Journal of Engineering Technology and Applied Sciences, 2019 e-ISSN: 2548-0391
Received 12 June 2019, Accepted 23 August 2019 Vol. 4, No. 2, 71-83
Published online 26 August 2019 doi: 10.30931/jetas.614506

Citation: Kurulay, M., Senturk, I., "Numerical Solution of Time and Space Fractional Burger's Equation with Finite Difference Method". Journal of Engineering Technology and Applied Sciences 4 (2) 2019 : 71-83.

NUMERICAL SOLUTION OF TIME AND SPACE FRACTIONAL BURGER'S EQUATION WITH FINITE DIFFERENCE METHOD

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Abstract

In this study, fractional Burger's Equation, which has Dirichlet Boundary Conditions, is solved with the Finite Difference Method. Fractional Burger Equation is found by S. Momani, which is made with changing time and space terms with fractional terms. This equation is solved with the finite difference method and analysis of this scheme is discussed with examples. Stability and Uniqueness are discussed with using matrix method. We compare analytical and numerical solutions with error analysis of them.

Keywords: Finite difference method, Burger's equation, fractional derivative

1. Introduction

Burgers' equation [1] is a famous non-linear equation for physics problems. The problem has Dirichlet boundary conditions. With changing the order of differential terms of the equation with fractional order, we can achieve the fractional Burger's Equation [12] which was formulated by S. Momani. The following equation is fractional Burger's Equation:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u\frac{\partial u}{\partial x} = v\frac{\partial^\beta u}{\partial x^\beta} \qquad a \le x \le b \;,\; 0 < t \le T \;, \qquad 0 < \alpha < 1 \,,\, 1 < \beta < 2 \tag{1}$$

The problem has the following conditions:

$$
\begin{aligned}
&\text{boundary conditions:}\\
&u(a,t) = f_1(t), \quad u(b,t) = f_2(t)\\
&\text{initial condition:}\\
&u(x,0) = f(x)
\end{aligned} \tag{1}
$$

where $v > 0$ is the viscosity constant, $f_1(x)$, $f_2(x)$ and $f(x)$ are the functions of x. There are many studies about the solving of Burger's and Fractional Burger's equations. Some finite difference approximations are found in the literature[16-17]. For example Zhang and Wang, Kutluay and Bahadir and Özdes, Pandey and Verma studied on finite difference method for burger's equation [3, 5 and 6], Varöglu and LiamFinn studied with finite elements method for solve Burger's equation [4]. Momani and Kurulay have studies about time and space fractional solution of Burger's equations [14-15]. Asaithambi, Hon, Mao, Asaithambi and Mena studied about Burger's equations with using different methods [9, 10 and 11].

2. Numerical method

Fractional Calculus:

We can define the fractional calculus as the expand of differential and integral terms with non-integer orders. Caputo and Riemann-Liouville fractional derivatives are used in the approximation of the solving of partial differential equations [2,19].

Gamma Function:

The gamma function is the expand of factorial to real numbers. The following expression is general form of gamma function;

$$\Gamma(z) = \int_0^\infty e^{-u} u^{z-1} du \quad , \quad z \in R \tag{3}$$

Riemann-Liouville fractional derivative:

We know the following expression as Cauchy integral

$$\int_a^x dt_1 \int_a^{t_1} dt_2 \ldots \int_a^{t_{n-1}} f(t_n) dt_n = \frac{1}{\Gamma(n)} \int_a^x (x-t)^{n-1} f(t) dt \tag{4}$$

If we change the n term with q, the q can be a real number; we can achieve the Riemann-Liouville fractional integral;

$$\frac{d^q f}{d(x-a)^q} = \frac{1}{\Gamma(-q)} \int_a^x (x-t)^{-q-1} f(t) dt \qquad ; q < 0 \tag{5}$$

With some changings in the Equation (5) we can achieve the Riemann-Liouville fractional derivative:

$$\frac{d^q f}{d(x-a)^q} = \frac{d^n}{dx^n}\left[\frac{1}{\Gamma(n-q)}\int_a^x (x-t)^{-(q-n)-1} f(t)dt\right] \tag{6}$$

In this expression $n > q$ and $q \geq 0$.

Caputo Fractional derivative:

The other approach of fractional calculus is caputo's approach. If we want to use physical conditions effectively we can use Caputo fractional derivative. The physical conditions are same in integer orders between Caputo and normal derivative. The following equation is a general form of Caputo fractional derivative:

$${}_a^C D_t^\alpha = \frac{1}{\Gamma(\alpha-n)}\int_a^t \frac{f^{(n)}(\tau)d\tau}{(t-\tau)^{\alpha+1-n}}, \qquad (n-1<\alpha<n) \tag{7}$$

Finite Difference Method:

In our approximation we use the following forms of Caputo and Riemann-Liouville fractional derivatives:

Caputo fractional derivative:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\eta)^{-\alpha}\frac{\partial u(x,\eta)}{\partial \eta}d\eta$$

Riemann-Liouville derivative:

$$\frac{\partial^\beta u(x,t)}{\partial t^\beta} = \frac{1}{\Gamma(2-\beta)}\frac{\partial^2}{\partial x^2}\int_0^t \frac{u(\xi,t)}{(x-\xi)^{\beta-1}}d\xi \tag{8}$$

To find the finite difference scheme of fractional burgers equations, we choose grid size to Δx for the space of this problem and we can find, then we can find integration time as $\tau = \frac{t}{n}$. $0 < t_k < T$ for this problem, ($k = 0,1,.....,n$) and $x_i = ih$ for this problem $(i = 0,1,.....,m)$. In the scheme we write U_i^k for $U(x_i,t_k)$. For writing the scheme we change time derivative term to time fractional derivative term;

$$\begin{aligned}\frac{\partial^\alpha u_i^{k+1}}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{u_i^{j+1}-u_i^j}{\tau}\int_{j\tau}^{(j+1)\tau}\frac{d\gamma}{(t_{k+1}-\gamma)^\alpha}+o(\tau) \\ &= \frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{k}\frac{u_i^{k+1-j}-u_i^{k-j}}{\tau}[(j+1)^{1-\alpha}-j^{1-\alpha}]+O(\tau).\end{aligned} \tag{9}$$

Then with using Riemann-Liouville fractional derivative, we can find the space fractional term of these problem as:

$$\frac{\partial^\beta u_i^{k+1}}{\partial x^\beta} = \frac{1}{h^\beta}\sum_{j=0}^{i+1} g_j u_{i-(j-1)h}^{k+1} + O(\tau+h) \tag{10}$$

In (10) g is a special function of β *and* j, writing in (12). Finally with applying (9) and (10) to fractional Burger's equation [7-8], we can write the following finite difference scheme;

$$\frac{\Delta t}{\Gamma(2-\alpha)}\sum_{j=0}^{k}\frac{u_i^{k+1-j}-u_i^{k-j}}{\Delta x}[(j+1)^{1-\alpha}-j^{1-\alpha}]=-u_i^k\,\frac{u_i^k-u_{i-1}^k}{\Delta x}+\frac{v}{(\Delta x)^{\beta}}\sum_{j=0}^{i+1}g_i u_{i-j+1}^{k+1}. \tag{11}$$

Take some terms as special terms for convenience[18].

$$\begin{aligned}
&\sigma_j=(j+1)^{1-\alpha}-j^{1-\alpha}\\
&p_i=\frac{\Gamma(2-\alpha)(\Delta t)^{\alpha}}{\Delta x}\\
&r_i=\frac{v}{(\Delta x)^{\beta}}(\Delta t)^{\alpha}\Gamma(2-\alpha)\\
&g_j=(-1)^j\,\frac{\beta\cdot(\beta-1)\cdot(\beta-2)......(\beta-j+1)}{j!},\qquad g_0=1,\quad g_1=-\beta.
\end{aligned} \tag{12}$$

If we write this changing, we can achieve the following scheme;

$$\sum_{j=0}^{k}\sigma_j(u_i^{k+1-j}-u_i^{k-j})=-p_i(u_i^k(u_i^k-u_{i-1}^k))+r_i\sum_{j=0}^{i+1}g_j u_{i-j+1}^{k+1}. \tag{13}$$

For $k=0$ we can get this expression;

$$u_i^1-u_i^0+p_i(u_i^0(u_i^0-u_{i-1}^0))=r_i\sum_{j=0}^{i+1}g_i u_{i-j+1}^1. \tag{14}$$

$$u_i^1-u_i^0+p_i(u_i^0(u_i^0-u_{i-1}^0))=r_i u_{i+1}^1+r_i u_i^1 g_1+r_i g_2 u_{i-1}^1+r_i\sum_{j=3}^{i+1}g_i u_{i-j+1}^1. \tag{15}$$

$$-r_i u_{i+1}^1+(1-r_i g_1)u_i^1-r_i g_2 u_{i-1}^1-r_i\sum_{j=3}^{i+1}g_i u_{i-j+1}^1 u_i^1=-p_i(u_i^0(u_i^0-u_{i-1}^0))+u_i^0. \tag{16}$$

Then for $k>0$, we can get this expression;

$$\begin{aligned}
\sigma_0(u_i^{k+1}-u_i^k)+\sum_{j=1}^{k}\sigma_j u_i^{k+j-1}-u_i^{k-j}+p_i(u_i^k(u_i^k-u_{i-1}^k)=r_i u_{i+1}^{k+1}+r_i g_1 u_i^{k+1}+r_i g_2 u_{i-1}^{k+1}\\
+r_i\sum_{j=3}^{i+1}g_j u_{i-j+1}^{k+1}.
\end{aligned} \tag{17}$$

If we take $p_i(u_i^k(u_i^k-u_{i-1}^k)$ and do some regulations,

$$-r_i u_{i+1}^{k+1} + (1-r_i g_1)u_i^{k+1} - r_i g_2 u_{i-1}^{k+1} - r_i \sum_{j=3}^{i+1} g_j u_{i-j+1}^{k+1} = u_i^k - \sum_{j=1}^{k} \sigma_j (u_i^{k+j-1} - u_i^{k-j}) - P$$

$$= u_i^k - \sum_{j=1}^{k} \sigma_j u_i^{k+j-1} + \sum_{j=1}^{k} \sigma_j u_i^{k-j} - P \qquad (18)$$

$$= u_i^k - \sum_{j=0}^{k-1} \sigma_{j+1} u_i^{k+j} + \sum_{j=1}^{k} \sigma_j u_i^{k-j} - P.$$

$$-r_i u_{i+1}^{k+1} + (1-r_i g_1)u_i^{k+1} - r_i g_2 u_{i-1}^{k+1} - r_i \sum_{j=3}^{i+1} g_j u_{i-j+1}^{k+1} = u_i^k - \sigma_1 u_i^k + \sigma_k u_i^0 + \sum_{j=1}^{k-1} (-\sigma_{j+1} + \sigma_j) u_i^{k-j} - P$$

$$= \sigma_k u_i^0 + (2 - 2^{1-\alpha}) u_i^k \qquad (19)$$

$$+ \sum_{j=1}^{k-1} u_i^{k-j} (2(j+1)^{1-\alpha} - (j+2)^{1-\alpha} - j^{1-\alpha}) - P.$$

Finally for $k > 0$ our difference scheme is;

$$-r_i u_{i+1}^{k+1} + (1-r_i g_1)u_i^{k+1} - r_i g_2 u_{i-1}^{k+1} - r_i \sum_{j=3}^{i+1} g_j u_{i-j+1}^{k+1} = \sigma_k u_i^0 + (2 - 2^{1-\alpha}) u_i^k$$

$$+ \sum_{j=1}^{k-1} u_i^{k-j} (2(j+1)^{1-\alpha} - (j+2)^{1-\alpha} - j^{1-\alpha}) \qquad (20)$$

$$- p_i (u_i^k (u_i^k - u_{i-1}^k).$$

If we want to write this difference scheme as a algebraic equation system;

$$\begin{cases} AU^1 &= -p_i u_i^0 (u_i^0 - u_{i-1}^0) + u_i^0 \\ AU^{k+1} &= d_1 U^k + d_2 U^{k-1} + ... + d_k U^1 + \sigma_k U^0 - p_i (u_i^k (u_i^k - u_{i-1}^k), \quad k > 0 \\ U^0 &= f \end{cases} \qquad (21)$$

In this system;

$$\begin{aligned} d_j &= 2j^{1-\alpha} - (j+1)^{1-\alpha} - (j-1)^{1-\alpha}, \qquad j = 1,2,.....,k \\ \sigma_j &= (j+1)^{1-\alpha} - j^{1-\alpha} \\ p_i &= \frac{\Gamma(2-\alpha)(\Delta t)^\alpha}{\Delta x} \end{aligned} \qquad (22)$$

A is a matrix which has the coefficient of unknown terms for our problem;

$$A_{ij} = \begin{cases} -r_i g_{i-j+1}, & j < i-1 \\ -r_i g_2, & j = i-1 \\ 1 - r_i g_1, & j = i \\ -r_i, & j = i+1 \\ 0, & j > i+1 \end{cases} \qquad (23)$$

In this matrix;

$$r_i = \frac{v}{(\Delta x)^{\beta}}(\Delta t)^{\alpha}\Gamma(2-\alpha)$$
$$g_j = (-1)^j \frac{\beta\cdot(\beta-1)\cdot(\beta-2)......(\beta-j+1)}{j!}, \qquad g_0 = 1, \quad g_1 = -\beta \tag{24}$$

An example for Aij matrix for i and j from 1 to 10;

$$A_{ij} = \begin{pmatrix}
1-r_ig_1 & -r_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-r_ig_2 & 1-r_ig_1 & -r_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-r_ig_3 & -r_ig_2 & 1-r_ig_1 & -r_i & 0 & 0 & 0 & 0 & 0 & 0 \\
-r_ig_4 & -r_ig_3 & -r_ig_2 & 1-r_ig_1 & -r_i & 0 & 0 & 0 & 0 & 0 \\
-r_ig_5 & -r_ig_4 & -r_ig_3 & -r_ig_2 & 1-r_ig_1 & -r_i & 0 & 0 & 0 & 0 \\
-r_ig_6 & -r_ig_5 & -r_ig_4 & -r_ig_3 & -r_ig_2 & 1-r_ig_1 & -r_i & 0 & 0 & 0 \\
-r_ig_7 & -r_ig_6 & -r_ig_5 & -r_ig_4 & -r_ig_3 & -r_ig_2 & 1-r_ig_1 & -r_i & 0 & 0 \\
-r_ig_8 & -r_ig_7 & -r_ig_6 & -r_ig_5 & -r_ig_4 & -r_ig_3 & -r_ig_2 & 1-r_ig_1 & -r_i & 0 \\
-r_ig_9 & -r_ig_8 & -r_ig_7 & -r_ig_6 & -r_ig_5 & -r_ig_4 & -r_ig_3 & -r_ig_2 & 1-r_ig_1 & -r_i \\
-r_ig_{10} & -r_ig_9 & -r_ig_8 & -r_ig_7 & -r_ig_6 & -r_ig_5 & -r_ig_4 & -r_ig_3 & -r_ig_2 & 1-r_ig_1
\end{pmatrix} \tag{25}$$

3. Stability and uniqueness

Theorem 3.1 The implicit system defined by the linear difference equations (19) and (20) has a unique solution and is unconditionally stable for all $0 < \alpha < 1,\ 1 < \beta < 2$.

Proof. By applying the Gerschgorin theorem we decided that each eigenvalue of matrix A had a magnitude greater than 1.

Note this $g_0 = 1,\ g_1 = -\beta,\ g_j = (-1)^j \frac{\beta(\beta-1)...(\beta-j+1)}{j!},\ j = 1,2,3,...,$ then for $1 < \beta \le 2$, and $j \ge 2$, we have $g_j \ge 0$. Also, with well-known results that for any $\gamma > 0$,

$$(1+z)^{\gamma} = \sum_{m=0}^{\infty} \binom{\gamma}{m} z^m \ , \ |z| \le 1, \tag{26}$$

Substituting $z = -1$ into (26) yields $\sum_{j=0}^{\infty} g_j = 0$, and then $-g_1 > \sum_{j=0, j\ne 1}^{\infty} g_j$,i.e. $\sum_{j=0}^{\infty} g_j < 0$ for any $I = 1,2,3, \ldots\ldots m$. According to the Gerschgorin theorem, the eigenvalues of the matrix A are in the disks centered at $A_{i,i} = 1 - r_ig_1 = 1 + r_i\beta$ with radius

$$R_i = \sum_{j=1, j\ne 1}^{m-1} |A_{i,j}| = \sum_{j=1, j\ne 1}^{i+1} |A_{i,j}| = \sum_{j=1}^{i-2} |-r_ig_{i-j+1}| + |-r_ig_2| + |-r_i| = r_i \sum_{j=0, j\ne 1}^{i-2} g_j < -r_ig_1 \le r_i\beta.$$

Hence, each eigenvalue λ of the matrix A has a real part which is greater than one, and therefore has a magnitude greater than one. Therefore, the spectral radius of A^{-1} is less than one. This proves that the scheme has a unique solution.

To prove unconditional stability of (19) and (20) let $u_i^k, \tilde{u}_i^k (i=1,2,3,...,m-1, k=1,2,...n-1)$ be the solution of (19) and (20) with initial value $u_i^0, \tilde{u}_i^0$ respectively, the computation of $q_i^k (i=1,2,...,m-1, k=1,2,...,n-1)$ is exact. Then error $\varepsilon_i^k = \tilde{u}_i^k - u_i^k$ satisfies if k=0,

$$-r_i\varepsilon_{i+1}^1 + (1-r_ig_1)\varepsilon_i^1 - (r_ig_2)\varepsilon_{i-1}^1 - r_i\sum_{j=3}^{i+1} g_i\varepsilon_{i-j+1}^1 = \varepsilon_i^0$$

if $k > 0$,

$$-r_i\varepsilon_{i+1}^{k+1} + (1-r_ig_1)\varepsilon_i^{k+1} - (r_ig_2)\varepsilon_{i-1}^{k+1} - r_i\sum_{j=3}^{i+1} g_i\varepsilon_{i-j+1}^{k+1} = d_1\varepsilon_i^k + \sum_{j=1}^{k-1} d_{j+1}\varepsilon_i^{k-j} + \sigma_k\varepsilon_i^0.$$

Equivalent to the following matrix form:

$$AE^1 = E^0,\ AE^{k+1} = d_1E^k + d_2E^{k-1} + ... + d_kE^1 + \sigma_kE^0,\ k>0$$

where $E^k = (\varepsilon_1^k, \varepsilon_2^k, ..., \varepsilon_{m-1}^k)^T$. Let us use mathematical induction to prove $\| E^k \|_\infty \le \| E^0 \|_\infty, k=1,2,...$ In fact, if k=1, suppose $|\varepsilon_l^1| = \max_{1\le i\le m-1} |\varepsilon_i^1|$, note that $r_i, p_i > 0$ and for any integer number $N, \sum_{j=0}^{\infty} g_j < 0$, we have

$$\begin{aligned}\| E^1 \|_\infty &= |\varepsilon_l^1| \le |\varepsilon_l^1| + p_l(|\varepsilon_l^1| - |\varepsilon_{l-1}^1|) - r_l\left(\sum_{j=0}^{l+1} g_j\right)|\varepsilon_l^1| \\ &\le -r_l|\varepsilon_l^1| + (1-r_lg_1)|\varepsilon_l^1| - (r_lg_2)|\varepsilon_{l-1}^1| - r_l\left(\sum_{j=3}^{l+1} g_j\right)|\varepsilon_{l-j+1}^1| \\ &\le -r_l\varepsilon_l^1 + (1-r_lg_1)\varepsilon_l^1 - (r_lg_2)\varepsilon_{l-1}^1 - r_l\left(\sum_{j=3}^{l+1} g_j\right)\varepsilon_{l-j+1}^1 | = |\varepsilon_l^0| \le \| E^0 \|.\end{aligned}$$

Therefore $\| E^1 \|_\infty \le \| E^0 \|_\infty$. Suppose if $k \le s, \| E^s \|_\infty \le \| E^0 \|_\infty$ hold, then when $k = s+1$, let $|\varepsilon_l^{s+1}| = \max_{1\le i\le m-1} |\varepsilon_i^{s+1}|$, notice that $\sum_{j=0}^{i} g_j < 0, i=1,2,...,m$ similar to previous estimate, we have

$$\|E^{s+1}\|_\infty = |\varepsilon_l^{s+1}| \le -r_l|\varepsilon_{l+1}^{s+1}| + (1-r_lg_1)|\varepsilon_l^{s+1}| - (r_lg_2)|\varepsilon_{l-1}^{s+1}| - r_l\sum_{j=3}^{l+1} g_j|\varepsilon_{l-j+1}^{s+1}|$$

$$\le -r_l\varepsilon_{l+1}^{s+1} + (1-r_lg_1)\varepsilon_l^{s+1} - (r_lg_2)\varepsilon_{l-1}^{s+1} - r_l\sum_{j=3}^{l+1} g_j\varepsilon_{l-j+1}^{s+1}| \le \|AE^{s+1}\|_\infty$$

$$\le d_1|\varepsilon_l^s| + \sum_{j=1}^{s-1} d_{j+1}|\varepsilon_l^{s-j}| + |\varepsilon_l^0|\cdot\left[(s+1)^{1-\alpha} - s^{1-\alpha}\right]$$

$$\le d_1\|E^{s-j}\|_\infty + \sum_{j=1}^{s-1} d_{j+1}\|E^{s-j}\|_\infty + \left[(s+1)^{1-\alpha} - s^{1-\alpha}\right]\cdot\|E^0\|_\infty$$

$$\le \left(d_1 + \sum_{j=1}^{s-1} d_{j+1} + \left[(s+1)^{1-\alpha} - s^{1-\alpha}\right]\right)\cdot\|E^0\|_\infty = \|E^0\|_\infty$$

Hence, $\|E^{s+1}\|_\infty \le \|E^0\|_\infty$ so the implicit scheme defined by the linear difference equations (19) and (20) is unconditionally stable and Theorem 3.1 completes the proof.
Denote $e_i^k = u(x_i,t^k) - u_i^k$ and $e^k = (e_1^k, e_2^k, \ldots, e_{m-1}^k)^T$, we have Theorem 3.2.

Theorem 2. Suppose that $u(x_i,t_k)$ is the exact solution of (1) at grid point (x_i,t_k), u_i^k is the difference solution of (19), (20), then there exists positive constant *M*, such that

$$\|e^k\|_\infty \le \sigma_{k-1}^{-1} M(\tau^{1+\alpha} - \tau^\alpha h),\ k = 1,2,\ldots,n \tag{27}$$

where $\|e^k\|_\infty = \max_{1\le i\le m-1}|e_i^k|$, *M* is a constant independent of *h* and τ .

Proof. Since $u_i^k = u(x_i,t^k) - e_i^k$, notice that $e^0 = 0$, we have from (19) and (20), if *k*=0,

$$-r_ie_{i+1}^1 + (1-r_ig_1)e_i^1 - (r_ig_2)e_{i-1}^1 - r_i\sum_{j=3}^{i+1} g_ie_{i-j+1}^1 = R_i^1$$

if $k > 0$,

$$-r_ie_{i+1}^{k+1} + (1-r_ig_1)e_i^{k+1} - (r_ig_2)e_{i-1}^{k+1} - r_i\sum_{j=3}^{i+1} g_ie_{i-j+1}^{k+1} = d_1e_i^k + \sum_{j=1}^{k-1} d_{j+1}e_i^{k-j} + R_i^{k+1} \ ,$$

where $\|R_i^{k+1}\|_\infty \le M(\tau^{1+\alpha} - \tau^\alpha h),\ i = 1,2,\ldots,m-1,\ k = 1,2,\ldots,n-1,$ *M* is positive constant independent of *h* and τ . Let's use mathematical induction to prove the theorem. If *k*=1, suppose $\|e_l^1\|_\infty = \max_{1\le i\le m-1}|e_l^1|$, we have

$$\| e^1 \|_\infty = | e_l^1 | \le -r_l | e_{l+1}^1 | + (1 - r_l g_1) | e_l^1 | - (r_l g_2) | e_{l-1}^1 | - r_l \sum_{j=3}^{l+1} g_j | e_{l-j+1}^1 |$$

$$\le -r_l e_{l+1}^1 + (1 - r_l g_1) e_l^1 - (r_l g_2) e_{l-1}^1 - r_l \sum_{j=3}^{l+1} g_j e_{l-j+1}^1 | = | R_l^1 | \le M(\tau^{1+\alpha} + \tau^\alpha h) = \sigma_0^{-1} M(\tau^{1+\alpha} + \tau^\alpha h).$$

Suppose that if $k \le s$, $\| e^s \|_\infty \le \sigma_{s-1}^{-1} M(\tau^{1+\alpha} + \tau^\alpha h)$ hold, then when $k = s+1$, let $| e_l^{s+1} | = \max_{1 \le i \le m-1} | e_i^{s+1} |$, notice that $\sigma_j^{-1} < \sigma_k^{-1}$, $j = 0,1,2,...,k$ and $\sum_{j=0}^{i} g_j < 0, i = 1,2,...,m$ Therefore

$$\| e^{s+1} \|_\infty = | e_l^{s+1} | \le d_1 \| e^s \|_\infty + \sum_{j=1}^{s-1} d_{j+1} \| e^{s-j} \|_\infty + M(\tau^{1+\alpha} + \tau^\alpha h) = \sum_{j=0}^{s-1} d_{j+1} \| e^{s-j} \|_\infty + M(\tau^{1+\alpha} + \tau^\alpha h)$$

$$\le (d_1 \sigma_{s-1}^{-1} + d_2 \sigma_{s-2}^{-1} + ... + d_s \sigma_0^{-1} + 1) M(\tau^{1+\alpha} + \tau^\alpha h)$$

$$\le \sigma_s^{-1} \left(\sum_{i=0}^{s-1} d_{i+1} + \sigma_s \right) M(\tau^{1+\alpha} + \tau^\alpha h) = \sigma_s^{-1} M(\tau^{1+\alpha} + \tau^\alpha h).$$

Therefore Theorem 3.2 is proved.

4. Numerical examples

Example 4.1

$$\frac{\partial^\alpha u}{\partial t^\alpha} + u \frac{\partial u}{\partial x} = v \frac{\partial^\beta u}{\partial x^\beta} \qquad 0 \le x \le 1 \;,\; 0 < t \le 0,1 \;,$$

$0 < \alpha < 1$, $1 < \beta < 2$

boundary condutions:

$u(0,t) = 0, \quad u(1,t) = 0$ (28)

initial condution:

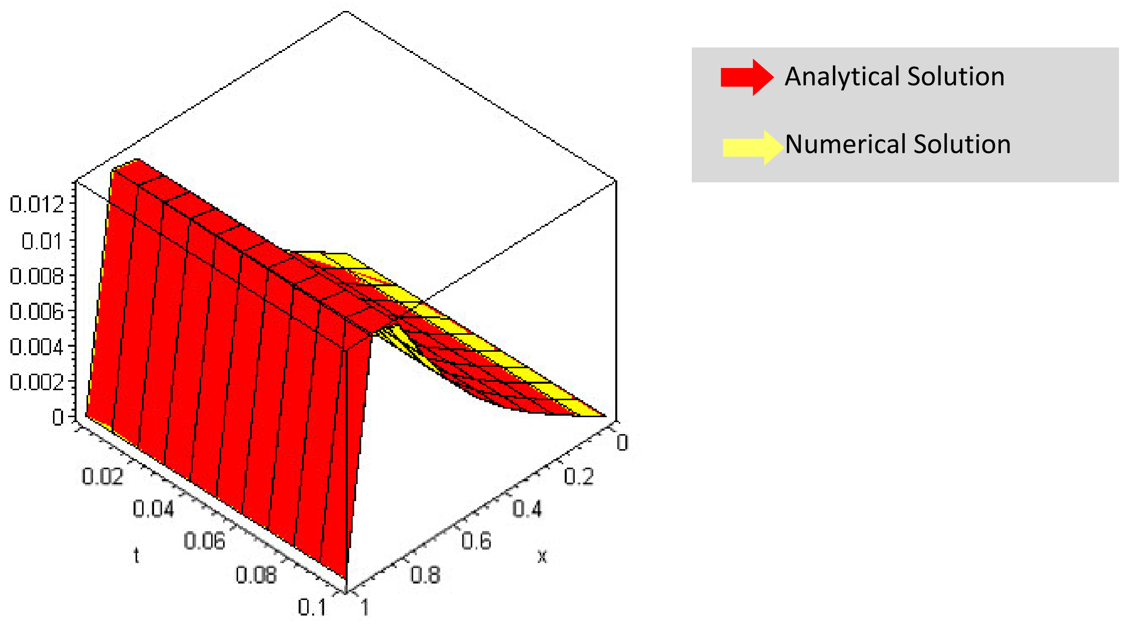
$$u(x,0) = \frac{2 v \pi \sin(\pi x)}{a + \cos(\pi x)}, \qquad a > 1$$

We can solve numerically this problem in following conditions,

$$v = 0.001, \quad a = 1.1, \quad \alpha = 0.95, \quad \beta = 1.95$$

Then the analytical solution of Example 4.1 is:

$$u(x,t) = \frac{2 v \pi \exp(-\pi^2 v t) \sin(\pi x)}{a + \exp(-\pi^2 v t) \cos(\pi x)} \qquad (29)$$



Finite difference solution for Example 4.1 for $\Delta t = 0.01,\ \ \Delta x = 0.1,\ \ \alpha = 0.95,\ \ \beta = 1.95$

X	Numerical Solution	Analytical Solution	Error
0	0.0	0.0	0.0
0.1	0.946184518e-3	0.94613841e-3	0.461089e-7
0.2	0.1924465410e-2	0.1933489085e-2	0.9023675e-5
0.3	0.2991456650e-2	0.3009822748e-2	0.18366098e-4
0.4	0.4209350415e-2	0.4237749682e-2	0.28399267e-4
0.5	0.5666712399e-2	0.5706351919e-2	0.3963952e-4
0.6	0.7491462912e-2	0.7544371558e-2	0.52908646e-4
0.7	0.9832247233e-2	0.99029697e-1	0.70722543e-4
0.8	0.1253842757e-1	0.1264482419e-1	0.10639662e-3
0.9	0.1278084698e-1	0.1294151322e-1	0.16066624e-3
1	0.0	0.0	0.0

Example 4.2

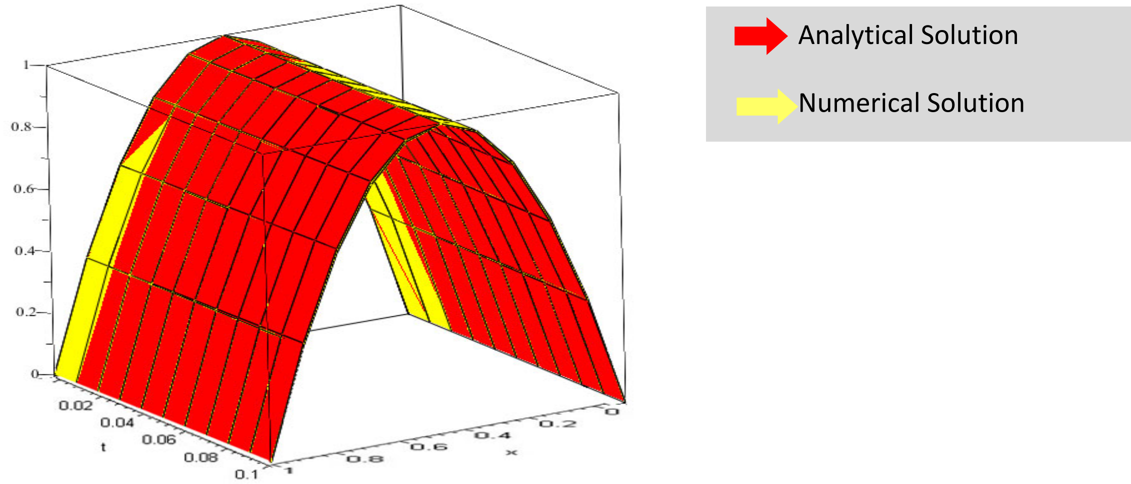
$$
\begin{aligned}
&\frac{\partial^\alpha u}{\partial t^\alpha} + u\frac{\partial u}{\partial x} = v\frac{\partial^\beta u}{\partial x^\beta} \qquad 0 \le x \le 1\ ,\ 0 < t \le 0{,}01\ ,\\
&0 < \alpha < 1\ ,\ 1 < \beta < 2\\
&\text{boundary condutions:}\\
&u(0,t) = 0, \quad u(1,t) = 0\\
&\text{initial condution:}\\
&u(x,0) = 4x(1-x)
\end{aligned}
\tag{30}
$$

We can solve numerically this problem in following conditions,

$$v = 0.001, \qquad \alpha = 0.95, \quad \beta = 1.95$$

Then the analytical solution of Example 4.2 is:

$$u(x,t) = \frac{2v\pi \sum_{k=1}^{\infty} kA_k \exp(-k^2\pi^2 vt)\sin(k\pi x)}{A_0 + 2v\pi \sum_{k=1}^{\infty} kA_k \exp(-k^2\pi^2 vt)\cos(k\pi x)}, \left(\begin{array}{c} A_0 = \int_0^1 \exp\{-x^2(3v)^{-1}(3-2x)\}dx \\ A_k = \int_0^1 \exp\{-x^2(3v)^{-1}(3-2x)\}\cos(k\pi x)dx, \ k \geq 1 \end{array} \right) \qquad (31)$$



Finite difference solution for Example 4.2 for $\Delta t = 0.001, \ \Delta x = 0.1, \ \alpha = 0.95, \ \beta = 1.95$

X	Numerical Solution	Analytical Solution	Error
0	0.0	0.0	0.0
0.1	0.3473711502	0.3486899313	0.13187811e-2
0.2	0.6220256101	0.6247583412	0.27327311e-2
0.3	0.8226195405	0.8264408177	0.38212772e-2
0.4	0.9474252601	0.9519845695	0.45593094e-2
0.5	0.9946485647	0.9995746590	0.49260943e-2
0.6	0.9624307453	0.9673304143	0.48996690e-2
0.7	0.8488455983	0.8533022617	0.44566634e-2
0.8	0.6518961906	0.6554683894	0.35721988e-2
0.9	0.3695116587	0.3717305024	0.22188437e-2
1	0.0	0.0	0.0

Acknowledgments

Authors was partially supported by the Office of Scientific Research Projects of Yildiz Technical University (Project no: 2013-01-03-YL01).

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