Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (2) (2020), 754 – 765
DOI : 10.15672/hujms.623990

Research Article

Computation of Zagreb indices and Zagreb polynomials of Sierpiński graphs

Hafiz Muhammad Afzal Siddiqui

Department of Mathematics, COMSATS University Islamabad Lahore Campus, Pakistan

Abstract

The Sierpiński fractal or Sierpiński gasket and generalized Sierpiński graphs are objects of great interest in dynamical systems and probability. In this paper, we consider the Sierpiński gasket graph S_n, the generalized Sierpiński graphs $S(n, C_3)$ and $S(n, C_4)$. We provide explicit computing formulae for Zagreb indices, multiple Zagreb indices and Zagreb polynomials of Sierpiński graphs.

Mathematics Subject Classification (2010). 05C12, 05C90

Keywords. Sierpiński gasket graph, generalized Sierpiński graph, topological indices, Zagreb index, augmented Zagreb index, Zagreb polynomials

1. Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote the *order* of G by $|V(G)|$ and *size* of G by $|E(G)|$. An edge in $E(G)$ with end vertices u and v is denoted by uv. Two vertices u and v are called *adjacent* if there is an edge between them. The *neighborhood* of u, denoted by $N(u)$, is the set of all vertices adjacent to u. The *degree* of u is denoted by d_u and equals $|N(u)|$.

Graphs of Sierpiński type appear naturally in many different areas of mathematics as well as in several other scientific fields. One of the most important family of such graphs is the Sierpiński graphs S_n, obtained after a finite number of iterations that in the limit give the Sierpiński gasket [23]. More simply, S_{n+1} consists of three attached copies of S_n which are referred as the top, bottom left and bottom right components of S_{n+1}. These graphes had been already introduced in 1944 by Scorer, Grundy and Smith [26]. They play an important role in dynamic systems and probability [21], as well as in psychology [26]. The generalized Sierpiński graph, $S(n, G)$ is constructed by copying $|G|$ times $S(n-1, G)$ and adding one edge between copy x and copy y of $S(n-1, G)$, whenever xy is an edge of G.

The Sierpiński graphs $S(n, k)$ and $S(n, G)$ are defined as follows:

S(n,k) has vertex set $\{1, 2, \cdots, k\}^n$, and there is an edge between two vertices $u = (u_1, u_2 \cdots, u_n)$ and $v = (v_1, v_2, \cdots, v_n)$ iff there is an $h \in \{1, 2, \cdots, n\}$ such that:

- $u_j = v_j$ for $j = 1, \cdots, h-1$;
- $u_h \neq v_h$; and
- $u_j = v_h; v_j = u_h$ for $j = h+1, \cdots, n$.

Email address: hmasiddiqui@gmail.com
Received: 31.08.2016; Accepted: 25.03.2019

The generalized Sierpiński graph of dimension n denoted by $S(n;G)$ is the graph with vertex set $\{1,2,\cdots,k\}$ and edge set defined by: $\{u,v\}$ is an edge if and only if there exists $i \in \{1,2,\cdots,n\}$ such that:

- $u_j = v_j$ if $j < i$,
- $u_i \neq v_i$ and $(u_i, v_i) \in E(G)$,
- $u_j = v_i$ and $v_j = u_i$ if $j > i$.

The topological indices are the objects of great importance in quantitative structure-activity research ($QSAR$) and structure-property relationships research ($QSPR$) study. The graph invariants, known as the first and second Zagreb indices, were the first vertex-degree-based structure descriptors [18,19]. The terms, $\sum_{v\in V(G)} d_v^2$, $\sum_{uv\in E(G)} d_u d_v$ and $\sum_{v\in V(G)} d_v^3$ were first appeared in the topological formula for total π-energy of conjugated molecules that was derived in 1972 by Gutman and Trinajstić [18]. Ten years later, Balaban et al. included

$$M_1(G) = \sum_{v\in V(G)} d_v^2 = \sum_{uv\in E(G)} (d_u + d_v) \tag{1.1}$$

and

$$M_2(G) = \sum_{uv\in E(G)} d_u d_v \tag{1.2}$$

among topological indices and named them "Zagreb group indices" [2]. The name "Zagreb group indices" was abbreviated to "Zagreb indices" and now we call $M_1(G)$, the first Zagreb index and $M_2(G)$, the second Zagreb index. Afterwards these indices have been used as branching indices [5]. Later on the Zagreb indices found applications in QSPR and QSAR studies [16, 29]. These indices have been used to study molecular complexity, chirality, ZE-isomorphism and hetero-systems. For further study on chemical applications and mathematical properties see the following papers [1,4,6,10,13–15,17,20,22,25,28,30,31].

The term, $\sum_{v\in V(G)} (d_v)^3$ was ignored for more than forty years. Recently, Furtula and Gutman proved that this term have a very promising application potential [8]. They proposed that this term should be named the forgotten topological index or shortly the F-index that is defined as

$$F(G) = \sum_{v\in V(G)} (d_v)^3 = \sum_{uv\in E(G)} [(d_u)^2 + (d_v)^2]. \tag{1.3}$$

They discovered a remarkable fact that the linear combination $M_1 + \lambda F$ yields a highly accurate mathematical model of certain physico-chemical properties of alkanes [8].

Another important graph invariant that is necessarily encountered within the studies of difference between two Zagreb indices [3], is the reduced second Zagreb index, defined as

$$RM_2(G) = \sum_{uv\in E(G)} (d_u - 1) \times (d_v - 1). \tag{1.4}$$

The augmented Zagreb index of G proposed by Furtula et al. in 2010 [9] is defined as

$$AZI(G) = \sum_{uv\in E(G)} (\frac{d_u d_v}{d_u + d_v - 2})^3. \tag{1.5}$$

This graph invariant has proven to be a valuable predictive index in the study of the heat of formation in octanes and heptanes [9]. Noting that if instead of the exponent 3 we would set -0.5, then we would arrive at the ordinary ABC index. Preliminary studies [9, 12, 17] indicate that AZI has an even better correlation potential than ABC index.

The third Zagreb index was introduced by Shirdel in 2013 [27], defined as

$$M_3(G) = \sum_{uv \in E(G)} (d_u + d_v)^2. \tag{1.6}$$

Clearly, this index is a combination of F-index and the second Zagreb index i.e.

$$M_3(G) = F(G) + 2M_2(G). \tag{1.7}$$

Because of the above mentioned relation, we studied the F-index with the indices of the Zagreb family.

The degree product $P(G) = \prod_{v \in V(G)} d_v$ of a graph G was introduced and studied by Narumi and Katayama for the first time. The Narumi-Katayama index was proposed in 1984, by Narumi and Katayama [24]. It is defined as

$$NK(G) = \prod_{v \in V(G)} d_v. \tag{1.8}$$

The first and second multiple Zagreb indices were introduced by Ghorbani and Azimi in 2012 [11], defined as

$$PM_1(G) = \prod_{uv \in V(G)} (d_u + d_v) = \prod_{v \in V(G)} (d_v)^2 \tag{1.9}$$

and

$$PM_2(G) = \prod_{uv \in V(G)} (d_u d_v). \tag{1.10}$$

Clearly, the first multiple Zagreb index is the square of Narumi-Katayama index.

In 2009, Fath-Tabar [7] put forward the first and the second Zagreb polynomials of the graph G, defined respectively as

$$ZG_1(G, x) = \sum_{uv \in E(G)} x^{d_u + d_v} \tag{1.11}$$

and

$$ZG_2(G, x) = \sum_{uv \in E(G)} x^{d_u d_v}, \tag{1.12}$$

where x is a dummy variable.

In this paper, we compute the above mentioned topological indices for S_n, $S(n, C_3)$ and $S(n, C_4)$.

2. Zagreb indices and Zagreb polynomials for the Sierpiński gasket graph, S_n

The Sierpiński gasket graphs for $n = 1, 2, 3$ are given in Figure 1. The order of S_n is $\frac{1}{2}(3^n + 3)$ and the size of S_n is 3^n. There are two kinds of edges corresponding to their degrees of end vertices for $n > 1$. The edge partition of edge set of S_n is shown in Table 1.

Table 1. Edge partition of edge set of S_n

(d_u, d_v)	$(2, 4)$	$(4, 4)$
Number of Edges	6	$3^n - 6$

There are two kinds of vertices in the set $V(G)$ corresponding to their degrees. Table 2 shows such a partition of the set $V(G)$ of S_n.

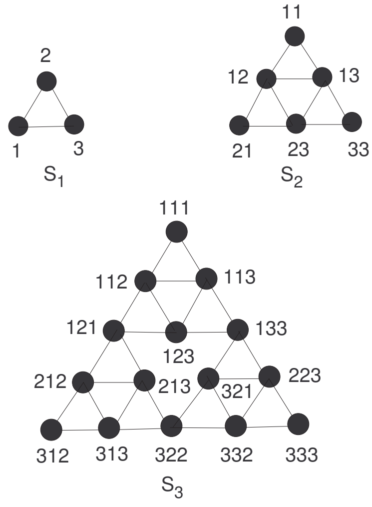

Figure 1. The graph of S_1, S_2 and S_3

Table 2. The partition of $V(G)$ of S_n

d_v	2	4
Number of Vertices	3	$\frac{1}{2}(3^n - 3)$

The following theorems present the analytically closed formulae of Zagreb indices and Zagreb polynomials for S_n .

Theorem 2.1. *The first and second Zagreb indices for $G = S_n$ are given by*

$$M_1(G) = 8 \times 3^n - 12 \text{ and } M_2(G) = 16 \times 3^n - 48.$$

Proof. Using Equation 1.1 and Table 1, we have

$$\begin{aligned} M_1(G) &= \sum_{uv \in E(G)} (d_u + d_v) \\ &= 6(2+4) + (3^n - 6)(4+4). \end{aligned}$$

After simplification, we get the required result, $M_1(G) = 8 \times 3^n - 12$.

Similarly, Using Equation 1.2 and Table 1, we have

$$\begin{aligned} M_2(G) &= \sum_{uv \in E(G)} (d_u \times d_v) \\ &= 6(2 \times 4) + (3^n - 6)(4 \times 4) = 16 \times 3^n - 48. \end{aligned}$$

□

Theorem 2.2. *The reduced second Zagreb index for $G = S_n$ is given by*

$$RM_2(G) = 9 \times 3^n - 36.$$

Proof. Using Equation 1.4 and Table 1, we find

$$\begin{aligned} RM_2(G) &= \sum_{uv \in E(G)} (d_u - 1) \times (d_v - 1) \\ &= 6(1 \times 3) + (3^n - 6)(3 \times 3). \end{aligned}$$

After simplification, we get the required result, $RM_2(G) = 9 \times 3^n - 36$. □

Theorem 2.3. *The third Zagreb index for $G = S_n$ is given by*

$$M_3(G) = 8(8 \times 3^n - 21).$$

Proof. Using Equation 1.6 and Table 1, we get

$$M_3(G) = \sum_{uv \in E(G)} (d_u + d_v)^2 = 6(6)^2 + (3^n - 6)(8)^2.$$

After simplification, we get the required result, $M_3(G) = 8(8 \times 3^n - 21)$. □

Theorem 2.4. *The F-index for $G = S_n$ is given by*

$$F(G) = 32 \times 3^n - 72.$$

Proof. Using Equation 1.7, Theorem 2.1 and Theorem 2.3, we get

$$F(G) = 8(8 \times 3^n - 21) - 2(16 \times 3^n - 48) = 32 \times 3^n - 72.$$

□

Theorem 2.5. *The augmented Zagreb index for $G = S_n$ is given by*

$$AZI(G) = 16(32 \times 3^{n-3} - \tfrac{37}{9}).$$

Proof. Using Equation 1.5, Table 1 and simplifying, we have

$$\begin{aligned} AZI(G) &= \sum_{uv \in E(G)} [\frac{d_u d_v}{d_u + d_v - 2}]^3 \\ &= 6(2)^3 + (3^n - 6)(\frac{8}{3})^3 \\ &= 16(32 \times 3^{n-3} - \frac{37}{9}). \end{aligned}$$

□

Theorem 2.6. *The first and second multiple Zagreb indices for $G = S_n$ are given by*

$$PM_1(G) = 32(3^{n+1} - 9) \text{ and } PM_2(G) = 4^4(3^{n+1} - 18).$$

Proof. Using Equation 1.9 and Table 2, we get

$$PM_1(G) = \prod_{v \in V(G)} (d_v)^2 = (2)^2 \times 3 \times 8(3^n - 3),$$

which after simplification gives $PM_1(G) = 32(3^{n+1} - 9)$.

Similarly, using Equation 1.10 and Table 1 and after simplification, we have

$$\begin{aligned} PM_2(G) &= \prod_{uv \in E(G)} d_u d_v \\ &= 8 \times 6 \times (16)(3^n - 6) = 4^4(3^{n+1} - 18). \end{aligned}$$

□

Corollary 2.7. *The Narumi-Katayama index for $G = S_n$ is given by*

$$NK(G) = \sqrt{PM_1(G)} = 12\sqrt{4(3^{n-1} - 1)}.$$

Theorem 2.8. *The first Zagreb polynomial for $G = S_n$ is given by*

$$ZG_1(G, x) = 6 \times x^6 + (3^n - 6) \times x^8.$$

Proof. Using Equation 1.11 and Table 1, we have

$$ZG_1(G, x) = \sum_{uv \in E(G)} x^{d_u + d_v} = 6 \times x^6 + (3^n - 6) \times x^8$$

. □

Theorem 2.9. *The second Zagreb polynomial for $G = S_n$ is given by*

$$ZG_2(G, x) = 6 \times x^8 + (3^n - 6) \times x^{16}.$$

Proof. Using Equation 1.12 and Table 1, we get

$$ZG_2(G,x) = \sum_{uv\in E(G)} x^{d_u d_v} = 6\times x^8 + (3^n-6)\times x^{16}.$$

□

3. Zagreb indices and Zagreb polynomials for $S(n, C_3)$

The generalized Sierpiński graphs, $S(1, C_3)$, $S(2, C_3)$ and $S(3, C_3)$ are shown in Figure 2.

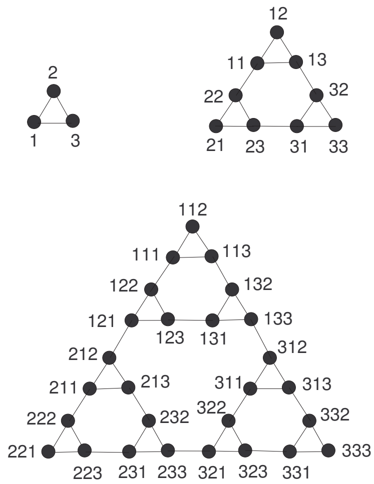

Figure 2. The graphs $S(1, C_3)$, $S(2, C_3)$ and $S(3, C_3)$

The order and size of $S(n, C_3)$ are 3^n and $\frac{3}{2}(3^n - 1)$, respectively. There are two kinds of edges corresponding to their degrees of end vertices for $n > 1$. The edge partition of edge set of $S(n, C_3)$ is shown in Table 3.

Table 3. Edge partition of edge set of $S(n, C_3)$

(d_u, d_v)	$(2,3)$	$(3,3)$
Number of Edges	6	$\frac{3}{2}(3^n-5)$

There are two kinds of vertices in the set $V(G)$ corresponding to their degrees. Table 4 shows such a partition of the set $V(G)$ of $S(n, C_3)$.

Table 4. The partition of $V(G)$ of $S(n, C_3)$

d_v	2	3
Number of Vertices	3	3^n-3

The following theorems present the analytically closed formulae of Zagreb indices and Zagreb polynomials for $S(n, C_3)$ for $n > 1$.

Theorem 3.1. *The first and second Zagreb indices for $G = S(n, C_3)$ are given by*

$$M_1(G) = 9\times 3^n - 15 \text{ and } M_2(G) = \tfrac{9}{2}(3^{n+1} - 7).$$

Proof. Using Equation 1.1 and Table 3, we have

$$\begin{aligned} M_1(G) &= \sum_{uv \in E(G)} (d_u + d_v) \\ &= 6(2+3) + \frac{3}{2}(3^n - 5)(3+3). \end{aligned}$$

After simplification, we get the required result, $M_1(G) = 9 \times 3^n - 15$.

Similarly, Using Equation 1.2 and Table 3, we have

$$\begin{aligned} M_2(G) &= \sum_{uv \in E(G)} (d_u \times d_v) \\ &= 6(2 \times 3) + \frac{3}{2}(3^n - 5)(3 \times 3) = \frac{9}{2}(3^{n+1} - 7). \end{aligned}$$

□

Theorem 3.2. *The reduced second Zagreb index for $G = S(n, C_3)$ is given by*

$$RM_2(G) = 6(3^n - 3).$$

Proof. Using Equation 1.4 and Table 3, we find

$$\begin{aligned} RM_2(G) &= \sum_{uv \in E(G)} (d_u - 1) \times (d_v - 1) \\ &= 6(1 \times 2) + \frac{3}{2}(3^n - 5)(2 \times 2). \end{aligned}$$

After simplification, we get the required result, $RM_2(G) = 6(3^n - 3)$. □

Theorem 3.3. *The third Zagreb index for $G = S(n, C_3)$ is given by*

$$M_3(G) = 2(3^{n+3} - 20).$$

Proof. Using Equation 1.6 and Table 3, we get

$$M_3(G) = \sum_{uv \in E(G)} (d_u + d_v)^2 = 6(5)^2 + \frac{3}{2}(3^n - 5)(6)^2.$$

After simplification, we get the required result, $M_3(G) = 2(3^{n+3} - 20)$. □

Theorem 3.4. *The F-index for $G = S(n, C_3)$ is given by*

$$F(G) = 3(3^{n+2} - 23).$$

Proof. Using Equation 1.7, Theorem 3.1 and Theorem 3.3, we get

$$F(G) = 6(9 \times 3^n - 20) - 2 \times \frac{9}{2}(3^{n+1} - 7) = 3(3^{n+2} - 23).$$

□

Theorem 3.5. *The augmented Zagreb index for $G = S(n, C_3)$ is given by*

$$AZI(G) = \tfrac{1}{2^7}(3^{n+7} + 4791).$$

Proof. Using Equation 1.5, Table 3 and simplifying, we have

$$\begin{aligned} AZI(G) &= \sum_{uv \in E(G)} [\frac{d_u d_v}{d_u + d_v - 2}]^3 \\ &= 6(2)^3 + \frac{3}{2}(3^n - 5)(\frac{9}{4})^3 \\ &= \frac{1}{2^7}(3^{n+7} + 4791). \end{aligned}$$

□

Theorem 3.6. *The first and second multiple Zagreb indices for $G = S(n, C_3)$ are given by*

$$PM_1(G) = 4(3^{n+3} - 81) \text{ and } PM_2(G) = 2(3^{n+5} - 1215).$$

Proof. Using Equation 1.9 and Table 4, we get

$$PM_1(G) = \prod_{v \in V(G)} (d_v)^2 = 4 \times 3 \times 9(3^n - 3),$$

which after simplification gives $PM_1(G) = 4(3^{n+3} - 81)$.

Similarly, using Equation 1.10 and Table 3 and after simplification, we have

$$PM_2(G) = \prod_{uv \in E(G)} d_u d_v$$
$$= 6 \times 6 \times 9(\tfrac{3}{2}(3^n - 5)) = 2(3^{n+5} - 1215).$$

□

Corollary 3.7. *The Narumi-Katayama index for $G = S(n, C_3)$ is given by*

$$NK(G) = \sqrt{PM_1(G)} = 2\sqrt{3^{n+3} - 81}.$$

Theorem 3.8. *The first Zagreb polynomial for $G = S(n, C_3)$ is given by*

$$ZG_1(G, x) = 6 \times x^5 + \tfrac{1}{2}(3^{n+1} - 15)x^6.$$

Proof. Using Equation 1.11 and Table 3, we have

$$ZG_1(G, x) = \sum_{uv \in E(G)} x^{d_u + d_v} = 6 \times x^5 + \frac{3}{2}(3^n - 5) \times x^6$$
$$= 6 \times x^5 + \frac{1}{2}(3^{n+1} - 15)x^6.$$

□

Theorem 3.9. *The second Zagreb polynomial for $G = S(n, C_3)$ is given by*

$$ZG_2(G, x) = 6 \times x^6 + \tfrac{1}{2}(3^{n+1} - 15)x^9.$$

Proof. Using Equation 1.12 and Table 3, we get

$$ZG_2(G, x) = \sum_{uv \in E(G)} x^{d_u d_v} = 6 \times x^6 + \frac{3}{2}(3^n - 5) \times x^9$$
$$= 6 \times x^6 + \frac{1}{2}(3^{n+1} - 15)x^9.$$

□

4. Zagreb indices and Zagreb polynomials for $S(n, C_4)$

The generalized Sierpiński graphs, $S(1, C_4)$, $S(2, C_4)$ and $S(3, C_4)$ are shown in Figure 3.

The order and size of $S(n, C_4)$ are 4^n and $\frac{4}{3}(4^n - 1)$, respectively. There are two kinds of edges corresponding to their degrees of end vertices for $n > 1$. The edge partition of edge set of $S(n, C_4)$ is shown in Table 5.

Table 5. Edge partition of edge set of $S(n, C_4)$

(d_u, d_v)	$(2, 3)$	$(3, 3)$
Number of Edges	$\frac{2}{3}(4^n + 8)$	$\frac{2}{3}(4^n - 10)$

There are two kinds of vertices in the set $V(G)$ corresponding to their degrees. Table 6 shows such a partition of the set $V(G)$ of $S(n, C_4)$.

The following theorems present the analytically closed formulae of Zagreb indices and Zagreb polynomials for $S(n, C_4)$, for $n > 1$.

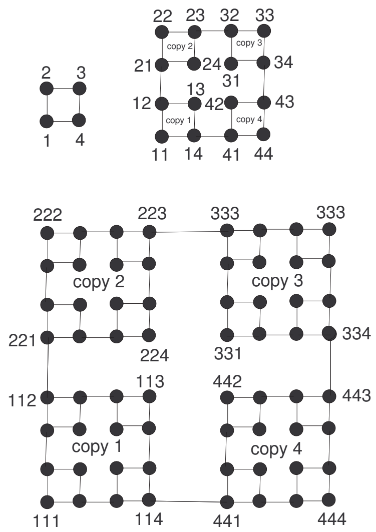

Figure 3. The graphs $S(1, C_4)$, $S(2, C_4)$ and $S(3, C_4)$

Table 6. The partition of $V(G)$ of $S(n, C_4)$

d_v	2	3
Number of Vertices	$\frac{1}{3}(4^n + 8)$	$\frac{2}{3}(4^n - 4)$

Theorem 4.1. *The first and second Zagreb indices for $G = S(n, C_4)$ are given by*

$$M_1(G) = \tfrac{2}{3}(11 \times 4^n - 20) \text{ and } M_2(G) = 2(5 \times 4^n - 14).$$

Proof. Using Equation 1.1 and Table 5, we have

$$\begin{aligned} M_1(G) &= \sum_{uv \in E(G)} (d_u + d_v) \\ &= \frac{2}{3}(4^n + 8)(2 + 3) + \frac{2}{3}(4^n - 10)(3 + 3). \end{aligned}$$

After simplification, we get the required result, $M_1(G) = \frac{2}{3}(11 \times 4^n - 20)$.

Similarly, using Equation 1.2 and Table 5, we have

$$\begin{aligned} M_2(G) &= \sum_{uv \in E(G)} (d_u \times d_v) \\ &= \frac{2}{3}(4^n + 8)(2 \times 3) + \frac{2}{3}(4^n - 10)(3 \times 3) = 2(5 \times 4^n - 14). \end{aligned}$$

□

Theorem 4.2. *The reduced second Zagreb index for $G = S(n, C_4)$ is given by*

$$RM_2(G) = 4^{n+1} - 16.$$

Proof. Using Equation 1.4 and Table 5, we find

$$\begin{aligned} RM_2(G) &= \sum_{uv \in E(G)} (d_u - 1) \times (d_v - 1) \\ &= \frac{2}{3}(4^n + 8)(1 \times 2) + \frac{2}{3}(4^n - 10)(2 \times 2). \end{aligned}$$

After simplification, we get the required result, $RM_2(G) = 4^{n+1} - 16$. □

Theorem 4.3. *The third Zagreb index for $G = S(n, C_4)$ is given by*

$$M_3(G) = \tfrac{2}{3}(61 \times 4^n - 160).$$

Proof. Using Equation 1.6 and Table 5, we get

$$\begin{aligned} M_3(G) &= \sum_{uv \in E(G)} (d_u + d_v)^2 \\ &= 0(4)^2 + \frac{2}{3}(4^n + 8)(5)^2 + \frac{2}{3}(4^n - 10)(6)^2. \end{aligned}$$

After simplification, we get the required result, $M_3(G) = \frac{2}{3}(61 \times 4^n - 160)$. □

Theorem 4.4. *The F-index for $G = S(n, C_4)$ is given by*

$$F(G) = \tfrac{2}{3}(31 \times 4^n - 76).$$

Proof. Using Equation 1.7, Theorem 4.1 and Theorem 4.3, we get

$$F(G) = \frac{2}{3}(61 \times 4^n - 160) - 2 \times 2(5 \times 4^n - 14) = \frac{2}{3}(31 \times 4^n - 76).$$

□

Theorem 4.5. *The augmented Zagreb index for $G = S(n, C_4)$ is given by*

$$AZI(G) = \tfrac{1}{96}(1241 \times 4^n - 3194).$$

Proof. Using Equation 1.5, Table 5 and simplifying, we have

$$\begin{aligned} AZI(G) &= \sum_{uv \in E(G)} [\frac{d_u d_v}{d_u + d_v - 2}]^3 \\ &= \frac{2}{3}(4^n + 8)(2)^3 + \frac{2}{3}(4^n - 10)(\frac{9}{4})^3 \\ &= \frac{1}{96}(1241 \times 4^n - 3194). \end{aligned}$$

□

Theorem 4.6. *The first and second multiple Zagreb indices for $G = S(n, C_4)$ are given by*

$$PM_1(G) = 8(4^n + 8)(4^n - 4) \text{ and } PM_2(G) = 24(4^n + 8)(4^n - 10).$$

Proof. Using Equation 1.9 and Table 6, we get

$$PM_1(G) = \prod_{v \in V(G)} (d_v)^2 = 2^2 \times \frac{1}{3}(4^n + 8) \times 3^2 \times \frac{2}{3}(4^n - 4),$$

which after simplification gives $PM_1(G) = 8(4^n + 8)(4^n - 4)$.

Similarly, using Equation 1.10 and Table 5 and after simplification, we have

$$\begin{aligned} PM_2(G) &= \prod_{uv \in E(G)} d_u d_v \\ &= 6 \times \frac{2}{3}(4^n + 8) \times 9 \times \frac{2}{3}(4^n - 10) \\ &= 24(4^n + 8)(4^n - 10). \end{aligned}$$

□

As an immediate consequence of Theorem 4.6, we have the following result.

Corollary 4.7. *The Narumi-Katayama index for $G = S(n, C_4)$ is given by*

$$NK(G) = \sqrt{PM_1(G)} = 2\sqrt{(4^n + 8)(4^n - 4)}.$$

Theorem 4.8. *The first Zagreb polynomial for $G = S(n, C_4)$ is given by*

$$ZG_1(G, x) = \tfrac{2}{3}[(4^n + 8) \times x^5 + (4^n - 10) \times x^6].$$

Proof. Using Equation 1.11 and Table 5, we have

$$ZG_1(G,x) = \sum_{uv \in E(G)} x^{d_u+d_v} = \frac{2}{3}(4^n+8)\times x^5 + \frac{2}{3}(4^n-10)\times x^6$$
$$= \frac{2}{3}[(4^n+8)\times x^5 + (4^n-10)\times x^6].$$

□

Theorem 4.9. *The second Zagreb polynomial for $G = S(n, C_4)$ is given by*

$$ZG_2(G,x) = \tfrac{2}{3}[(4^n+8)\times x^6 + (4^n-10)\times x^9].$$

Proof. Using Equation 1.12 and Table 5, we get

$$ZG_2(G,x) = \sum_{uv \in E(G)} x^{d_u d_v} = \frac{2}{3}(4^n+8)\times x^6 + \frac{2}{3}(4^n-10)\times x^9$$
$$= \frac{2}{3}[(4^n+8)\times x^6 + (4^n-10)\times x^9].$$

□

5. Conclusion and general remarks

In this paper, we have conducted the study of Zagreb indices and Zagreb polynomials for the Sierpiński gasket graph S_n, the generalized Sierpiński graphs $S(n, C_3)$ and $S(n, C_4)$. We have computed the exact formulae of Zagreb indices and Zagreb polynomials for these structures. Various graph-theoretic parameters and certain distance based and counting related topological descriptors for the Sierpiński gasket graph S_n and the generalized Sierpiński graphs $S(n, C_3)$ and $S(n, C_4)$ can be considered for future study.

Acknowledgment. The author would like to thank the referee for his/her corrections, comments and useful criticism which improved the first version of this paper.

References

[1] H. Ali, A.Q. Baig and M.K. Shafiq, *On topological properties of hierarchical interconnection networks*, J. Appl. Math. Comput. **55** (1-2), 313–334, doi:10.1007/s12190-016-1038-3, 2016.

[2] A.T. Balaban, I. Motoc, D. Bonchev and O. Makenyan, *Topological indices for structure-activity correlations*, Topics Curr. Chem. **114**, 21–55, 1983.

[3] G. Caporossi, P. Hansen and D. Vukičević, *Comparing of Zagreb indices of cylic graphs*, MATCH Commun. Math. Comput. Chem. **63**, 441–451, 2010.

[4] K.C. Das and I. Gutman, *Some properties of the second Zagreb index*, MATCH Commun. Math. Comput. Chem. **52**, 103–112, 2004.

[5] M.V. Diudea, (Ed.), QSPR/QSAR *Studies by molecular descriptors*, NOVA, New York, 2001.

[6] M.R. Farahani, H.M.A. Siddiqui, Sh. Baby, M. Imran and M.K. Siddiqui, The Second and Second Sum connectivity Indices of TUC_4C_8 Nanutubes, J. Optoelectron. Bio. Mater. , **8** (3), 107–111, 2016.

[7] H. Fath-tabar, *Zagreb polynomials and PI indices of some nanostructures*, Digest. J. Nanomater. Bios. **4**, 189–191, 2009.

[8] B. Furtula and I. Gutman, *A forgotten topological index*, J. Math. Chem. **53**, 1184-1190, 2015.

[9] B. Furtula, A. Graovac and D. Vukičević, *Augmented Zagreb index*, J. Math. Chem. **48**, 370–380, 2010.

[10] B. Furtula, I. Gutman and M. Dehmer, *On structural-sensitivity of degree-based topological indices*, Appl.Math. Comput. **219**, 8973–8978, 2013.
[11] M. Ghorbani and N. Azimi, *Note on multiple Zagre indices*, Iran. J. Math. Chem. **3**, 137–143, 2012.
[12] I. Gutman, *Degree-based topological indices*, Croat. Chem. Acta, **86**, 351–361, 2013.
[13] I. Gutman, *An exceptional property of first Zagreb index*, MATCH Commun. Math. Comput. Chem. **72**, 733–740, 2014.
[14] I. Gutman and K.C. Das, *The first Zagreb index 30 years after*, MATCH Commun. Math. Comput. Chem. **50**, 83–92, 2004.
[15] I. Gutman and B. Furtula, Ž. K. Vukićević and G. Popivoda, *On Zagreb indices and coindices*, MATCH Commun. Math. Comput. Chem. **74**, 5–16, 2015.
[16] I. Gutman and O. Polansky, *Mathematical Concepts in Organic Chemistry*, Springer-Verlag, Berlin, 1986.
[17] I. Gutman and J. Tošović, *Testing the quality of molecular structures descriptors. Vertex-degree-based topological indices*, J. Serb. Chem. Soc. **78**, 805–810, 2013.
[18] I. Gutman and N. Trinajstić, *Graph theory and molecular orbitals. Total π-electron energy of alternate hydrocarbons*, Chem. Phy. Lett. **17**, 535–538, 1972.
[19] I. Gutman, B. Ruščić, N. Trinajstić and C.F. wilcox, *Graph theory and molecular orbitals. XII. Acyclic polyenes*, J. Chem. Phys. **62**, 1692–1704, 1975.
[20] S. Hayat and H.M.A. Siddiqui, *On bipartite edge frustration of carbon and boron nanotubes*, Studia UBB Chemia, **LXI(1)**, 283–290, 2016.
[21] A.M. Hinz and D. Parisse, *Coloring Hanoi and Sierpiński graphs*, Disc. Math. **312** (9), 1521–1535, 2012.
[22] M. Imran, A.Q. Baig, H. Ali and S.U. Rehman, *On topological properties of poly honeycomb networks*, Period. Math. Hungr. **73** (1), 100–119, 2016.
[23] A. Jonsson, *A trace theorem for the Drichlet form on the Sierpiński gasket*, Math. Z. **250**, 599–609, 2005.
[24] H. Narumi and H. Katayama, *Simple topological index, a newly devised index characterizing the topological nature of structural isomers of saturated hydrocarbons*, Mem. Fac. Engin. Hokkaido Univ. **16**, 209–214, 1984.
[25] S. Nikolić, G. Kovačević, A. Milič and N. Trinajstić, *The Zagreb indices 30 years after*, Croat. Chem. Acta, **76**, 113–124, 2003.
[26] R.S. Scorer, P.M. Grundy and C.A.B. Smith, *Some binary games*, Math. Gaz. **28**, 96–103, 1944.
[27] G.H. Shirdel, H. Rezapour and A.M. Sayadi, *The hyper-Zagreb index of graph operations*, Iran. J. Math. Chem. **4**, 213–220, 213.
[28] S. Wang, M.R. Farahani and A.Q. Baig, *The Sadhana polynomial and Sadhana index of polycyclic aromatic hydrocarbon PAHk*, J. Chem. Pharm. Res. **8** (6), 526–531, 2016.
[29] K. Xu and K.Ch. Das, *Zagreb indices and polynomials of $TUHRC_4$ and $TUSC_4C_8$ nanotubes*, MATCH Commun. Math. Comput. Chem. **68**, 257–272, 2012.
[30] L. Yang, X. Ai and L. Zhang, *The Zagreb coindices of a type of composite graph*, Hacettepe J. Math. Stat. **45** (4), 1135–1145, 2016.
[31] B. Zhou, *Zagreb indices*, MATCH Commun. Math. Comput. Chem. **52**, 113–118, 2004.