V = \emptyset$.*

If $(X, u, \leq)$ is T_2-ordered, then $(X, u, \leq)$ is T_1-ordered.

Theorem 2. *Let $(X, u, \leq)$ closure ordered space. Then the followings are equivalent,*

i) $(X, u, \leq)$ is T_2-ordered

ii) For each $a, b \in X \ni a \nleq b$, there exist $U \in \mathcal{N}(a), V \in \mathcal{N}(b) \ni$ if $x \in U$ and $y \in V$, then $x \nleq y$

iii) The graph of the partial order of X is closed in product closure space $X \times X$.

Proof. $i) \Rightarrow ii)$ and $ii) \Rightarrow iii)$ is clear, so we will only prove $iii) \Rightarrow i)$

Let $a, b \in X$ and $a \nleq b$. The graph of the partial order is $G_{\leq} = \{(x, y) : x \leq y\}$.

$a \nleq b \Rightarrow (a, b) \notin G_{\leq} \Rightarrow (a, b) \notin u(G_{\leq}) \Rightarrow U \in \mathcal{N}(a), V \in \mathcal{N}(b)$ such that $(U \times V) \cap G_{\leq} = \emptyset$. Let say $U' = i(U)$ and $V' = d(V)$, then U' and V' are increasing and decreasing neighbourhood of a and b, respectively.

$U' \cap V' = i(U) \cap d(V) = \emptyset$. To show that, suppose $i(U) \cap d(V) \neq \emptyset$. Then, there exists a such that $a \in i(U) \cap d(V) \Rightarrow a \in i(U)$ and $a \in d(V) \Rightarrow x \in U : x \leq a$ and $y \in U : a \leq y \Rightarrow Hence \leq$ is a transitive relation, $x \leq y$. But this contradicts with $(U \times V) \cap G_{\leq} = \emptyset$, so $i(U) \cap d(V) = \emptyset$. □

Proposition 5. *Every subspace of a T_2-ordered closure ordered space is a T_2-ordered space.*

Proof. It can be obtained smilar to Proposition 2. □

Proposition 6. *Let $(X, u, \leq)$ and $(Y, v, \leq')$ are closure ordered spaces and $f : (X, u, \leq) \rightarrow (Y, v, \leq')$ is a continuous and order-embedding function. If $(Y, v, \leq')$ T_2-ordered space, then $(X, u, \leq)$ is T_2-ordered space.*

Proof. It can be obtained similar to Proposition 3. □

Proposition 7. *Let $(X,u,\leq)$ closure ordered space. If $(X,t^{\uparrow},\leq)$ or $(X,t^{\downarrow},\leq)$ is T_2-space, then $(X,u,\leq)$ is T_2-ordered space.*

Proof. Let $(X,t^{\uparrow},\leq)$ be a $\mathrm{T}_2 - space$ and $a,b\in X$ such that $a\nleq b$, then $a\neq b$. Hence $(X,t^{\uparrow},\leq)$ is T_2-space, there exist disjoint open neighbourhoods U of a and V of b. Hence $U\in t^{\uparrow}$, U is an increasing neighbourhood of a and $d(V)$ is a decreasing neighbourhood of b such that $U\cap d(V)=\emptyset$. Otherwise, if $U\cap d(V)\neq\emptyset\Rightarrow\exists x\in U\cap d(V)\Rightarrow x\in U$ and $x\in d(V)$.

$x\in d(V)\Rightarrow\exists v\in V$ such that $x\leq v$. Hence U is increasing and $x\leq v$, $v\in U$, so $v\in U\cap V$ and $U\cap V\neq\emptyset$ which is a contradiction. Consequently, we found disjoint increasing neighbourhood U of a and decreasing neighbourhood $d(V)$ of b, so $(X,u,\leq)$ is T_2-ordered space. □

4. REGULAR–ORDERED CLOSURE ORDERED SPACES

In this section, we will give the definition of regular ordered topological space. Then, we will generalize McCartans $\tau - compatibly$ subspace definition and we will investigate some properties.

Definition 15. *Let $(X,u,\leq)$ be a closure ordered space.*

i) $(X,u,\leq)$ is called lower regular ordered if for each decreasing set $A\subseteq X$ and each element $x\notin u(A)$ there exist disjoint neighbourhoods U of x and V of A such that U is increasing and V is decreasing.

ii) $(X,u,\leq)$ is called upper regular ordered if for each increasing set $A\subseteq X$ and each $x\notin u(A)$ there exist disjoint neighbourhoods U of x and V of A such that U is decreasing and V is increasing.

If both of the conditions are satisfied, then $(X,u,\leq)$ will be called regular ordered closure ordered space.

$(X,u,\leq)$ is T_3-ordered$\Leftrightarrow$ $(X,u,\leq)$ regular ordered and T_1-ordered space.

Example 2. *Let $X=\{a,b,c\}$ and $\leq=\{(a,a),(b,b),(c,c),(a,b),(b,a)\}$ and*
$u:P(X)\to P(X)$ is defined such that,
$u(\{a\})=\{a\},u(\{b\})=\{a,b\},u(\{c\})=\{c\},u(\{b,c\})=X,u(\{a,c\})=\{a,c\}$,
$u(\{a,b\})=\{a,b\},u(X)=X,u(\emptyset)=\emptyset$. *$(X,u,\leq)$ is a regular ordered closure ordered space.*

Theorem 3. *Let $(X,u,\leq)$ be a closure ordered space. Then the followings are equivalent,*
i) $(X,u,\leq)$ is lower(upper) regular ordered closure space
ii) For each $x\in X$ and $U(V\in\mathcal{N}(x))\in\mathcal{N}(x)\ni U$ $(V)$is increasing (decreasing), there exists $U'(V')\in\mathcal{N}(x)$ increasing (decreasing) neighbouhoods such that $u(U')\subseteq U$ $(u(V')\subseteq V)$.

Proof. *i) $\Rightarrow$ ii)* Let $(X,u,\leq)$ be a lower regular ordered space and $x\in X$ and U be an increasing neighbourhood of x. $U\in\mathcal{N}(x)\Rightarrow x\in U\Rightarrow x\notin U^c\Rightarrow x\notin u(U^c)$.

Suppose $x \in u(U^c)$, then $U \in \mathcal{N}(x)$ and $U \cap U^c = \emptyset$ which is a contradiction. So, $x \notin u(U^c)$. Hence $(X, u, \leq)$ is lower regular *ordered*, there exist

an increasing neighbourhood V_1 of x and decreasing neighbourhood V_2 of U^c such that $V_1 \cap V_2 = \emptyset$. $V_2 \in \mathcal{N}(U^c) \Rightarrow U^c \subseteq int_u(V_2) = (u(V_2^c))^c \Rightarrow u(V_2^c) \subseteq U$. Hence $V_1 \cap V_2 = \emptyset$, $V_1 \subseteq V_2^c$ and $V_2^c \in \mathcal{N}(x)$ since $V_1 \in \mathcal{N}(x)$. Consequently, we found an increasing neighbourhood V_2^c of x such that $u(V_2^c) \subseteq U$.

It can be similarly shown for upper regular ordered space.

ii) $\Rightarrow$ *i*) We will show $(X, u, \leq)$ is lower regular ord*ered* space. Let $A \subseteq X$ be a decreasing set and $x \notin u(A)$. $x \notin u(A) \Rightarrow \exists U \in \mathcal{N}(x)$: $U \cap A = \emptyset$. Hence $U \in \mathcal{N}(x)$, $i(U)$ is an increasing neighbourhood of x. From *ii*) there exists an increasing neighbourhood V of x such that $u(V) \subseteq i(U)$. $u(V) \subseteq i(U) \Rightarrow (i(U))^c \subseteq (u(V))^c = int_u(V^c)$ and $A \subseteq (i(U))^c$, since $U \cap A = \emptyset \Rightarrow U \subseteq A^c$ and A^c is increasing set, so $i(U) \subseteq i(A^c) = A^c \Rightarrow A \subseteq (i(U))^c \subseteq int_u(V^c)$. We found a decreasing neighbourhood V^c of A and an increasing neighbourhood V of x such that $V^c \cap V = \emptyset$, so $(X, u, \leq)$ is lower regular ord*ered* space. It can be similarly shown for upper regular ordered space. □

Proposition 8. *If $(X, u, \leq)$ T_3-ordered closure ordered space, then $(X, u, \leq)$ is a T_2-ordered closure space.*

Proof. Let $(X, u, \leq)$ T_3-ordered closure ordered space and $a, b \in X$, $a \nleq b$ holds. Because of $(X, u, \leq)$ is T_1-ordered, $[\leftarrow, b]$ is closed, so $u([\leftarrow, b]) = [\leftarrow, b]$ and $a \notin u([\leftarrow, b])$. There exist $U \in \mathcal{N}(a)$ increasing neighbourhood and $V \in \mathcal{N}([\leftarrow, b])$ decreasing neighbourhood such that $U \cap V = \emptyset$. $V \in \mathcal{N}([\leftarrow, b]) \Rightarrow [\leftarrow, b] \subseteq int_u(V) \Rightarrow V \in \mathcal{N}(b)$, so $(X, u, \leq)$ is T_2-ordered space. □

Proposition 9. *Let $(X, u, \leq)$ and $(Y, v, \leq')$ are closure ordered spaces and $f : (X, u, \leq) \to (Y, v, \leq')$ is continuous, open, order-embedding and surjective function. If $(X, u, \leq)$ regular-ordered space, then $(Y, v, \leq')$ is regular$-$ordered space.*

Proof. Let $(X, u, \leq)$ be a regular-ordered space and $A \subseteq Y$ is a decreasing set and $f(x) \notin v(A)$.By continuity of f,

$f(u(f^{-1}(A)) \subseteq v(A) \Rightarrow u(f^{-1}(A)) \subseteq f^{-1}(v(A)) \Rightarrow x \notin u(f^{-1}(A))$ and because of $(X, u, \leq)$ is a regular-ordered space, $\exists U \in \mathcal{N}(x) \ni U$ is increasing, $\exists V \in \mathcal{N}(f^{-1}(A)) \ni V$ is decreasing and $U \cap V = \emptyset$. *Hence*, f is open map $f(U)$ and $f(V)$ are neighbourhoods of x and A such that $f(U) \cap f(V) = \emptyset$. So, $(X, u, \leq)$ is lower regular-ordered, it can be similarly shown for upper regularity. Consequently, $(Y, v, \leq')$ is *regular*$-$ordered space. □

Remark 1. *Every subspace of a regular ordered closure space may not be regular ordered closure space.*

Example 3. *Let* $X = \{a, b, c, d\}$,
$\leq = \{(a, a), (b, b), (c, c), (d, d), (a, b), (b, c), (a, c), (d, a)\}$ *and* $u : P(X) \to P(X)$, $u(\{a\}) = \{a\}, u(\{b\}) = \{a, b, c\}, u(\{c\}) = \{c\}, u(\{d\}) = \{d\}, u(\{a, b\}) = \{a, b, c\}$,

$u(\{a,c\}) = \{a,c\}, u(\{a,d\}) = \{a,d\}, u(\{b,c\}) = \{a,b,c\},$
$u(\{b,d\}) = X, u(\{c,d\}) = \{c,d\}, u(\{a,b,c\}) = \{a,b,c\}, u(\{a,c,d\}) = \{a,c,d\},$
$u(\{a,b,d\}) = X, u(\{b,c,d\}) = X, u(X) = X, u(\emptyset) = \emptyset.$

$(X,u,\leq)$ *is regular ordered space. Let* $A = \{a,b,c\}$ *and* $\leq_A = \{(a,a),(b,b),(c,c),(a,b),(b,c),(a,c)\}$, $(A,u_A,\leq_A)$ *is not a regular ordered space. Because,* $\{b,c\} \in \mathcal{N}(b)$ *and* $\{b,c\}$ *is an increasing set, but there is no increasing neighbourhood of* b *which is contained by* $u(\{b,c\})$.

We will give a definition which is a generalization of S.D.Mc.Cartan's definition of $\tau-$compatibly ordered subspace.

Definition 16. *Let* $(X,u,\leq)$ *be a closure ordered space.* $(A,u_A,\leq_A)$ *be a subspace of* $(X,u,\leq)$. *If for each* $B \subseteq A$ *increasing (decreasing) set, there exists* $B' \subseteq X$ *increasing (decreasing) set such that* $B = B' \cap A$. *Then* $(A,u_A,\leq_A)$ *will be called* $u-$ *compatibly ordered subspace.*

Theorem 4. *Every compatibly subspace of a regular ordered closure ordered space is a regular ordered space.*

Proof. Let $(X,u,\leq)$ be a regular ordered space and $(A,u_A,\leq_A)$ *be* a u-com*patibly* ordered subspace, $x \in A$. Let for each $B \subseteq A$ decreasing set, $x \notin u_A(B)$. Since $(A,u_A,\leq_A)$ *is* a u-com*patibly* ordered *subspace* there exists $B' \subseteq X$ decreasing and $B = B' \cap A$. $u_A(B) = u_A(B' \cap A) = u_A(B') \cap u_A(A) = u(B') \cap A$ and $x \notin u_A(B) \Rightarrow x \notin u(B') \Rightarrow$ There exist $U \in \mathcal{N}(x)$ an increasing neighbourhood and $V \in \mathcal{N}(B')$ decreasing neighbourhood, $U \cap V = \emptyset$. Hence, $B \subseteq B' \Rightarrow V \in \mathcal{N}(B)$.

$(A,u_A,\leq_A)$ is a lower regular ordered and it can be similarly shown for upper regularity. So, $(A,u_A,\leq_A)$ is regular ordered space. □

5. NORMALLY-ORDERED CLOSURE ORDERED SPACES

In this section we will give the definition of normally ordered closure ordered spaces and we will investigate some properties.

Definition 17. *Let* $(X,u,\leq)$ *be a closure ordered space.* $(X,u,\leq)$ *is called normally ordered* $\Leftrightarrow \forall F_1, F_2$ *disjoint closed subsets of* X, *such that* F_1 *is increasing,*F_2 *is decreasing, there exist an increasing neighbourhood of* F_1, *decreasing neighbourhood of* F_2 , *respectively* U_1, U_2 *and* $U_1 \cap U_2 = \emptyset$.

$(X,u,\leq)$ T_4-ordered $\Leftrightarrow$ $(X,u,\leq)$ is T_1-ordered and normally ordered space.

Example 4. *Let* $X = \{a,b,c\}$, $\leq = \{(a,a),(b,b),(c,c),(a,b)\}$ *and*
$u : P(X) \to P(X)$,
$u(\{a\}) = \{a\}$, $u(\{b\}) = \{a,b\}$, $u(\{c\}) = \{c\}$, $u(\{b,c\}) = X$,
$u(\{a,c\}) = \{a,c\}$, $u(\{a,b\}) = \{a,b\}$, $u(X) = X$, $u(\emptyset) = \emptyset$. *Then,* $(X,u,\leq)$ *is a normally ordered closure ordered space.*

Theorem 5. *Let* $(X,u,\leq)$ *be a closure ordered space. Then the followings are equivalent,* *i)* $(X, u, \leq)$ *is normally ordered*

ii) For each increasing(decreasing) closed set F *and each increasing(decreasing) open set* U *such that* $F \subseteq U$*, there exists an increasing(decreasing) neighbourhood of* F *such that* $u(V) \subseteq U$.

Proof. *i)* $\Rightarrow$ *ii)* Let $(X, u, \leq)$ be a normally ordered space and $F \subseteq X$ increasing closed set and $F \subseteq U$ such that U is an increasing open set, so $F \cap U^c = \emptyset$. Since $(X, u, \leq)$ is normally ordered, there exist an increasing neighbourhood of F and decreasing neighbourhood of U^c, respectively V_1, V_2 and $V_1 \cap V_2 = \emptyset$.

We can write $F \subseteq int_u(V_1)$ and $U^c \subseteq int_u(V_2) \Rightarrow (int_u(V_2))^c \subseteq U \Rightarrow u(V_2^c) \subseteq U$. We find an increasing neighbourhood of F and $u(V_2^c) \subseteq U$ holds.

ii) $\Rightarrow$ *i)* We will show that $(X, u, \leq)$ is normally ordered. Let F_1 and F_2 are disjoint closed sets such that F_1 is increasing and F_2 is decreasing. Hence, $F_1 \cap F_2 = \emptyset \Rightarrow F_1 \subseteq F_2^c$ and F_2^c is an increasing open set. From *ii)*, there exists an increasing neighbourhood U of F_1, such that $u(U) \subseteq F_2^c \Rightarrow F_2 \subseteq (u(U))^c = int_u(U^c)$, so U^c is a decreasing neighbourhood of F_2. We found an increasing neighbourhood U of F_1 and decreasing neighbourhood U^c of F_2 such that $U \cap U^c = \emptyset$. Consequently, $(X, u, \leq)$ is a normally ordered space. □

Proposition 10. *Let* $(X, u, \leq)$ *be a normally ordered space and* $Y \subseteq X$ *be a closed subspace and* $Y = i(Y) = d(Y)$*. Then,* $(Y, u_Y, \leq_Y)$ *is a normally ordered subspace.*

Proof. Let F_1, F_2 disjoint closed sets in Y such that F_1 is increasing, F_2 is decreasing. Then F_1 and F_2 are disjoint closed sets in X. Because of $(X, u, \leq)$ is normally ordered, there exist an increasing neighbourhood of F_1 and a decreasing neighbourhood of F_2, respectively U_1, U_2 and $U_1 \cap U_2 = \emptyset$. Then, $U_1 \cap Y$ is an increasing neighbourhood of F_1 in Y, $U_2 \cap Y$ is a decreasing neighbourhood of F_2 and $(U_1 \cap Y) \cap (U_2 \cap Y) = \emptyset$. Consequently, $(Y, u_Y, \leq_Y)$ is a normally ordered subspace. □

Proposition 11. *Let* $(X, u, \leq)$ *and* $(Y, v, \leq')$ *are closure ordered spaces and* $f : (X, u, \leq) \to (Y, v, \leq')$ *is a closed, continuous and order-embedding function. If* $(Y, v, \leq')$ *normally-ordered space, then* $(X, u, \leq)$ *is normally−ordered space.*

Proof. Let F_1 and F_2 be a disjoint closed subsets of X such that F_1 is an increasing set and F_2 is a decreasing set . Then, $f(F_1)$ and $f(F_2)$ are closed sets in Y such that $f(F_1)$ is increasing and $f(F_2)$ is decreasing. Because of $(Y, v, \leq')$ normally-ordered space, there exist an increasing neighbourhood of $f(F_1)$ and decreasing neighbourhood of $f(F_2)$, respectively U_1, U_2 such that $U_1 \cap U_2 = \emptyset$, so $f^{-1}(U_1) \in \mathcal{N}(F_1) \ni f^{-1}(U_1)$ is increasing, $f^{-1}(U_2) \in \mathcal{N}(F_2) \ni f^{-1}(U_2)$ is decreasing and $f^{-1}(U_1) \cap f^{-1}(U_2) = \emptyset$. Consequently, $(X, u, \leq)$ is *normally*−ordered space. □

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Current address: Irem Eroglu, Department of Mathematics, Faculty of Sciences, Ankara University Tandogan, Ankara, TURKEY

E-mail address: `ieroglu@ankara.edu.tr`

Current address: Erdal Guner, Department of Mathematics, Faculty of Science, Ankara University, Tandogan, Ankara, TURKEY

E-mail address: `guner@science.ankara.edu.tr`