Computation of Zagreb indices and Zagreb polynomials of Sierpiński graphs

Hafiz Muhammad Afzal Siddiqui

Department of Mathematics, COMSATS University Islamabad Lahore Campus, Pakistan

Abstract

The Sierpiński fractal or Sierpiński gasket and generalized Sierpiński graphs are objects of great interest in dynamical systems and probability. In this paper, we consider the Sierpiński gasket graph $S_n$, the generalized Sierpiński graphs $S(n, C_3)$ and $S(n, C_4)$. We provide explicit computing formulae for Zagreb indices, multiple Zagreb indices and Zagreb polynomials of Sierpiński graphs.

Mathematics Subject Classification (2010). 05C12, 05C90

Keywords. Sierpiński gasket graph, generalized Sierpiński graph, topological indices, Zagreb index; augmented Zagreb index, Zagreb polynomials.

1. Introduction

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. We denote the order of $G$ by $|V(G)|$ and size of $G$ by $|E(G)|$. An edge in $E(G)$ with end vertices $u$ and $v$ is denoted by $uv$. Two vertices $u$ and $v$ are called adjacent if there is an edge between them. The neighborhood of $u$, denoted by $N(u)$, is the set of all vertices adjacent to $u$. The degree of $u$ is denoted by $d_u$ and equals $|N(u)|$.

Graphs of Sierpiński type appear naturally in many different areas of mathematics as well as in several other scientific fields. One of the most important family of such graphs is the Sierpiński graphs $S_n$, obtained after a finite number of iterations that in the limit give the Sierpiński gasket [23]. More simply, $S_{n+1}$ consists of three attached copies of $S_n$ which are referred as the top, bottom left and bottom right components of $S_{n+1}$. These graphs had been already introduced in 1944 by Scorer, Grundy and Smith [26]. They play an important role in dynamic systems and probability [21], as well as in psychology [26]. The generalized Sierpiński graph, $S(n, G)$ is constructed by copying $|G|$ times $S(n-1, G)$ and adding one edge between copy $x$ and copy $y$ of $S(n-1, G)$, whenever $xy$ is an edge of $G$.

The Sierpiński graphs $S(n, k)$ and $S(n, G)$ are defined as follows:

$S(n, k)$ has vertex set $\{1, 2, \cdots, k\}^n$, and there is an edge between two vertices $u = (u_1, u_2, \cdots, u_n)$ and $v = (v_1, v_2, \cdots, v_n)$ iff there is an $h \in \{1, 2, \cdots, n\}$ such that:

- $u_j = v_j$ for $j = 1, \cdots, h - 1$;
- $u_h \neq v_h$; and
- $u_j = v_h; v_j = u_h$ for $j = h + 1, \cdots, n$.

Email addresses: hmasiddiqui@gmail.com
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The generalized Sierpiński graph of dimension $n$ denoted by $S(n; G)$ is the graph with vertex set $\{1, 2, \ldots, k\}$ and edge set defined by: $\{u, v\}$ is an edge if and only if there exists $i \in \{1, 2, \ldots, n\}$ such that:

- $u_j = v_j$ if $j < i$,
- $u_i \neq v_i$ and $(u_i, v_i) \in E(G)$,
- $u_j = v_i$ and $v_j = u_i$ if $j > i$.

The topological indices are the objects of great importance in quantitative structure-activity research (QSAR) and structure-property relationships research (QSPR) study. The graph invariants, known as the first and second Zagreb indices, were the first vertex-degree-based structure descriptors [17,19]. The terms, $\sum_{v \in V(G)} d_v^2$, $\sum_{uv \in E(G)} d_u d_v$ and $\sum_{v \in V(G)} d_v^3$ were first appeared in the topological formula for total $\pi$-energy of conjugated molecules that was derived in 1972 by Gutman and Trinajstić [19]. Ten years later, Balaban et al. included

$$M_1(G) = \sum_{v \in V(G)} d_v^2 = \sum_{uv \in E(G)} (d_u + d_v) \quad (1.1)$$

and

$$M_2(G) = \sum_{uv \in E(G)} d_u d_v \quad (1.2)$$

among topological indices and named them “Zagreb group indices” [2]. The name “Zagreb group indices” was abbreviated to “Zagreb indices” and now we call $M_1(G)$, the first Zagreb index and $M_2(G)$, the second Zagreb index. Afterwards these indices have been used as branching indices [5]. Later on the Zagreb indices found applications in QSPR and QSAR studies [16,29]. These indices have been used to study molecular complexity, chirality, $ZE$-isomorphism and hetero-systems. For further study on chemical applications and mathematical properties see the following papers [1,4,6,10,13–15,18,20,22,25,28,30,31].

The term, $\sum_{v \in V(G)} (d_v)^3$ was ignored for more than forty years. Recently, Furtula and Gutman proved that this term have a very promising application potential [9]. They proposed that this term should be named the forgotten topological index or shortly the F-index that is defined as

$$F(G) = \sum_{v \in V(G)} (d_v)^3 = \sum_{uv \in E(G)} [(d_u)^2 + (d_v)^2]. \quad (1.3)$$

They discovered a remarkable fact that the linear combination $M_1 + \lambda F$ yields a highly accurate mathematical model of certain physico-chemical properties of alkanes [9].

Another important graph invariant that is necessarily encountered within the studies of difference between two Zagreb indices [3], is the reduced second Zagreb index, defined as

$$RM_2(G) = \sum_{uv \in E(G)} (d_u - 1) \times (d_v - 1). \quad (1.4)$$

The augmented Zagreb index of $G$ proposed by Furtula et al. in 2010 [8] is defined as

$$AZI(G) = \sum_{uv \in E(G)} \left( \frac{d_u d_v}{d_u + d_v - 2} \right)^3. \quad (1.5)$$

This graph invariant has proven to be a valuable predictive index in the study of the heat of formation in octanes and heptanes [8]. Noting that if instead of the exponent 3 we would set $-0.5$, then we would arrive at the ordinary $ABC$ index. Preliminary studies [8,12,18] indicate that $AZI$ has an even better correlation potential than $ABC$ index.
Computation of Zagreb indices and Zagreb polynomials of Sierpiński graphs

The third Zagreb index was introduced by Shirdel in 2013 [27], defined as

\[ M_3(G) = \sum_{uv \in E(G)} (d_u + d_v)^2. \]  

(1.6)

Clearly, this index is a combination of F-index and the second Zagreb index i.e.

\[ M_3(G) = F(G) + 2M_2(G). \]  

(1.7)

Because of the above mentioned relation, we studied the F-index with the indices of the Zagreb family.

The degree product \( P(G) = \prod_{v \in V(G)} d_v \) of a graph \( G \) was introduced and studied by Narumi and Katayama for the first time. The Narumi-Katayama index was proposed in 1984, by Narumi and Katayama [24]. It is defined as

\[ NK(G) = \prod_{v \in V(G)} d_v. \]  

(1.8)

The first and second multiple Zagreb indices were introduced by Ghorbani and Azimi in 2012 [11], defined as

\[ PM_1(G) = \prod_{uv \in V(G)} (d_u + d_v) = \prod_{v \in V(G)} (d_v)^2 \]  

(1.9)

and

\[ PM_2(G) = \prod_{uv \in V(G)} (d_u d_v). \]  

(1.10)

Clearly, the first multiple Zagreb index is the square of Narumi-Katayama index.

In 2009, Fath-Tabar [7] put forward the first and the second Zagreb polynomials of the graph \( G \), defined respectively as

\[ ZG_1(G, x) = \sum_{uv \in E(G)} x^{d_u + d_v} \]  

(1.11)

and

\[ ZG_2(G, x) = \sum_{uv \in E(G)} x^{d_u d_v}, \]  

(1.12)

where \( x \) is a dummy variable.

In this paper, we compute the above mentioned topological indices for \( S_n \), \( S(n, C_3) \) and \( S(n, C_4) \).

2. Zagreb indices and Zagreb polynomials for the Sierpiński gasket graph, \( S_n \)

The Sierpiński gasket graphs for \( n = 1, 2, 3 \) are given in Figure 1. The order of \( S_n \) is \( \frac{1}{2}(3^n + 3) \) and the size of \( S_n \) is \( 3^n \). There are two kinds of edges corresponding to their degrees of end vertices for \( n > 1 \). The edge partition of edge set of \( S_n \) is shown in Table 1.

<table>
<thead>
<tr>
<th>( (d_u, d_v) )</th>
<th>( (2, 4) )</th>
<th>( (4, 4) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Edges</td>
<td>6</td>
<td>( 3^n - 6 )</td>
</tr>
</tbody>
</table>

Table 1. Edge partition of edge set of \( S_n \)

There are two kinds of vertices in the set \( V(G) \) corresponding to their degrees. Table 2 shows such a partition of the set \( V(G) \) of \( S_n \).
The following theorems present the analytically closed formulae of Zagreb indices and Zagreb polynomials for $S_n$.

**Theorem 2.1.** The first and second Zagreb indices for $G = S_n$ are given by

\[ M_1(G) = 8 \times 3^n - 12 \quad \text{and} \quad M_2(G) = 16 \times 3^n - 48. \]

**Proof.** Using Equation 1.1 and Table 1, we have

\[ M_1(G) = \sum_{uv \in E(G)} (d_u + d_v) = 6(2 + 4) + (3^n - 6)(4 + 4). \]

After simplification, we get the required result, $M_1(G) = 8 \times 3^n - 12$.

Similarly, Using Equation 1.2 and Table 1, we have

\[ M_2(G) = \sum_{uv \in E(G)} (d_u \times d_v) = 6(2 \times 4) + (3^n - 6)(4 \times 4) = 16 \times 3^n - 48. \]

\[ \square \]

**Theorem 2.2.** The reduced second Zagreb index for $G = S_n$ is given by

\[ RM_2(G) = 9 \times 3^n - 36. \]

**Proof.** Using Equation 1.4 and Table 1, we find

\[ RM_2(G) = \sum_{uv \in E(G)} (d_u - 1) \times (d_v - 1) = 6(1 \times 3) + (3^n - 6)(3 \times 3). \]

After simplification, we get the required result, $RM_2(G) = 9 \times 3^n - 36$. \[ \square \]

**Theorem 2.3.** The third Zagreb index for $G = S_n$ is given by

\[ M_3(G) = 8(8 \times 3^n - 21). \]
Proof. Using Equation 1.6 and Table 1, we get
\[ M_3(G) = \sum_{uv \in E(G)} (d_u + d_v)^2 = 6(6)^2 + (3^n - 6)(8)^2. \]
After simplification, we get the required result, \( M_3(G) = 8(8 \times 3^n - 21) \).

\[ \]

Theorem 2.4. The F-index for \( G = S_n \) is given by
\[ F(G) = 32 \times 3^n - 72. \]

Proof. Using Equation 1.7, Theorem 2.1 and Theorem 2.3, we get
\[ F(G) = 8(8 \times 3^n - 21) - 2(16 \times 3^n - 48) = 32 \times 3^n - 72. \]

\[ \]

Theorem 2.5. The augmented Zagreb index for \( G = S_n \) is given by
\[ AZI(G) = 16(32 \times 3^n - 3 - \frac{37}{9}). \]

Proof. Using Equation 1.5, Table 1 and simplifying, we have
\[ AZI(G) = \sum_{uv \in E(G)} \left[ \frac{d_ud_v}{d_u + d_v - 2} \right]^3 \]
\[ = 6(2)^3 + (3^n - 6)(\frac{8}{3})^3 \]
\[ = 16(32 \times 3^n - \frac{37}{9}). \]

\[ \]

Theorem 2.6. The first and second multiple Zagreb indices for \( G = S_n \) are given by
\[ PM_1(G) = 32(3^n+1 - 9) \]
\[ PM_2(G) = 4^4(3^n+1 - 18). \]

Proof. Using Equation 1.9 and Table 2, we get
\[ PM_1(G) = \prod_{v \in V(G)} (d_v)^2 = (2)^2 \times 3 \times 8(3^n - 3), \]
which after simplification gives \( PM_1(G) = 32(3^n+1 - 9) \).

Similarly, using Equation 1.10 and Table 1 and after simplification, we have
\[ PM_2(G) = \prod_{uv \in E(G)} d_ud_v \]
\[ = 8 \times 6 \times (16)(3^n - 6) = 4^4(3^n+1 - 18). \]

\[ \]

Corollary 2.7. The Narumi-Katayama index for \( G = S_n \) is given by
\[ NK(G) = \sqrt{PM_1(G)} = 12\sqrt{4(3^n-1)} \]

Theorem 2.8. The first Zagreb polynomial for \( G = S_n \) is given by
\[ ZG_1(G, x) = 6 \times x^6 + (3^n - 6) \times x^8. \]

Proof. Using Equation 1.11 and Table 1, we have
\[ ZG_1(G, x) = \sum_{uv \in E(G)} x^{d_u+d_v} = 6 \times x^6 + (3^n - 6) \times x^8. \]

\[ \]

Theorem 2.9. The second Zagreb polynomial for \( G = S_n \) is given by
\[ ZG_2(G, x) = 6 \times x^8 + (3^n - 6) \times x^{16}. \]
**Proof.** Using Equation 1.12 and Table 1, we get

\[ ZG_2(G, x) = \sum_{uv \in E(G)} x^{d_u d_v} = 6 \times x^8 + (3^n - 6) \times x^{16}. \]

\[ \square \]

3. Zagreb indices and Zagreb polynomials for \( S(n, C_3) \)

The generalized Sierpiński graphs, \( S(1, C_3) \), \( S(2, C_3) \) and \( S(3, C_3) \) are shown in Figure 2.

![Figure 2](image)

The order and size of \( S(n, C_3) \) are \( 3^n \) and \( \frac{3}{2}(3^n - 1) \), respectively. There are two kinds of edges corresponding to their degrees of end vertices for \( n > 1 \). The edge partition of edge set of \( S(n, C_3) \) is shown in Table 3.

<table>
<thead>
<tr>
<th>((d_u, d_v))</th>
<th>((2, 3))</th>
<th>((3, 3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Edges</td>
<td>6</td>
<td>(\frac{3}{2}(3^n - 5))</td>
</tr>
</tbody>
</table>

There are two kinds of vertices in the set \( V(G) \) corresponding to their degrees. Table 4 shows such a partition of the set \( V(G) \) of \( S(n, C_3) \).

<table>
<thead>
<tr>
<th>(d_v)</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Vertices</td>
<td>3</td>
<td>(3^n - 3)</td>
</tr>
</tbody>
</table>

The following theorems present the analytically closed formulae of Zagreb indices and Zagreb polynomials for \( S(n, C_3) \) for \( n > 1 \).

**Theorem 3.1.** The first and second Zagreb indices for \( G = S(n, C_3) \) are given by

\[ M_1(G) = 9 \times 3^n - 15 \] and \( M_2(G) = \frac{9}{2}(3^{n+1} - 7) \).
Proof. Using Equation 1.1 and Table 3, we have
\[
M_1(G) = \sum_{uv \in E(G)} (d_u + d_v) = 6(2 + 3) + \frac{3}{2}(3^n - 5)(3 + 3).
\]
After simplification, we get the required result, \(M_1(G) = 9 \times 3^n - 15\).

Similarly, Using Equation 1.2 and Table 3, we have
\[
M_2(G) = \sum_{uv \in E(G)} (d_u \times d_v) = 6(2 \times 3) + \frac{3}{2}(3^n - 5)(3 \times 3) = 9 \times \frac{3^n + 1 - 7}{2}.
\]

\[\square\]

Theorem 3.2. The reduced second Zagreb index for \(G = S(n, C_3)\) is given by
\[
RM_2(G) = 6(3^n - 3).
\]
Proof. Using Equation 1.4 and Table 3, we find
\[
RM_2(G) = \sum_{uv \in E(G)} (d_u - 1) \times (d_v - 1) = 6(1 \times 2) + \frac{3}{2}(3^n - 5)(2 \times 2).
\]
After simplification, we get the required result, \(RM_2(G) = 6(3^n - 3)\).

\[\square\]

Theorem 3.3. The third Zagreb index for \(G = S(n, C_3)\) is given by
\[
M_3(G) = 2(3^n + 3 - 20).
\]
Proof. Using Equation 1.6 and Table 3, we get
\[
M_3(G) = \sum_{uv \in E(G)} (d_u + d_v)^2 = 6(5)^2 + \frac{3}{2}(3^n - 5)(6)^2.
\]
After simplification, we get the required result, \(M_3(G) = 2(3^n + 3 - 20)\).

\[\square\]

Theorem 3.4. The F-index for \(G = S(n, C_3)\) is given by
\[
F(G) = 3(3^n + 2 - 23).
\]
Proof. Using Equation 1.7, Theorem 3.1 and Theorem 3.3, we get
\[
F(G) = 6(9 \times 3^n - 20) - 2 \times \frac{9}{2}(3^n + 1 - 7) = 3(3^n + 2 - 23).
\]

\[\square\]

Theorem 3.5. The augmented Zagreb index for \(G = S(n, C_3)\) is given by
\[
AZI(G) = \frac{1}{2} \left(3^n + 7 + 4791\right).
\]
Proof. Using Equation 1.5, Table 3 and simplifying, we have
\[
AZI(G) = \sum_{uv \in E(G)} \left[\frac{d_ud_v}{d_u + d_v - 2}\right]^3
\]
\[
= 6(2)^3 + \frac{3}{2}(3^n - 5)(\frac{9}{4})^3
\]
\[
= \frac{1}{2^7}(3^n + 7 + 4791).
\]

\[\square\]
Theorem 3.6. The first and second multiple Zagreb indices for \( G = S(n, C_3) \) are given by

\[
PM_1(G) = 4(3^{n+3} - 81) \quad \text{and} \quad PM_2(G) = 2(3^{n+5} - 1215).
\]

Proof. Using Equation 1.9 and Table 4, we get

\[
PM_1(G) = \prod_{v \in V(G)} (d_v)^2 = 4 \times 3 \times 9 \times 3^{n-3},
\]

which after simplification gives \( PM_1(G) = 4(3^{n+3} - 81) \).

Similarly, using Equation 1.10 and Table 3 and after simplification, we have

\[
PM_2(G) = \prod_{uv \in E(G)} d_u d_v = 6 \times 6 \times 9 \left( \frac{3}{2} (3^n - 5) \right) = 2(3^{n+5} - 1215).
\]

\[\square\]

Corollary 3.7. The Narumi-Katayama index for \( G = S(n, C_3) \) is given by

\[
NK(G) = \sqrt{PM_1(G)} = 2\sqrt{3^{n+3} - 81}.
\]

Theorem 3.8. The first Zagreb polynomial for \( G = S(n, C_3) \) is given by

\[
ZG_1(G, x) = 6 \times x^5 + \frac{3}{2} (3^n - 5) \times x^6.
\]

Proof. Using Equation 1.11 and Table 3, we have

\[
ZG_1(G, x) = \sum_{uv \in E(G)} x^{d_u + d_v} = 6 \times x^5 + \frac{3}{2} (3^n - 5) \times x^6,
\]

\[\square\]

Theorem 3.9. The second Zagreb polynomial for \( G = S(n, C_3) \) is given by

\[
ZG_2(G, x) = 6 \times x^6 + \frac{1}{2} (3^{n+1} - 15) \times x^9.
\]

Proof. Using Equation 1.12 and Table 3, we get

\[
ZG_2(G, x) = \sum_{uv \in E(G)} x^{d_u d_v} = 6 \times x^6 + \frac{3}{2} (3^n - 5) \times x^9,
\]

\[\square\]

4. Zagreb indices and Zagreb polynomials for \( S(n, C_4) \)

The generalized Sierpiński graphs, \( S(1, C_4) \), \( S(2, C_4) \) and \( S(3, C_4) \) are shown in Figure 3.

The order and size of \( S(n, C_4) \) are \( 4^n \) and \( \frac{4}{3}(4^n - 1) \), respectively. There are two kinds of edges corresponding to their degrees of end vertices for \( n > 1 \). The edge partition of edge set of \( S(n, C_4) \) is shown in Table 5.

<table>
<thead>
<tr>
<th>((d_u, d_v))</th>
<th>((2, 3))</th>
<th>((3, 3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Edges</td>
<td>(\frac{4}{3}(4^n + 8))</td>
<td>(\frac{2}{3}(4^n - 10))</td>
</tr>
</tbody>
</table>

There are two kinds of vertices in the set \( V(G) \) corresponding to their degrees. Table 6 shows such a partition of the set \( V(G) \) of \( S(n, C_4) \).

The following theorems present the analytically closed formulæ of Zagreb indices and Zagreb polynomials for \( S(n, C_4) \), for \( n > 1 \).
Computation of Zagreb indices and Zagreb polynomials of Sierpiński graphs

Figure 3. The graphs $S(1, C_4)$, $S(2, C_4)$ and $S(3, C_4)$

Table 6. The partition of $V(G)$ of $S(n, C_4)$

<table>
<thead>
<tr>
<th>Number of Vertices</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{1}{3}(4^n + 8)$</td>
<td>$\frac{2}{3}(4^n - 4)$</td>
<td></td>
</tr>
</tbody>
</table>

Theorem 4.1. The first and second Zagreb indices for $G = S(n, C_4)$ are given by $M_1(G) = \frac{2}{3}(11 \times 4^n - 20)$ and $M_2(G) = 2(5 \times 4^n - 14)$.

Proof. Using Equation 1.1 and Table 5, we have

$$M_1(G) = \sum_{uv \in E(G)} (d_u + d_v)$$

$$= \frac{2}{3}(4^n + 8)(2 + 3) + \frac{2}{3}(4^n - 10)(3 + 3).$$

After simplification, we get the required result, $M_1(G) = \frac{2}{3}(11 \times 4^n - 20)$.

Similarly, using Equation 1.2 and Table 5, we have

$$M_2(G) = \sum_{uv \in E(G)} (d_u \times d_v)$$

$$= \frac{2}{3}(4^n + 8)(2 \times 3) + \frac{2}{3}(4^n - 10)(3 \times 3) = 2(5 \times 4^n - 14).$$

Theorem 4.2. The reduced second Zagreb index for $G = S(n, C_4)$ is given by $RM_2(G) = 4^{n+1} - 16$.

Proof. Using Equation 1.4 and Table 5, we find

$$RM_2(G) = \sum_{uv \in E(G)} (d_u - 1) \times (d_v - 1)$$

$$= \frac{2}{3}(4^n + 8)(1 \times 2) + \frac{2}{3}(4^n - 10)(2 \times 2).$$

After simplification, we get the required result, $RM_2(G) = 4^{n+1} - 16$.

Theorem 4.3. The third Zagreb index for $G = S(n, C_4)$ is given by
$M_3(G) = \frac{2}{3}(61 \times 4^n - 160)$.

**Proof.** Using Equation 1.6 and Table 5, we get

$$M_3(G) = \sum_{uv \in E(G)} (d_u + d_v)^2 = 0(4)^2 + \frac{2}{3}(4^n + 8)(5)^2 + \frac{2}{3}(4^n - 10)(6)^2.$$  

After simplification, we get the required result, $M_3(G) = \frac{2}{3}(61 \times 4^n - 160)$. □

**Theorem 4.4.** The F-index for $G = S(n, C_4)$ is given by

$$F(G) = \frac{2}{3}(31 \times 4^n - 76).$$

**Proof.** Using Equation 1.7, Theorem 4.1 and Theorem 4.3, we get

$$F(G) = 2 \times 3(61 \times 4^n - 160) - 2 \times 2(5 \times 4^n - 14) = 2 \times 3(31 \times 4^n - 76).$$ □

**Theorem 4.5.** The augmented Zagreb index for $G = S(n, C_4)$ is given by

$$AZI(G) = \frac{1}{96}(1241 \times 4^n - 3194).$$

**Proof.** Using Equation 1.5, Table 5 and simplifying, we have

$$AZI(G) = \sum_{uv \in E(G)} \left(\frac{d_u d_v}{d_u + d_v - 2}\right)^3 = \frac{2}{3}(4^n + 8)(2)^3 + \frac{2}{3}(4^n - 10)(2)^3 = \frac{1}{96}(1241 \times 4^n - 3194).$$ □

**Theorem 4.6.** The first and second multiple Zagreb indices for $G = S(n, C_4)$ are given by

$$PM_1(G) = 8(4^n + 8)(4^n - 4)$$  

and

$$PM_2(G) = 24(4^n + 8)(4^n - 10).$$

**Proof.** Using Equation 1.9 and Table 6, we get

$$PM_1(G) = \prod_{v \in V(G)} (d_v)^2 = 2^2 \times \frac{1}{3}(4^n + 8) \times 3^2 \times \frac{2}{3}(4^n - 4),$$

which after simplification gives $PM_1(G) = 8(4^n + 8)(4^n - 4)$.

Similarly, using Equation 1.10 and Table 5 and after simplification, we have

$$PM_2(G) = \prod_{uv \in E(G)} d_u d_v$$

$$= 6 \times \frac{2}{3}(4^n + 8) \times 9 \times \frac{2}{3}(4^n - 10) = 24(4^n + 8)(4^n - 10).$$ □

As an immediate consequence of Theorem 4.6, we have the following result.

**Corollary 4.7.** The Narumi-Katayama index for $G = S(n, C_4)$ is given by

$$NK(G) = \sqrt{PM_1(G)} = 2\sqrt{(4^n + 8)(4^n - 4)}.$$
Proof. Using Equation 1.11 and Table 5, we have

\[ Z_{G1}(G, x) = \sum_{uv \in E(G)} x^{d_u + d_v} = \frac{2}{3}(4^n + 8) \times x^5 + \frac{2}{3}(4^n - 10) \times x^6 \]

\[ = \frac{2}{3}[(4^n + 8) \times x^5 + (4^n - 10) \times x^6]. \]

\[ \square \]

Theorem 4.9. The second Zagreb polynomial for \( G = S(n, C_4) \) is given by

\[ Z_{G2}(G, x) = \frac{2}{3}[(4^n + 8) \times x^6 + (4^n - 10) \times x^9]. \]

Proof. Using Equation 1.12 and Table 5, we get

\[ Z_{G2}(G, x) = \sum_{uv \in E(G)} x^{d_u d_v} = \frac{2}{3}(4^n + 8) \times x^6 + \frac{2}{3}(4^n - 10) \times x^9 \]

\[ = \frac{2}{3}[(4^n + 8) \times x^6 + (4^n - 10) \times x^9]. \]

\[ \square \]

5. Conclusion and general remarks

In this paper, we have conducted the study of Zagreb indices and Zagreb polynomials for the Sierpiński gasket graph \( S_n \), the generalized Sierpiński graphs \( S(n, C_3) \) and \( S(n, C_4) \). We have computed the exact formulae of Zagreb indices and Zagreb polynomials for these structures. Various graph-theoretic parameters and certain distance based and counting related topological descriptors for the Sierpiński gasket graph \( S_n \) and the generalized Sierpiński graphs \( S(n, C_3) \) and \( S(n, C_4) \) can be considered for future study.

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References


