Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 49 (2) (2020), 808 – 821
DOI : 10.15672/hujms.624000

Research Article

On $\mathscr{C}$-coherent rings, strongly $\mathscr{C}$-coherent rings and $\mathscr{C}$-semihereditary rings

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Abstract

Let R be a ring and $\mathscr{C}$ be a class of some finitely presented left R-modules. A left R-module M is called *$\mathscr{C}$-injective* if $\mathrm{Ext}^1_R(C, M) = 0$ for every $C \in \mathscr{C}$; a left R-module M is called $\mathscr{C}$-projective if $\mathrm{Ext}^1_R(M, E) = 0$ for any $\mathscr{C}$-injective module E. R is called left $\mathscr{C}$-coherent if every $C \in \mathscr{C}$ is 2-presented; R is called left strongly $\mathscr{C}$-coherent, if whenever $0 \to K \to P \to C \to 0$ is exact, where $C \in \mathscr{C}$ and P is finitely generated projective, then K is $\mathscr{C}$-projective; a ring R is called left $\mathscr{C}$-semihereditary, if whenever $0 \to K \to P \to C \to 0$ is exact, where $C \in \mathscr{C}$, P is finitely generated projective, then K is projective. In this paper, we give some new characterizations and properties of left $\mathscr{C}$-coherent rings, left strongly $\mathscr{C}$-coherent rings and left $\mathscr{C}$-semihereditary rings.

Mathematics Subject Classification (2010). 16D40, 16D50, 16E60, 16P70.

Keywords. $\mathscr{C}$-coherent ring, strongly $\mathscr{C}$-coherent ring, $\mathscr{C}$-semihereditary ring.

1. Introduction

Recall that a ring R is said to be *left coherent* [1, 19] if every finitely generated left ideal of R is finitely presented, a ring R is said to be *left semihereditary* if every finitely generated left ideal of R is projective. Coherent rings, semihereditary rings and their generalizations have been studied extensively by many authors (see, for example, [1, 2, 4, 6, 11, 13–15, 19, 24, 26]). In [27], we introduced the concepts of *left $\mathscr{C}$-coherent rings* and *left $\mathscr{C}$-semihereditary rings*, and in [28], we introduced the concept of *left strongly $\mathscr{C}$-coherent rings*. Let $\mathscr{C}$ be a class of some finitely presented left R-modules. Following [27], a ring R is called left $\mathscr{C}$-coherent if every $C \in \mathscr{C}$ is 2-presented; a ring R is called *left $\mathscr{C}$-semihereditary*, if whenever $0 \to K \to P \to C \to 0$ is exact , where $C \in \mathscr{C}$, P is finitely generated projective, then K is projective. To characterize left $\mathscr{C}$-coherent rings and left $\mathscr{C}$-semihereditary rings , in [27], we also introduced the concepts of *$\mathscr{C}$-injective modules* and *$\mathscr{C}$-flat modules*. According to [27], a left R-module M is called *$\mathscr{C}$-injective* if $\mathrm{Ext}^1_R(C, M) = 0$ for every $C \in \mathscr{C}$, a right R-module M is called *$\mathscr{C}$-flat* if $\mathrm{Tor}^R_1(M, C) = 0$ for every $C \in \mathscr{C}$. In [28], we introduced the concepts of *$\mathscr{C}$-projective modules* and *left strongly $\mathscr{C}$-coherent rings*. Following [28], a left R-module M is called $\mathscr{C}$-projective if $\mathrm{Ext}^1_R(M, E) = 0$ for any $\mathscr{C}$-injective module E; a ring R is called left strongly $\mathscr{C}$-coherent, if whenever $0 \to K \to P \to C \to 0$ is exact, where $C \in \mathscr{C}$ and P is finitely generated

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Received: 30.05.2018; Accepted: 05.04.2019

projective, then K is $\mathscr{C}$-projective. We shall denote the class of $\mathscr{C}$-flat (resp., $\mathscr{C}$-injective, $\mathscr{C}$-projective) modules by $\mathscr{C}\mathcal{F}$ (resp., $\mathscr{C}\mathcal{I}$, $\mathscr{C}\mathcal{P}$).

In this article, we continues to study left $\mathscr{C}$-coherent rings, left strongly $\mathscr{C}$-coherent rings and left $\mathscr{C}$-semihereditary rings. Series characterizations and properties of these rings will be given respectively.

Next, we recall some known notions and facts needed in the sequel.

Given a class $\mathscr{L}$ of R-modules, we shall denote by $\mathscr{L}^{\perp} = \{M : \mathrm{Ext}^1_R(L, M) = 0, L \in \mathscr{L}\}$ the right orthogonal class of $\mathscr{L}$, and by ${}^{\perp}\mathscr{L} = \{M : \mathrm{Ext}^1_R(M, L) = 0, L \in \mathscr{L}\}$ the left orthogonal class of $\mathscr{L}$.

Let $\mathcal{F}$ be a class of R-modules and M an R-module. Following [9], we say that a homomorphism $\varphi : M \to F$ where $F \in \mathcal{F}$ is an $\mathcal{F}$-preenvelope of M if for any morphism $f : M \to F'$ with $F' \in \mathcal{F}$, there is a $g : F \to F'$ such that $g\varphi = f$. An $\mathcal{F}$-preenvelope $\varphi : M \to F$ is said to be an $\mathcal{F}$-envelope if every endomorphism $g : F \to F$ such that $g\varphi = \varphi$ is an isomorphism. Dually, we have the definitions of $\mathcal{F}$-precovers and $\mathcal{F}$-covers. $\mathcal{F}$-envelopes ($\mathcal{F}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism. It is easy to see that every $\mathscr{C}$-injective preenvelope is monic, and every $\mathscr{C}$-projective precover is epic.

Following [9], a pair $(\mathscr{A}, \mathscr{B})$ of classes of R-modules is called a *cotorsion pair* if $\mathscr{A}^{\perp} = \mathscr{B}$ and ${}^{\perp}\mathscr{B} = \mathscr{A}$. A cotorsion pair $(\mathscr{A}, \mathscr{B})$ is called *hereditary* [10, Definition 1.1] if whenever $0 \to A' \to A \to A'' \to 0$ is exact with $A, A'' \in \mathscr{A}$ then A' is also in $\mathscr{A}$. By [10, Proposition 1.2], a cotorsion pair $(\mathscr{A}, \mathscr{B})$ is hereditary if and only if whenever $0 \to B' \to B \to B'' \to 0$ is exact with $B', B \in \mathscr{B}$ then B'' is also in $\mathscr{B}$. A cotorsion pair $(\mathscr{A}, \mathscr{B})$ is called *perfect* [10] if every R-module has an $\mathscr{A}$-cover and a $\mathscr{B}$-envelope. A cotorsion pair $(\mathscr{A}, \mathscr{B})$ is called *complete* (see [9, Definition 7.16] and [20, Lemma 1.13]) if for any R-module M, there are exact sequences $0 \to M \to B \to A \to 0$ with $A \in \mathscr{A}$ and $B \in \mathscr{B}$, and $0 \to B' \to A' \to M \to 0$ with $A' \in \mathscr{A}$ and $B' \in \mathscr{B}$.

Throughout this paper, R is an associative ring with identity and all modules considered are unitary, $\mathscr{C}$ is a class of some finitely presented left R-modules. For any R-module M, $E(M)$ will denote the injective envelope of M, $M^+ = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ will be the character module of M and $M^* = \mathrm{Hom}(M, R)$ will be the dual module of M.

2. $\mathscr{C}$-coherent rings

Theorem 2.1. *The following statements are equivalent for a ring R:*

(1) *R is a left $\mathscr{C}$-coherent ring.*
(2) *For any projective left R-module P, P^* is $\mathscr{C}$-flat.*
(3) *For any free left R-module F, F^* is $\mathscr{C}$-flat.*

Proof. (1)⇒(2). For any projective left R-module P, there is an index set I and an R-module Q such that $P \oplus Q \cong R^{(I)}$. So we have $P^* \oplus Q^* \cong (R^{(I)})^* \cong R^I$, and thus P^* is $\mathscr{C}$-flat by [27, Theorem 3.3(4) and Proposition 2.6].

(2)⇒(3). It is clear.

(3)⇒(1). Let I be any index set. Then by (3), $R^I \cong (R^{(I)})^*$ is $\mathscr{C}$-flat, and so R is $\mathscr{C}$-coherent by [27, Theorem 3.3(4)]. □

Recall that a left R-module M is said to be *FP-injective* [19] if $\mathrm{Ext}^1_R(A, M) = 0$ for every finitely presented left R-module A; a left R-module M is said to be *P-injective* [16] if every homomorphism from a principal left ideal of R to M can be extended to a homomorphism of R to M, it is easy to see that a left R-module M is P-injective if and only if $\mathrm{Ext}^1_R(R/Ra, M) = 0$ for any $a \in R$. We recall also that a left R-module M is said to be *FI-injective* [13] (resp., *D-injective* [14], *copure injective* [8]) if $\mathrm{Ext}^1_R(G, M) = 0$ for every FP-injective (resp., P-injective, injective) left R-module G; a right R-module N is said to be *FI-flat* [13] (resp., *D-flat* [14], *copure flat* [8]) if $\mathrm{Tor}^R_1(N, G) = 0$ for every

FP-injective (resp., P-injective, injective) left R-module G. Inspired by these concepts, we have the following concepts.

Definition 2.2. A left R-module M is said to be $\mathscr{C}$I-injective if $\mathrm{Ext}^1_R(G,M)=0$ for every $\mathscr{C}$-injective left R-module G; a right R-module F is said to be $\mathscr{C}$I-flat if $\mathrm{Tor}^R_1(F,G)=0$ for every $\mathscr{C}$-injective left R-module G.

Proposition 2.3. *The following statements are equivalent for a left R-module M:*

(1) *M is $\mathscr{C}$I-injective.*
(2) *For every exact sequence $0\to M\to E\to L\to 0$ with E $\mathscr{C}$-injective, $E\to L$ is a $\mathscr{C}$-injective precover of L.*
(3) *M is the kernel of a $\mathscr{C}$-injective precover $f:E\to L$ with E injective.*
(4) *M is injective with respect to every exact sequence $0\to A\to B\to C\to 0$ with C $\mathscr{C}$-injective.*

Proof. (1)⇒(2) and (1)⇒(4) are clear.

(2)⇒(3). It follows from the exact sequence $0\to M\to E(M)\to E(M)/M\to 0$.

(3)⇒(1). Let M be the kernel of a $\mathscr{C}$-injective precover $f:E\to L$ with E injective. Then $f:E\to \mathrm{im}(f)$ is a $\mathscr{C}$-injective precover, so, for any $\mathscr{C}$-injective module N, the map $\mathrm{Hom}(N,E)\to\mathrm{Hom}(N,\mathrm{im}(f))$ is epic and hence the map $\mathrm{Hom}(N,E)\to\mathrm{Hom}(N,E/M)$ is epic. Thus, by the exactness of the sequence $0\to\mathrm{Hom}(N,E)\to\mathrm{Hom}(N,E/M)\to\mathrm{Ext}^1_R(N,M)\to 0$, we have $\mathrm{Ext}^1_R(N,M)=0$.

(4) ⇒ (1). For any $\mathscr{C}$-injective module N, there exists an exact sequence $0\to K\to P\to N\to 0$, where P is projective. Hence we get an exact sequence $\mathrm{Hom}(P,M)\to\mathrm{Hom}(K,M)\to\mathrm{Ext}^1_R(N,M)\to\mathrm{Ext}^1_R(P,M)=0$, and thus $\mathrm{Ext}^1_R(N,M)=0$ by (4). Therefore, M is $\mathscr{C}$I-injective. □

Remark 2.4. Since the class of all $\mathscr{C}$-injective modules is closed under extensions, by Wakamutsu's Lemma (see [23, Lemma 2.1.1]), any kernel of a $\mathscr{C}$-injective cover is $\mathscr{C}$I-injective .

Recall that a left R-module M is called reduced [9] if M has no nonzero injective submodules.

Proposition 2.5. *Let R be a left $\mathscr{C}$-coherent ring. Then the following statements are equivalent for a left R-module M:*

(1) *M is a reduced $\mathscr{C}$I-injective module.*
(2) *M is the kernel of a $\mathscr{C}$-injective cover $f:E\to L$ with E injective.*

Proof. (1)⇒(2). Since M is $\mathscr{C}$I-injective, by proposition 2.3, the natural mapping $\pi: E(M)\to E(M)/M$ is a $\mathscr{C}$-injective precover. Since R is left $\mathscr{C}$-coherent, by [27, Corollary 3.7], $E(M)/M$ has a $\mathscr{C}$-injective cover. Note that there is no nonzero summand K of $E(M)$ contained in M as M is reduced, by [23, Corollary 1.2.8], $\pi: E(M)\to E(M)/M$ is a $\mathscr{C}$-injective cover.

(2)⇒(1). Let M be the kernel of a $\mathscr{C}$-injective cover $f:E\to L$ with E injective. Then by proposition 2.3(3), M is a $\mathscr{C}$I-injective module. Now let K be an injective submodule of M. Suppose $E=K\oplus N, p:E\to N$ is the projective and $i:N\to E$ is the inclusion for some submodule N of M. It is easy to see that $f(ip)=f$ since $f(K)=0$. So ip is an isomorphism since f is a cover. Thus i is epic and hence $E=N,K=0$. Therefore M is reduced. □

Recall that a submodule A of left R-module B is said to be a pure submodule if for all right R-module M, the induced map $M\otimes_R A\to M\otimes_R B$ is monic, or equivalently, every finitely presented left R-module is projective with respect to the exact sequence $0\to A\to B\to B/A\to 0$. In this case, the exact sequence $0\to A\to B\to B/A\to 0$

is called *pure exact*. An exact sequence $0 \to A \to B \to L \to 0$ is called *RD-exact* [14] if, for any $a \in R$, R/Ra is projective with respect to this sequence. We call a short exact sequence of left R-modules $0 \to A \to B \to L \to 0$ *$\mathscr{C}$-pure exact* if every $C \in \mathscr{C}$ is projective with respect to this sequence. Let A be a submodule of B, if the short exact sequence of left R-modules $0 \to A \to B \to B/A \to 0$ is $\mathscr{C}$-pure exact, then we call A a $\mathscr{C}$-pure submodule of B and B/A a $\mathscr{C}$-pure quotient module of B.

Next, we give some characterizations of $\mathscr{C}$-injective modules.

Theorem 2.6. *Let M be a left R-module, then the following statements are equivalent:*

(1) *M is $\mathscr{C}$-injective.*
(2) *M is injective with respect to every exact sequence $0 \to A \to B \to C \to 0$ of left R-modules with $C \in \mathscr{C}$.*
(3) *M is injective with respect to every exact sequence $0 \to K \to P \to C \to 0$ of left R-modules with $C \in \mathscr{C}$ and P finitely generated projective.*
(4) *Every exact sequence $0 \to M \to M' \to M'' \to 0$ is $\mathscr{C}$-pure.*
(5) *There exists a $\mathscr{C}$-pure exact sequence $0 \to M \to M' \to M'' \to 0$ of left R-modules with M' injective.*
(6) *There exists a $\mathscr{C}$-pure exact sequence $0 \to M \to M' \to M'' \to 0$ of left R-modules with M' FP-injective.*
(7) *There exists a $\mathscr{C}$-pure exact sequence $0 \to M \to M' \to M'' \to 0$ of left R-modules with M' $\mathscr{C}$-injective.*

Proof. (1) $\Rightarrow$ (2). It follows from the exact sequence

$$\mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M) \to \mathrm{Ext}^1_R(C, M) = 0.$$

(2) $\Rightarrow$ (3). It is obvious.

(3) $\Rightarrow$ (1). It follows from the exact sequence

$$\mathrm{Hom}(P, M) \to \mathrm{Hom}(K, M) \to \mathrm{Ext}^1_R(C, M) \to \mathrm{Ext}^1_R(P, M) = 0.$$

(1) $\Rightarrow$ (4). Assume (1). Then we have an exact sequence $\mathrm{Hom}(C, M') \to \mathrm{Hom}(C, M'') \to \mathrm{Ext}^1_R(C, M) = 0$ for every $C \in \mathscr{C}$, and so (4) follows.

(4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (7) is obvious.

(7) $\Rightarrow$ (1). By (7), we have a $\mathscr{C}$-pure exact sequence $0 \to M \to M' \xrightarrow{f} M'' \to 0$ of left R-modules where M' is $\mathscr{C}$-injective, and so, for each $C \in \mathscr{C}$, we have an exact sequence $\mathrm{Hom}(C, M') \xrightarrow{f_*} \mathrm{Hom}(C, M'') \to \mathrm{Ext}^1_R(C, M) \to \mathrm{Ext}^1_R(C, M') = 0$ with f_* epic. Which implies that $\mathrm{Ext}^1_R(C, M) = 0$, and (1) follows. □

Recall that a left R-module M is called *pure injective* [9, Definition 5.3.6] if it is injective with respect to every pure exact sequence of left R-modules; a left R-module M is called *RD-injective* [14] if it is injective with respect to every RD-exact sequence of left R-modules. We call a left R-module M *$\mathscr{C}$-pure injective* if it is injective with respect to every $\mathscr{C}$-pure exact sequence of left R-modules.

Proposition 2.7. *Let R be a left $\mathscr{C}$-coherent ring. Then every $\mathscr{C}$-pure injective module M has a $\mathscr{C}$-injective cover $f : N \to M$ with N injective. Moreover,* $\mathrm{Ker}(f)$ *is a reduced $\mathscr{C}$I-injective left R-module.*

Proof. By [27, Corollary 3.7], M has a $\mathscr{C}$-injective cover $f : N \to M$. Since N is $\mathscr{C}$-injective, by Theorem 2.6(4), the exact sequence $0 \to N \xrightarrow{i} E(N) \to E(N)/N \to 0$ is $\mathscr{C}$-pure exact, and so there exists $g : E(N) \to M$ such that $gi = f$. Note that f is a cover, there exists $h : E(N) \to N$ such that $fh = g$. Thus $fhi = f$ and hence hi is an isormorphism. It follows that N is isomorphic to a direct summand of $E(N)$ and so N is injective. By Proposition 2.5, $\mathrm{Ker}(f)$ is a reduced $\mathscr{C}$I-injective left R-module. □

Theorem 2.8. *Let R be a left $\mathscr{C}$-coherent ring. Then a left R-module M is $\mathscr{C}$I-injective if and only if M is a direct sum of an injective left R-module and a reduced $\mathscr{C}$I-injective left R-module.*

Proof. " $\Leftarrow$ ". It is clear.

" $\Rightarrow$ ". Let M be a $\mathscr{C}$I-injective left R-module . Then by Proposition 2.3, $E(M) \to E(M)/M$ is a $\mathscr{C}$-injective precover. Since R is left $\mathscr{C}$-coherent, $E(M)/M$ has a $\mathscr{C}$-injective cover $L \xrightarrow{g} E(M)/M$ by [27, Corollary 3.7], so we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \xrightarrow{f} & L & \xrightarrow{g} & E(M)/M & \longrightarrow & 0 \\ & & \downarrow{\phi} & & \downarrow{\gamma} & & \| & & \\ 0 & \longrightarrow & M & \xrightarrow{\alpha} & E(M) & \xrightarrow{\pi} & E(M)/M & \longrightarrow & 0 \\ & & \downarrow{\sigma} & & \downarrow{\beta} & & \| & & \\ 0 & \longrightarrow & K & \xrightarrow{f} & L & \xrightarrow{g} & E(M)/M & \longrightarrow & 0 \end{array}$$

where K is a reduced $\mathscr{C}$I-injective left R-module by Proposition 2.5. Note that $g = g(\beta\gamma)$, we have that $\beta\gamma$ is an isomorphism, so $E(M) = \mathrm{Ker}(\beta) \oplus im(\gamma)$, and thus $\mathrm{Ker}(\beta)$ is injective. Since $\sigma\phi$ is an isomorphism by the Five Lemma, we have that $M = \mathrm{Ker}(\sigma) \oplus im(\phi)$ and $im(\phi) \cong K$. Moreover, by the Snake Lemma [17, Theorem 6.5], we have that $\mathrm{Ker}(\sigma) \cong \mathrm{Ker}(\beta)$ is injective. This completes the proof. □

Proposition 2.9. *Let M be a right R-module. Then M is $\mathscr{C}$I-flat if and only if M^+ is $\mathscr{C}$I-injective.*

Proof. It follows from the isomorphism $\mathrm{Tor}_1^R(M, G)^+ \cong \mathrm{Ext}_R^1(G, M^+)$. □

Corollary 2.10. *A pure submodule of a $\mathscr{C}$I-flat module is $\mathscr{C}$I-flat.*

Proof. Let M be a $\mathscr{C}$I-flat module and M_1 a pure submodule of M, then the pure exact sequence $0 \to M_1 \to M \to M/M_1 \to 0$ induces a split exact sequence $0 \to (M/M_1)^+ \to M^+ \to M_1^+ \to 0$. By Proposition 2.9, M^+ is $\mathscr{C}$I-injective, so M_1^+ is $\mathscr{C}$I-injective, and hence M_1 is $\mathscr{C}$I-flat by Proposition 2.9 again. □

Proposition 2.11. *Let R be a ring and $\mathscr{C}$ be a class of some finitely presented left R-modules.*

(1) *If M is a finitely presented $\mathscr{C}$I-flat module, then it is a cokernet of a $\mathscr{C}$-flat preenvelope.*
(2) *If R is left $\mathscr{C}$-coherent and L is the cokernet of a $\mathscr{C}$I-flat preenvelope $f : M \to F$, then L is $\mathscr{C}$I-flat.*

Proof. (1). Let M be a finitely presented $\mathscr{C}$I-flat module. Then there exists an exact sequence of right R-modules $0 \to K \to P \to M \to 0$ with P finitely generated projective and K finitely generated. We claim that $K \to P$ is a $\mathscr{C}$-flat preenvelope. In fact, for any $\mathscr{C}$-flat module F, we have F^+ is $\mathscr{C}$-injective by [27, Theorem 2.7], and so $\mathrm{Tor}_1^R(M, F^+) = 0$ since M is $\mathscr{C}$I-flat. Hence, we have the following commutative diagram with α monic:

$$\begin{array}{ccc} K \otimes F^+ & \xrightarrow{\alpha} & P \otimes F^+ \\ {\scriptstyle\tau_1}\downarrow & & \downarrow{\scriptstyle\tau_2} \\ \mathrm{Hom}(K, F)^+ & \xrightarrow{\beta} & \mathrm{Hom}(P, F)^+ \end{array}$$

Since K is finitely generated and P is finitely presented, by [3, Lemma 2], τ_1 is epic and τ_2 is an isomorphism, this follows that β is monic, and hence $\mathrm{Hom}(P, F) \to \mathrm{Hom}(K, F)$ is epic, as required.

(2). There is an exact sequence $0 \to \text{im}(f) \xrightarrow{i} F \to L \to 0$. We claim that $i : \text{im}(f) \to F$ is a $\mathscr{C}$-flat preenvelope. In fact, for any $\mathscr{C}$-flat module F_1 and any homomorphism $\varphi : \text{im}(f) \to F_1$, φf is a homomorphism from M to F_1. Since $f : M \to F$ is a $\mathscr{C}$-flat preenvelope, there exists a $\psi : F \to F_1$ such that $\varphi f = \psi f$. Now, for any $y \in \text{im}(f)$, write $y = f(x)$. Then $\varphi f(x) = \psi i f(x)$, i.e., $\varphi(y) = \psi i(y)$. It shows that $\varphi = \psi i$, and so $i : \text{im}(f) \to F$ is a $\mathscr{C}$-flat preenvelope. Let N be any $\mathscr{C}$-injective module. Since R is left $\mathscr{C}$-coherent, N^+ is $\mathscr{C}$-flat by [27, Theorem 3.3(8)], and so, the mapping $\text{Hom}(F, N^+) \to \text{Hom}(\text{im}(f), N^+)$ is epic. Then, from the following commutative diagram :

$$\begin{array}{ccc} \text{Hom}(F, N^+) & \xrightarrow{\alpha} & \text{Hom}(\text{im}(f), N^+) \\ \sigma_1 \downarrow & & \downarrow \sigma_2 \\ (F \otimes N)^+ & \xrightarrow{\beta} & (\text{im}(f) \otimes N)^+ \end{array}$$

where σ_1 and σ_2 are isomorphisms, we have that the mapping $(F \otimes N)^+ \to (\text{im}(f) \otimes N)^+$ is epic. Thus, the mapping $\text{im}(f) \otimes N \to F \otimes N$ is monic. But the $\mathscr{C}$I-flatness of F implies the exactness of $0 \to \text{Tor}_1^R(L, N) \to \text{im}(f) \otimes N \to F \otimes N$, and therefore $\text{Tor}_1^R(L, N) = 0$. □

3. Strongly $\mathscr{C}$-coherent rings

Theorem 3.1. *The following statements are equivalent for a ring R:*

(1) *R is a left strongly $\mathscr{C}$-coherent ring.*
(2) *If $0 \to K \to E \to L \to 0$ is an exact sequence of left R-modules with K $\mathscr{C}$-injective and E FP-injective, then L is $\mathscr{C}$-injective.*
(3) *If $0 \to K \to E \to L \to 0$ is an exact sequence of left R-modules with K $\mathscr{C}$-injective and E injective, then L is $\mathscr{C}$-injective.*
(4) *R is left $\mathscr{C}$-coherent, and if $0 \to N \to M \to Q \to 0$ is an exact sequence of right R-modules with M and Q $\mathscr{C}$-flat, then N is $\mathscr{C}$-flat.*
(5) *R is left $\mathscr{C}$-coherent, and if $0 \to N \to M \to Q \to 0$ is an exact sequence of right R-modules with M flat and Q $\mathscr{C}$-flat, then N is $\mathscr{C}$-flat.*
(6) *R is left $\mathscr{C}$-coherent, and if $0 \to N \to P \to Q \to 0$ is an exact sequence of right R-modules with P projective and Q $\mathscr{C}$-flat, then N is $\mathscr{C}$-flat.*

Proof. (1)⇒(2). It follows from [28, Theorem 1(7)].

(2)⇒(3); and (4)⇒ (5) ⇒ (6) are trivial.

(3)⇒(1). Let M be a $\mathscr{C}$-injective left R-module. Then by (2), $E(M)/M$ is $\mathscr{C}$-injective. And so R is left strongly $\mathscr{C}$-coherent by [28, Theorem 1(8)].

(1)⇒(4). It follows from [28, Theorem 1(9)] and [27, Proposition 3.11(2)].

(6)⇒(1). For any $\mathscr{C}$-flat right R-module N, there exists an exact sequence $0 \to K \to P \to N \to 0$ with P projective. So K is $\mathscr{C}$-flat by (6), and thus $\text{Tor}_2^R(N, C) \cong \text{Tor}_1^R(K, C) = 0$ for any $C \in \mathscr{C}$. Therefore R is left strongly $\mathscr{C}$-coherent by [28, Theorem 1(11)]. □

Proposition 3.2. *Let R be a left strongly $\mathscr{C}$-coherent ring. Then the following statements are equivalent for a left R-module M:*

(1) *M is injective.*
(2) *M is both $\mathscr{C}$-injective and $\mathscr{C}$I-injective.*
(3) *There exists a $\mathscr{C}$-injective cover $f : M \to N$ with N $\mathscr{C}$I-injective.*

Proof. (1)⇒(2). It is trivial.

(2)⇒(3). It is clear because $M \to M$ is a $\mathscr{C}$-injective cover of M.

(3)⇒(1). Consider the exact sequence $0 \to M \xrightarrow{i} E(M) \to E(M)/M \to 0$. Since R is a left strongly $\mathscr{C}$-coherent ring, by [28, Theorem 1(7)], $E(M)/M$ is $\mathscr{C}$-injective, so $\text{Ext}_R^1(E(M)/M, N) = 0$. Thus there exists a homomorphism $g : E(M) \to N$ such that

$f = gi$. Since f is a cover, there exists a homomorphism $h : E(M) \to M$ such that $g = fh$. Hence $f(hi) = f$, and so hi is an isomorphism, this follows that i is left split, and therefore $M = E(M)$ is injective. □

Theorem 3.3. *The following statements are equivalent for a ring R:*

(1) *R is a left strongly $\mathscr{C}$-coherent ring.*
(2) *R is left $\mathscr{C}$-coherent, and every $\mathscr{C}$-injective $\mathscr{C}$I-injective left R-module is injective.*
(3) *Each left R-module has a $\mathscr{C}$-injective cover, and every $\mathscr{C}$-injective $\mathscr{C}$I-injective left R-module is injective.*
(4) *R is left $\mathscr{C}$-coherent, and for every $\mathscr{C}$I-injective left R-module L, there there exists a $\mathscr{C}$-injective cover $E \to L$ with E injective.*
(5) *Each left R-module has a $\mathscr{C}$-injective cover, and for every $\mathscr{C}$I-injective left R-module L, there there exists a $\mathscr{C}$-injective cover $E \to L$ with E injective.*
(6) *Every $\mathscr{C}$-pure quotient of a $\mathscr{C}$-injective left R-module has a $\mathscr{C}$-injective cover, and for every $\mathscr{C}$I-injective left R-module L, there exists a $\mathscr{C}$-injective cover $E \to L$ with E injective.*
(7) *Every $\mathscr{C}$-pure quotient of a $\mathscr{C}$-injective left R-module has a $\mathscr{C}$-injective cover, and every $\mathscr{C}$-injective $\mathscr{C}$I-injective left R-module is injective.*

Proof. (1)⇒(2). Since R is left strongly $\mathscr{C}$-coherent, by [28, Theorem 1(10)], it is left $\mathscr{C}$-coherent. Moreover, by Proposition 3.2, every $\mathscr{C}$-injective $\mathscr{C}$I-injective left R-module is injective.

(2)⇒(3). It follows from [27, Corollary 3.7].

(1)⇒(4). It is clear that R is left $\mathscr{C}$-coherent. Let L be any $\mathscr{C}$I-injective left R-module. Then by [27, Corollary 3.7], L has a $\mathscr{C}$-injective cover $f : E \to L$, and by Proposition 3.2, E is injective.

(4)⇒(5). It follows from [27, Corollary 3.7].

(3)⇒(7), and (5)⇒(6) are trivial.

(6)⇒(7). Let M be a $\mathscr{C}$-injective $\mathscr{C}$I-injective left R-module. Then by (6), there exists a $\mathscr{C}$-injective cover $f : E \to M$ with E injective. Note that $1_M : M \to M$ is also a $\mathscr{C}$-injective cover of M, we have that $M \cong E$, and hence M is injective.

(7)⇒(1). Let $0 \to N \xrightarrow{i} E \xrightarrow{f} L \to 0$ be an exact sequence of left R-modules with N $\mathscr{C}$-injective and E injective. Then by Theorem 2.6(4), this exact sequence is $\mathscr{C}$-pure, and so L has a $\mathscr{C}$-injective cover $\varphi : E' \to L$. Thus there exists a homomorphism $g : E \to E'$ such that $f = \varphi g$. Since f is epic, φ is also epic. Now, forming a pullback we obtain the following commutative diagram with exact rows and columns (see [21, 10.3(1)]).

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & P & \xrightarrow[h_2]{} & E' & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle\varphi} & & \\
0 & \longrightarrow & N & \longrightarrow & E & \xrightarrow[f]{} & L & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

where $P=\{(x,y)\in E'\oplus E\mid \varphi(x)=f(y)\}$, $K=\mathrm{Ker}(\varphi)$, $\alpha: K\to P, k\mapsto (k,0)$, $h_1(x,y)=x, h_2(x,y)=y$. Let $\beta: P\to E', (x,y)\mapsto x-g(y)$. Then $\varphi\beta(x,y)=\varphi(x)-\varphi g(y)=\varphi(x)-f(y)=0$, so $\beta(x,y)\in K$, and hence β is a homomorphism from P to K. Note that $\beta\alpha(k)=\beta(k,0)=k-g(0)=k$, we have that $\beta\alpha=1_K$. Since N and E' are both $\mathscr{C}$-injective, P is also $\mathscr{C}$-injective, and so K is $\mathscr{C}$-injective. Note that K is $\mathscr{C}$I-injective by [9, Corollary 7.2.3], we have that K is injective by conditions, so L is $\mathscr{C}$-injective, and hence R is a left strongly $\mathscr{C}$-coherent ring by Theorem 3.1(3). □

Let $\mathfrak{F}$ be a class of R-modules. According to [5], an $\mathfrak{F}$-cover $\phi: F\to M$ is said to have the unique mapping property if for any homomorphism $f: F'\to M$ with $F'\in\mathfrak{F}$, there is a unique homomorphism $g: F'\to F$ such that $f=\phi g$.

Theorem 3.4. *The following statements are equivalent for a ring R:*

(1) *Every left R-module is $\mathscr{C}$-projective.*
(2) *Every nonzero left R-module has a nonzero $\mathscr{C}$-projective submodule.*
(3) *R is left strongly $\mathscr{C}$-coherent, and every ($\mathscr{C}$-injective) left R-module has a $\mathscr{C}$-projective cover with the unique mapping property.*

Proof. (1)⇒(2) and (1)⇒ (3) are obvious.

(2)⇒(1). Assume (2). To prove (1), we need only to prove that every $\mathscr{C}$-injective module E is injective by [28, Theorem 6(3)].

Let I be a left ideal of R , $i: I\to R$ be the inclusion map and $f: I\to E$ be any homomorphism. It suffices to show that there is $g: R\to E$ that extends f. Let $\mathscr{A}$ consist of all pair (I',g'), where $I\subseteq I'\subseteq R$ and $g': I'\to E$ extends f. Since $(I,f)\in\mathscr{A}$, $\mathscr{A}\neq\phi$. $\mathscr{A}$ is a partially set by saying $(I',g')\leq(I'',g'')$ if $I'\subseteq I''$ and g'' extends g'. By Zorn's Lemma, there is a maximal element (I_0,g_0) in $\mathscr{A}$. If $I_0\neq R$, then $R/I_0\neq 0$. By (2), there is a nonzero $\mathscr{C}$-projective submodule K/I_0 of R/I_0. Note that $\mathrm{Ext}^1_R(K/I_0,E)=0$, we have that g_0 can be extended to K, this contradicts to the maximality of (I_0,g_0). Thus, $I_0=R$ and E is injective, as required.

(3)⇒(1). Assume (3). To prove (1), we need only to prove that every $\mathscr{C}$-injective module E is $\mathscr{C}$-projective by [28, Theorem 6(4)]. By (3), E has a $\mathscr{C}$-projective cover $\phi: P\to E$ with the unique mapping property. Let $K=\mathrm{Ker}(\phi)$, $i: K\to P$ be the inclusion map and $\varphi: P'\to K$ be a $\mathscr{C}$-projective cover of K. Then $\phi i\varphi=0=\phi 0$, and so $i\varphi=0$ by the unique mapping property. Since every $\mathscr{C}$-projective cover is epic, φ and ϕ are epic, so ϕ is an isomorphism, and thus E is $\mathscr{C}$-projective. This completes the proof. □

According to [28], the $\mathscr{C}$-injective dimension of a module ${}_RM$ is defined by

$$\mathscr{C}\mathfrak{I}\text{-}dim({}_RM)=inf\{n:\mathrm{Ext}^{n+1}_R(C,M)=0 \text{ for every } C\in\mathscr{C}\};$$

the $\mathscr{C}$-injective global dimension of a ring R is defined by

$$\mathscr{C}\mathfrak{I}\text{-GLD(R)=sup}\{\mathscr{C}\mathfrak{I}\text{-dim}(M):\ M \text{ is a left } R\text{-module}\};$$

the $\mathscr{C}$-flat dimension of a module M_R is defined by

$$\mathscr{C}\mathfrak{F}\text{-}dim(M_R)=inf\{n:\mathrm{Tor}^R_{n+1}(M,C)=0 \text{ for every } C\in\mathscr{C}\};$$

the $\mathscr{C}$-weak global dimension of a ring R is defined by

$$\mathscr{C}\text{-WD(R)=sup}\{\mathscr{C}\mathfrak{F}\text{-dim}(M):\ M \text{ is a right } R\text{-module}\}.$$

Theorem 3.5. *Let R be a left strongly $\mathscr{C}$-coherent ring, M a left R-module and n a nonnegative integer. Then the following statements are equivalent:*

(1) *$\mathscr{C}\mathfrak{I}$-dim$({}_RM)\leq n$.*
(2) *$\mathrm{Ext}^{n+k}_R(P,M)=0$ for all $\mathscr{C}$-projective module P and all positive integers k.*
(3) *$\mathrm{Ext}^{n+1}_R(P,M)=0$ for all $\mathscr{C}$-projective module P.*

Proof. (1)⇒(2). Assume (1). Then since R is left strongly $\mathscr{C}$-coherent, by [28, Theorem 2], there exists an exact sequence of left R-modules $0 \to M \stackrel{\varepsilon}{\to} E_0 \stackrel{d_0}{\to} \cdots \to E_{n-1} \stackrel{d_{n-1}}{\to} E_n \to 0$ such that $E_0, \cdots, E_{n-1}, E_n$ are $\mathscr{C}$-injective. Thus, by [28, Theorem 1(12)], we have $\mathrm{Ext}_R^{n+1}(P, M) \cong \mathrm{Ext}_R^{n}(P, im(d_0)) \cong \mathrm{Ext}_R^{n-1}(P, im(d_1)) \cong \cdots \cong \mathrm{Ext}_R^{1}(P, im(d_{n-1})) = \mathrm{Ext}_R^{1}(P, E_n) = 0$ for any $\mathscr{C}$-projective module P, and $\mathrm{Ext}_R^{n+k}(P, M) \cong \mathrm{Ext}_R^{1}(P, 0) = 0$ for any $k > 1$. So (2) follows.

(2)⇒ (3) ⇒(1). It is trivial. □

Corollary 3.6. *Let R be a left strongly $\mathscr{C}$-coherent ring and $0 \to A \to B \to C \to 0$ an exact sequence of left R-modules. If two of $\mathscr{C}\mathfrak{I}$-dim(A),$\mathscr{C}\mathfrak{I}$-dim(B),$\mathscr{C}\mathfrak{I}$-dim(C) are finite, then so is the third. Moreover:*

1. *$\mathscr{C}\mathfrak{I}$-dim(B)≤ sup{ $\mathscr{C}\mathfrak{I}$-dim(A), $\mathscr{C}\mathfrak{I}$-dim(C)}.*
2. *$\mathscr{C}\mathfrak{I}$-dim(A)≤ sup{ $\mathscr{C}\mathfrak{I}$-dim(B), $\mathscr{C}\mathfrak{I}$-dim$(C)+1$}.*
3. *$\mathscr{C}\mathfrak{I}$-dim(C)≤ sup{ $\mathscr{C}\mathfrak{I}$-dim(B), $\mathscr{C}\mathfrak{I}$-dim$(A)-1$}.*

In particular, $\mathscr{C}\mathfrak{I}$-dim$(A \oplus C)$= sup{ $\mathscr{C}\mathfrak{I}$-dim(A), $\mathscr{C}\mathfrak{I}$-dim(C)}.

Let n be a positive integer. then according to [4], a left R-module M is said to be n-presented in case there is an exact sequence of left R-modules $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ in which every F_i is finitely generated free. It is easy to see that a left R-module M is n-presented if and only if there exists an exact sequence of left R-modules $0 \to K_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ such that $F_0, \cdots, F_{n-1}$ are finitely generated free and K_n is finitely generated.

Lemma 3.7. *Let R be a left strongly $\mathscr{C}$-coherent ring. Then every $C \in \mathscr{C}$ is n-presented for any positive integer n.*

Proof. Use induction on n. If $n = 1$, then it is clear that the result holds. Assume that every $C \in \mathscr{C}$ is n-presented. Then for any $C \in \mathscr{C}$ and any FP-injective module N, we have $\mathrm{Ext}_R^{n+1}(C, N) = 0$ by [28, Theorem 1(5)] because R is left strongly $\mathscr{C}$-coherent. Let $0 \to K_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to C \to 0$ be an exact sequence of left R-modules with $F_0, \cdots, F_{n-1}$ finitely generated free left R–modules and K_n finitely generated. Then $\mathrm{Ext}_R^{1}(K_n, N) \cong \mathrm{Ext}_R^{n+1}(C, N) = 0$, so K_n is finitely presented by [7], and hence C is $(n+1)$-presented. □

Theorem 3.8. *Let R be a left strongly $\mathscr{C}$-coherent ring and M a left R-module. Then $\mathscr{C}\mathfrak{I}$-dim(M)=$\mathscr{C}\mathfrak{F}$-dim(M^+).*

Proof. Let n be a positive integer, $C \in \mathscr{C}$. Since R is left strongly $\mathscr{C}$-coherent, by Lemma 3.7, C is $(n+2)$-presented. So, by [2, Lemma 2.7(2)], we have $\mathrm{Tor}_{n+1}^R(M^+, C) \cong \mathrm{Ext}_R^{n+1}(C, M)^+$. Consequently, $\mathscr{C}\mathfrak{I}$-dim(M) =$\mathscr{C}\mathfrak{F}$-dim(M^+) by [28, Theorem 2, Theorem 3]. □

Theorem 3.9. *Let R be left strongly $\mathscr{C}$-coherent and ${}_RR$ be $\mathscr{C}$-injective. If ${}_RM$ is $\mathscr{C}$-projective with finite projective dimension, then ${}_RM$ is projective.*

Proof. Suppose that ${}_RM$ is $\mathscr{C}$-projective with $pd(M) = n < \infty$. Then by [28, Theorem 5], there exists an exact sequence of left R-modules

$$0 \to P_n \stackrel{d_n}{\to} P_{n-1} \stackrel{d_{n-1}}{\to} \cdots \to P_1 \stackrel{d_1}{\to} P_0 \stackrel{d_0}{\to} M \to 0$$

such that $P_0, \cdots, P_{n-1}, P_n$ are projective. Since ${}_RR$ is $\mathscr{C}$-injective and direct sums and direct summands of $\mathscr{C}$-injective modules are $\mathscr{C}$-injective by [28, Proposition 2.5], each P_i is $\mathscr{C}$-injective for $i = 0, 1, \cdots, n$. Clearly, $im(d_n) \cong P_n$ is $\mathscr{C}$-injective. Note that R is left strongly $\mathscr{C}$-coherent , by [28, Theorem 1(7)], $im(d_{n-1})$ is $\mathscr{C}$-injective. Continues in this way, one can get that $im(d_1)$ is $\mathscr{C}$-injective, so $\mathrm{Ext}_R^1(M, im(d_1)) = 0$, and thus the exact sequence $0 \to im(d_1) \to P_0 \to M \to 0$ is split, this follows that ${}_RM$ is projective, as required. □

Recall that, by [28, Example 1], a left $\mathscr{C}$-coherent ring need not be left strongly $\mathscr{C}$-coherent. As the end of this section, we give another example which shows that even if R is a left artinian ring, it need not be left strongly $\mathscr{C}$-coherent.

Example 3.10. Let K be a field and L be a proper subfield of K such that $\rho : K \to L$ is an isomorphism. Let $K[x;\rho]$ be the ring of twisted right polynomials over K where $kx = x\rho(k)$ for all $k \in K$. Set $R = K[x;\rho]/(x^2)$, and $\mathscr{C} = \{R/Ra : a \in R\}$. If b_1, b_2 is a basis for K as a vector space over L, then R is left artinian and hence left $\mathscr{C}$-coherent, but it is not left strongly $\mathscr{C}$-coherent.

Proof. Since K has finite vector space dimension over L, by [18, Example 1], R is left artinian . Since the only proper right ideal of R is $\mathbf{r}_R(x) = xR = xK$, it is readily verified that $\mathbf{r}_R\mathbf{l}_R(a) = aR$ for any $a \in R$, so ${}_RR$ is P-injective by [16, Lemma 1.1]. Now, we define $f : Rxb_1 + Rxb_2 \to R$ by $f(r_1xb_1 + r_2xb_2) = r_1x + r_2x$, then it is easy to see that f is a left R-homomorphism. We claim that this homomorphism can not be extended to an endomorphism of R. Otherwise, there exists a $c = k_0 + xk_0' \in R$ such that $f = \cdot c$. Clearly, $k_0 \neq 0$. Thus, $f(xb_1 - xb_2) = (xb_1 - xb_2)(k_0 + xk_0')$, and so $0 = x - x = (xb_1 - xb_2)k_0$, this follows that $b_1 = b_2$, a contradiction. Observing that $\mathbf{l}_R(x) = xK = xR = Rxb_1 + Rxb_2$, we have $\mathrm{Ext}^1_R(Rx, R) \cong \mathrm{Ext}^1_R(R/(Rxb_1 + Rxb_2), R) \neq 0$, and hence R is not left strongly $\mathscr{C}$-coherent. □

4. $\mathscr{C}$-semihereditary rings

We begin with the following definition.

Definition 4.1. A ring R is called weakly $\mathscr{C}$-semihereditary, if whenever $0 \to K \to P \to C \to 0$ is exact , where $C \in \mathscr{C}$, P is finitely generated projective , then K is flat.

Recall that a ring R is called *left weakly n-semihereditary* [25] if every n-generated left ideal is flat; a ring R is called a *left p.f ring* [11] if every principal left ideal of R is flat. By [11, Theorem 2.2], a ring R is left p.f if and only if it is right p.f; a ring R is called a *left FS-ring* [12, 22] if $Soc({}_RR)$ is flat.

Example 4.2. (1). Let $\mathscr{C} = \{R/I : I$ is an n-generated left ideal of $R\}$. Then the ring R is weakly $\mathscr{C}$-semihereditary if and only if R is left weakly n-semihereditary.

(2). Let $\mathscr{C} = \{R/Ra : a \in R\}$. Then the ring R is weakly $\mathscr{C}$-semihereditary if and only if R is left p.f.

(3). Let $\mathscr{C} = \{R/Ra : Ra$ is a minimal left ideal of $R\}$. Then the ring R is weakly $\mathscr{C}$-semihereditary if and only if every minimal left ideal of R is flat, if and only if R is a left FS-ring .

Theorem 4.3. *The following statements are equivalent for a ring R:*

(1) *R is a left weakly $\mathscr{C}$-semihereditary ring.*
(2) *Every submodule of a $\mathscr{C}$-flat right R-module is $\mathscr{C}$-flat.*
(3) *Every submodule of a flat right R-module is $\mathscr{C}$-flat.*
(4) *Every submodule of a projective right R-module is $\mathscr{C}$-flat.*
(5) *Every submodule of a free right R-module is $\mathscr{C}$-flat.*
(6) *Every finitely generated right ideal of R is $\mathscr{C}$-flat.*

Proof. (2)⇒(3)⇒ (4)⇒ (5)⇒ (6) is trivial.

(1)⇒(2). Assume (1). Let A be a submodule of a $\mathscr{C}$-flat right R-module B and let $C \in \mathscr{C}$. Then there exists an exact sequence of left R-modules $0 \to K \to P \to C \to 0$, where P is finitely generated projective. By (1), K is flat. Then the exactness of $0 = \mathrm{Tor}^R_2(B/A, P) \to \mathrm{Tor}^R_2(B/A, C) \to \mathrm{Tor}^R_1(B/A, K) = 0$ implies that $\mathrm{Tor}^R_2(B/A, C) = 0$. And thus from the exactness of the sequence $0 = \mathrm{Tor}^R_2(B/A, C) \to \mathrm{Tor}^R_1(A, C) \to \mathrm{Tor}^R_1(B, C) = 0$ we have $\mathrm{Tor}^R_1(A, C) = 0$. It shows that A is $\mathscr{C}$-flat.

(6)⇒(1). Let $C \in \mathscr{C}$. There exists an exact sequence of left R-modules $0 \to K \to P \to C \to 0$, where P is finitely generated projective. For any finitely generated right ideal I of R, we have an exact sequence $0 \to \mathrm{Tor}_2^R(R/I, C) \to \mathrm{Tor}_1^R(I, C) = 0$ since I is $\mathscr{C}$-flat. So $\mathrm{Tor}_2^R(R/I, C) = 0$, and hence we obtain an exact sequence $0 = \mathrm{Tor}_2^R(R/I, C) \to \mathrm{Tor}_1^R(R/I, K) \to 0$. Thus, $\mathrm{Tor}_1^R(R/I, K) = 0$. And so K is flat. □

Proposition 4.4. *If R is a left weakly $\mathscr{C}$-semihereditary ring, then $\mathscr{C}$-WD(R)≤ 1.*

Proof. Let M be any right R-module and let $C \in \mathscr{C}$. Then there exists an exact sequence of left R-modules $0 \to K \to P \to C \to 0$, where P is finitely generated projective. Since R is left weakly $\mathscr{C}$-semihereditary, K is flat. So $\mathrm{Tor}_2^R(M, C) \cong \mathrm{Tor}_1^R(M, K) = 0$. It shows that $\mathscr{C}$-WD(R)≤ 1. □

Lemma 4.5. *Let $\mathcal{F}$ be a class of some right R-modules. If $N \overset{f_1}{\to} N_1$ and $N \overset{f_2}{\to} N_2$ are $\mathcal{F}$-preenvelopes , then $N_1 \oplus N_2/f_2(N) \cong N_2 \oplus N_1/f_1(N)$.*

Proof. Let $\varepsilon_i : N_i \to N_1 \oplus N_2$ be the injections, $i = 1, 2$. We obtain a morphism $q^* = \varepsilon_1 f_1 + \varepsilon_2 f_2 : N \to N_1 \oplus N_2$. Let $\overline{\varepsilon_1} : N_1 \to Coker(q^*); n_1 \mapsto (n_1, 0) + im(q^*)$, $\overline{\varepsilon_2} : N_2 \to Coker(q^*); n_2 \mapsto (0, n_2) + im(q^*)$ and $Q = Coker(q^*)$. Then we get the following pushout diagram:

$$\begin{array}{ccc} N & \xrightarrow{f_2} & N_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle \overline{\varepsilon_2}} \\ N_1 & \xrightarrow{\overline{\varepsilon_1}} & Q \end{array}$$

And so, by the proof of [21, 10.6(1)(i)], we have the following commutative diagram with exact rows, where $g : Q \to N_2/f_2(N); (n_1, n_2) + im(q^*) \mapsto n_2 + f_2(N)$:

$$\begin{array}{ccccccc} N & \xrightarrow{f_2} & N_2 & \longrightarrow & N_2/f_2(N) & \longrightarrow & 0 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle \overline{\varepsilon_2}}\downarrow & & {\scriptstyle 1}\downarrow & & \\ N_1 & \xrightarrow{\overline{\varepsilon_1}} & Q & \xrightarrow{g} & N_2/f_2(N) & \longrightarrow & 0 \end{array}$$

Since $N \overset{f_2}{\to} N_2$ is an $\mathcal{F}$-preenvelope and $N_1 \in \mathcal{F}$, there exists a homomorphism $\alpha : N_2 \to N_1$ such that $f_1 = \alpha f_2$. If $\overline{\varepsilon_1}(n_1) = 0$, then $(n_1, 0) = q^*(n) = (f_1(n), f_2(n))$ for some $n \in N$, so $f_2(n) = 0, f_1(n) = n_1$, and hence $n_1 = f_1(n) = \alpha f_2(n) = 0$. It shows that $\overline{\varepsilon_1}$ is monic. Now, we define $h : Q \to N_1$ by $(n_1, n_2) + im(q^*) \mapsto n_1 - \alpha(n_2)$. Then h is well-defined, and $h\overline{\varepsilon_1}(n_1) = h((n_1, 0) + im(q^*)) = n_1 - \alpha(0) = n_1$ for each $n_1 \in N_1$, so $h\overline{\varepsilon_1} = 1_{N_1}$, and then $\overline{\varepsilon_1}$ is left split. Thus, we have $Q \cong N_1 \oplus N_2/f_2(N)$. Similarly, we have also that $Q \cong N_2 \oplus N_1/f_1(N)$ and so $N_1 \oplus N_2/f_2(N) \cong N_2 \oplus N_1/f_1(N)$. □

Next, we give some new characterizations of left $\mathscr{C}$-semihereditary rings.

Theorem 4.6. *The following statements are equivalent for a ring R:*

(1) *R is left $\mathscr{C}$-semihereditary.*

(2) *R is left $\mathscr{C}$-coherent and left weakly $\mathscr{C}$-semihereditary.*

(3) *R is left strongly $\mathscr{C}$-coherent and every $\mathscr{C}$-projective left R-module has a monic $\mathscr{C}$-injective cover.*

(4) *Every $\mathscr{C}$-projective left R-module has projective dimension at most 1.*

(5) *R is left $\mathscr{C}$-coherent and every $\mathscr{C}$I-injective module is injective.*

(6) *Every left R-module has a $\mathscr{C}$-injective cover and every $\mathscr{C}$I-injective module is injective.*

(7) *Every $\mathscr{C}$-pure quotient of a $\mathscr{C}$-injective left R-module has a $\mathscr{C}$-injective cover and every $\mathscr{C}$I-injective module is injective.*

(8) *R is left strongly $\mathscr{C}$-coherent and every $\mathscr{C}$I-injective module is $\mathscr{C}$-injective.*

(9) *R is left strongly $\mathscr{C}$-coherent and the kernel of any $\mathscr{C}$-injective precover of a left R-module is $\mathscr{C}$-injective.*

(10) *R is left strongly $\mathscr{C}$-coherent and the kernel of any $\mathscr{C}$-injective cover of a left R-module is $\mathscr{C}$-injective.*

(11) *R is left strongly $\mathscr{C}$-coherent and the cokernel of any $\mathscr{C}$-injective preenvelope of a left R-module is $\mathscr{C}$-injective.*

(12) *R is left strongly $\mathscr{C}$-coherent and the kernel of any $\mathscr{C}$-flat precover of a right R-module is $\mathscr{C}$-flat.*

(13) *R is left strongly $\mathscr{C}$-coherent and the kernel of any $\mathscr{C}$-flat cover of a right R-module is $\mathscr{C}$-flat.*

(14) *R is left strongly $\mathscr{C}$-coherent and the cokernel of any $\mathscr{C}$-flat preenvelope of a right R-module is $\mathscr{C}$-flat.*

Proof. (1)$\Leftrightarrow$ (2). It follows from [27, Theorem 4.3(2)] and Theorem 4.3(2).

(1)$\Rightarrow$(3). Suppose that R is left $\mathscr{C}$-semihereditary. Then it is left strongly $\mathscr{C}$-coherent by [28, Theorem 4]. Moreover, by [27, Theorem 4.3(7)], every $\mathscr{C}$-projective left R-module has a monic $\mathscr{C}$-injective cover.

(3)$\Rightarrow$(1). Let E be any injective left R-module and K any submodule of E. By [27, Theorem 4.3(6)], we need only to prove that E/K is $\mathscr{C}$-injective. In fact, since $(\mathscr{C}\mathcal{P}, \mathscr{C}\mathcal{I})$ is a complete cotorsion pair by [27, Theorem 2.10(1)], there exists an exact sequences $0 \to K \to E_1 \xrightarrow{f} P \to 0$ with P $\mathscr{C}$-projective and E_1 $\mathscr{C}$-injective. By (3), P has a monic $\mathscr{C}$-injective cover $\varphi : E_2 \to P$. So, there exists a homomorphism $g : E_1 \to E_2$ such that $f = \varphi g$. Thus φ is epic, and hence φ is an isomorphism. This implies that P is $\mathscr{C}$-injective. For any $C \in \mathscr{C}$, we have the exact sequence

$$0 = \mathrm{Ext}^1_R(C, P) \to \mathrm{Ext}^2_R(C, K) \to \mathrm{Ext}^2_R(C, E_1).$$

But R is left strongly $\mathscr{C}$-coherent, by [28, Theorem 1(6)], $\mathrm{Ext}^2_R(C, E_1) = 0$, and so $\mathrm{Ext}^2_R(C, K) = 0$. On the other hand, the short exact sequence $0 \to K \to E \to E/K \to 0$ induces the exact sequence

$$0 = \mathrm{Ext}^1_R(C, E) \to \mathrm{Ext}^1_R(C, E/K) \to \mathrm{Ext}^2_R(C, K) = 0.$$

so, we have $\mathrm{Ext}^1_R(C, E/K) = 0$, and hence E/K is $\mathscr{C}$-injective. Consequently, R is left $\mathscr{C}$-semihereditary by [27, Theorem 4.3(6)].

(1)$\Rightarrow$(4). Let M be a $\mathscr{C}$-projective module and N be any left R-module. Since R is left $\mathscr{C}$-semihereditary, by [27, Theorem 4.3(6)], $E(N)/N$ is $\mathscr{C}$-injective. So, by the exactness of the sequence

$$0 = \mathrm{Ext}^1_R(M, E(N)/N) \to \mathrm{Ext}^2_R(M, N) \to \mathrm{Ext}^2_R(M, E(N)) = 0.$$

We have $\mathrm{Ext}^2_R(M, N) = 0$, and hence M has projective dimension at most 1.

(4)$\Rightarrow$(1). Let $C \in \mathscr{C}$ and $0 \to K \to P \to C \to 0$ be exact, where P is finitely generated projective. Note that C is $\mathscr{C}$-projective, by (4), $pd(C) \leq 1$, and so K is projective by Schanuel's Lemma.

(1)$\Rightarrow$(5). Since R is left $\mathscr{C}$-semihereditary, by [27, Theorem 4.3], R is left $\mathscr{C}$-coherent and every quotient module of an injective left R-module is $\mathscr{C}$-injective . Let M be a $\mathscr{C}$I-injective left R-module. Then $E(M)/M$ is $\mathscr{C}$-injective, so M is injective with respect to the exact sequence $0 \to M \to E(M) \to E(M)/M \to 0$ by Proposition 2.3, and hence $M = E(M)$ is injective.

(5)$\Rightarrow$(6). It follows from [27, Corollary 3.7].

(6) $\Rightarrow$ (1). Let M be a quotient of an injective left R-module. By (6), M has a $\mathscr{C}$-injective cover. Suppose $f : F \to M$ is a $\mathscr{C}$-injective cover of M. Then f is epic. By Remark 2.4, $\mathrm{Ker}(f)$ is $\mathscr{C}$I-injective, and so it is injective by (6). Thus, M is isomorphic to a direct summand of F and hence it is $\mathscr{C}$-injective. Hence, by [27, Theorem 4.3(6)], R is left $\mathscr{C}$-semihereditary.

(6) $\Rightarrow$ (7). It is obvious.

(7)$\Rightarrow$(8). It follows from Theorem 3.3(7).

(8) $\Rightarrow$ (5). Assume (8). Then by [28, Theorem 1(10)], R is left $\mathscr{C}$-coherent. Let M be a $\mathscr{C}$I-injective module. Then by (8), M is $\mathscr{C}$-injective. But R is left strongly $\mathscr{C}$-coherent, by [28, Theorem 1(7)], $E(M)/M$ is $\mathscr{C}$-injective. Thus, by Proposition 2.3(4), M is injective .

(1) $\Rightarrow$ (9). Clearly, R is left strongly $\mathscr{C}$-coherent . Let $f : F \to M$ be a $\mathscr{C}$-injective precover and $K = \mathrm{Ker}(f)$. Since R is left $\mathscr{C}$-semihereditary, by [27, Theorem 4.3(7)], there exists a monic $\mathscr{C}$-injective cover $\varphi : G \to M$. Thus, by [9, Lemma 8.6.3], we have $K \oplus G \cong F$, and so K is $\mathscr{C}$-injective.

(9)$\Rightarrow$(10). It is obvious.

(10) $\Rightarrow$ (1). Let M be a quotient of a $\mathscr{C}$-injective left R-module. Since R is left $\mathscr{C}$-coherent, by [27, Corollary 3.7], M has a $\mathscr{C}$-injective cover $f : F \to M$. Clearly, f is epic. So, by (10), we have that $\mathrm{Ker}(f)$ is $\mathscr{C}$-injective , this implies that M is also $\mathscr{C}$-injective by [28, Theorem 1(7)] as R is left strongly $\mathscr{C}$-coherent. Therefore, by [27, Theorem 4.3(5)], R is left $\mathscr{C}$-semihereditary.

(1) $\Rightarrow$ (11). Clearly, R is left strongly $\mathscr{C}$-coherent. And by [27, Theorem 4.3(5)], every quotient module of a $\mathscr{C}$-injective module is $\mathscr{C}$-injective, so the cokernel of any $\mathscr{C}$-injective preenvelope of a left R-module is $\mathscr{C}$-injective.

(11) $\Rightarrow$ (1). Let M be any left R-module. Since the class of all $\mathscr{C}$-injective left R-modules is closed under pure submodules , isomorphisms and direct product, by [29, Theorem 2.6], M has a $\mathscr{C}$-injective preenvelope $f : M \to E$. By (11), $E/im(f)$ is $\mathscr{C}$-injective . It is easy to see that f is monic. Since R is left strongly $\mathscr{C}$-coherent, by [28, Theorem 2(5)], $\mathscr{C}\mathfrak{I}\text{-}dim({}_RM) \leq 1$. And so , $\mathscr{C}\mathfrak{I}$-GLD(R)≤ 1. Therefore, by [28, Theorem 4(2)], R is left $\mathscr{C}$-semihereditary.

(1) $\Rightarrow$ (12). Clearly, R is left strongly $\mathscr{C}$-coherent. And by [27, Theorem 4.3(2)], the kernel of any $\mathscr{C}$-flat precover of a right R-module is $\mathscr{C}$-flat.

(12)$\Rightarrow$(13). It is obvious.

(13) $\Rightarrow$ (1). Let N be any right R-module. Then by [27, Theorem 2.10(2)], N has a $\mathscr{C}$-flat cover $f : F \to N$. Clearly, f is epic. By (13), we have that $\mathrm{Ker}(f)$ is $\mathscr{C}$-flat. But R is left strongly $\mathscr{C}$-coherent, by [28, Theorem 3(5)], $\mathscr{C}\mathfrak{F}\text{-}dim(N_R) \leq 1$. Thus, $\mathscr{C}$-WD(R)≤ 1. Consequently, by [28, Theorem 4(3)], we have that R is left $\mathscr{C}$-semihereditary.

(1) $\Rightarrow$ (14). Clearly, R is left strongly $\mathscr{C}$-coherent. Let $\varphi : N \to F$ be a $\mathscr{C}$-flat preenvelope of a right R-module N and $L = coker(\varphi)$. Since R is left $\mathscr{C}$-semihereditary, by [27, Theorem 4.3(8)], N has an epic $\mathscr{C}$-flat envelope $\phi : N \to G$. Hence, by Lemma 4.5, we have $F \cong G \oplus L$, and so L is $\mathscr{C}$-flat.

(14) $\Rightarrow$ (1). Let N be a submodule of a $\mathscr{C}$-flat module. Since R is left $\mathscr{C}$-coherent, by [27, Theorem 3.3(12)], N has a $\mathscr{C}$-flat preenvelope $f : N \to F$. It is easy to see that f is monic. By (14), $F/im(f)$ is $\mathscr{C}$-flat. Note that R is left strongly $\mathscr{C}$-coherent, by Theorem 3.1(4), N is $\mathscr{C}$-flat. Therefore, by [27, Theorem 4.3(2)], R is left $\mathscr{C}$-semihereditary. □

Acknowledgment. The authors would like to thank the referee for the useful suggestions and comments. This research was supported by the Natural Science Foundation of Zhejiang Province, China(LY18A010018).

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