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DIRECT PRODUCTS OF ROUGH SUBGROUPS

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ABSTRACT

In this research, the direct products of the rough approximations and rough subgroups in a group by direct product of normal subgroups are studied. In addition, some basic properties and homomorphic images of these structures are examined.

Keywords: Rough sets, Rough group, Rough subgroup, Direct product

1. INTRODUCTION

Z. Pawlak [1] first defined the theory of rough set as a significant mathematical instrument for forming and transaction the unfinished data in [1]. Since it was defined, the rough set theory evolved in various ways and was applied in different areas [2-4]. Some authors examined algebraic properties of rough sets, for example, Bonikowski [5], Pomykala and Pomykala [6] and Iwinski [7]. In [8], Kuroki and Wang studied the fundamental features of the rough approximations according to the normal subgroups. In recent years, many authors have studied the features of the rough approximations and subgroups according to the normal subgroups [9-14]. On the other hand, Biswas and Nanda [15] introduced a new definition of the notions of rough group. which were based only on the upper approximation. After this, Miao et al. [16] developed these concepts and proved their some new features. In addition, Bağırmaz and Özcan [17] gave new definition of rough semigroups groups on rough approximation spaces.

This study was configured in five sections. In section 2, the fundamental notions and results to be referred throughout the article were included. In section 3, the direct product of the rough approximations in a group was defined. We also gave some examples and examined their properties. In section 4, the direct product of the rough subgroups in a group was defined and some characteristics were given. Finally, in section 5 we examined the homomorphic images of the direct product of the rough subgroups.

2. PRELIMINARIES

In this section, we remember a few fundamental definitions and properties about rough approximations and direct product of groups to be used in this study. Throughout the study, G indicates a finite group with identity e.

Definition 1 [8] *Let N be a normal subgroup of G and A a non-empty subset of G. Let*

$$\overline{N}(A) = \{x \in G: xN \cap A \neq \emptyset\}, \quad \underline{N}(A) = \{x \in G: xN \subseteq A\}.$$

Then $\overline{N}(A)$ and $\underline{N}(A)$ are called upper and lower approximations of A with respect to the normal subgroup N, respectively.

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Definition 2 [8] *Let N be a normal subgroup of a group G and A a non-empty subset of G. Then A is called an upper rough subgroup (respectively, normal subgroup) of G if $\overline{N}(A)$ is a subgroup (respectively, normal subgroup) of G. Similarly, A is called a lower rough subgroup (respectively, normal subgroup) of G if $\underline{N}(A)$ is a subgroup (respectively, normal subgroup) of G.*

Theorem 3 [11] *Let N be a normal subgroup of a group G and A a non-empty subset of G. Then*

$$\overline{N}(A) = AN.$$

Theorem 4 [11] *Let N be a normal subgroup of a group G and A a subgroup of G. If $N \not\subset A$, then $\underline{N}(A) = \emptyset$; if $N \subseteq A$, then $\underline{N}(A) = A$.*

Theorem 5 [11] *Let G_1 and G_2 be two groups. Let f be an epimorphism from G_1 to G_2, N a normal subgroup of G_1 and A a subgroup of G_1. Then*

$$(1)\ f(\underline{N}(A)) = \underline{f(N)}(f(A)),$$
$$(2)\ f(\overline{N}(A)) = (\overline{f(N)})(f(A)).$$

Theorem 6 [11] *Let G_1 and G_2 be two groups. Let f be an epimorphism from G_1 to G_2, N' a normal subgroup of G_2 and A' a subgroup of G_2. Then*

$$(1)\ f^{-1}(\underline{N'}(A')) = \underline{f^{-1}(N')}(f^{-1}(A')),$$
$$(2)\ f^{-1}(\overline{N'}(A')) = \overline{f^{-1}(N')}(f^{-1}(A')).$$

Theorem 7 [11] *Let G_1 and G_2 be two groups. Let f be an epimorphism from G_1 to G_2, N and A normal subgroups of G_1. Then*

$$(1)\ If\ kerf \subseteq N \subseteq A, then\ G_1/\underline{N}(A) \cong G_2/\underline{f(N)}(f(A)),$$
$$(2)\ If\ kerf \subseteq N, then\ G_1/\overline{N}(A) \cong G_2/\overline{f(N)}(f(A)).$$

Theorem 8 [19] *Let N_1 and N_2 be normal subgroups of groups G_1 and G_2, respectively. Then $N_1 \times N_2$ normal subgroups of groups $G_1 \times G_2$ and $G_1 \times G_2/N_1 \times N_2 = G_1/N_1 \times G_2/N_2$.*

3. DIRECT PRODUCT OF ROUGH APPROXIMATIONS

In this section, we will present direct product of rough approximations in a group according to the direct product of normal subgroups.

Definition 9 *Let N_1 and N_2 be normal subgroups of groups G_1 and G_2, respectively. Let A_1 and A_2 be non-empty subsets of G_1 and G_2, respectively. Let*

$$\overline{(N_1 \times N_2)}(A_1 \times A_2) = \{(g_1, g_2) \in G_1 \times G_2 : (g_1, g_2)(N_1 \times N_2) \cap (A_1 \times A_2) \neq \emptyset\},$$

$$\underline{(N_1 \times N_2)}(A_1 \times A_2) = \{(g_1, g_2) \in G_1 \times G_2 : (g_1, g_2)(N_1 \times N_2) \subseteq (A_1 \times A_2)\}.$$

Then $\overline{(N_1 \times N_2)}(A_1 \times A_2)$ and $\underline{(N_1 \times N_2)}(A_1 \times A_2)$ are called upper and lower approximations of $A_1 \times A_2 \subseteq G_1 \times G_2$ according to the normal subgroup $N_1 \times N_2$ of $G_1 \times G_2$, respectively.

Let's explain this definition with the example below.

Example 10 Consider the groups $G_1 = S_3 = \{e, (12), (13), (23), (123), (132)\}$ and $G_2 = (Z_4, \oplus) = \{0,1,2,3\}$. Let $N_1 = \{e, (123), (132)\}$ and $N_2 = \{0,2\}$. Then N_1 and N_2 are normal subgroups of G_1 and G_2, respectively. A classification of $G_1 \times G_2$ with respect to the normal subgroup $N_1 \times N_2$ given below:

$$\begin{aligned}(e,0)(N_1 \times N_2) &= \{(e,0),(e,2),((123),0),((123),2),((132),0),((132),2)\},\\ (e,1)(N_1 \times N_2) &= \{(e,1),(e,3),((123),1),((123),3),((132),1),((132),3)\},\\ ((12),0)(N_1 \times N_2) &= \{((12),0),((12),2),((13),0),((13),2),((23),0),((23),2)\},\\ ((12),1)(N_1 \times N_2) &= \{((12),1),((12),3),((13),1),((13),3),((23),1),((23),3)\}.\end{aligned}$$

Let $A_1 = \{e, (12), (123), (132)\}$ be a subset of G_1 and $A_2 = \{0,2,3\}$ be a subset of G_2. Thus

$$\underline{(N_1 \times N_2)}(A_1 \times A_2) = \{(e,0),(e,2),((123),0),((123),2),((132),0),((132),2)\}$$

and

$$\overline{(N_1 \times N_2)}(A_1 \times A_2) = G_1 \times G_2.$$

Using the definition of direct product of the rough approximations in a group, we can list several key features of direct product of the rough approximations in a group which are similar to properties of the rough approximations [8]. The proofs of these properties are all straightforward.

Proposition 11 *Let N_1 and N_2 be normal subgroups of groups G_1 and G_2, respectively. Let A_1 and B_1 be non-empty subsets of G_1 and A_2 and B_2 be non-empty subsets of G_2. Let $A = A_1 \times A_2$ and $B = B_1 \times B_2$. Then*

1 $\underline{(N_1 \times N_2)}(A) \subseteq A \subseteq \overline{(N_1 \times N_2)}(A)$,
2 $\underline{(N_1 \times N_2)}(A \cap B) = \underline{(N_1 \times N_2)}(A) \cap \underline{(N_1 \times N_2)}(B)$,
3 $\overline{(N_1 \times N_2)}(A \cup B) = \overline{(N_1 \times N_2)}(A) \cup \overline{(N_1 \times N_2)}(B)$,
4 $A \subseteq B \Rightarrow \underline{(N_1 \times N_2)}(A) \subseteq \underline{(N_1 \times N_2)}(B)$,
5 $A \subseteq B \Rightarrow \overline{(N_1 \times N_2)}(A) \subseteq \overline{(N_1 \times N_2)}(B)$,
6 $\underline{(N_1 \times N_2)}(A \cup B) \supseteq \underline{(N_1 \times N_2)}(A) \cup \underline{(N_1 \times N_2)}(B)$,
7 $\overline{(N_1 \times N_2)}(A \cap B) \subseteq \overline{(N_1 \times N_2)}(A) \cap \overline{(N_1 \times N_2)}(B)$.

Proposition 12 *Let N_1 and N_2 be normal subgroups of groups G_1 and G_2, respectively. Let A_1 and A_2 be non-empty subsets of G_1 and G_2, respectively. Then*

$$\underline{(N_1 \times N_2)}(A_1 \times A_2) = \underline{N_1}(A_1) \times \underline{N_2}(A_2).$$

Proof. *For any $(g_1, g_2) \in \underline{(N_1 \times N_2)}(A_1 \times A_2)$, we have*

$$\begin{aligned}\underline{(N_1 \times N_2)}(A_1 \times A_2) &= \{(g_1,g_2) \in G_1 \times G_2 : (g_1,g_2)(N_1 \times N_2) \subseteq A_1 \times A_2\}\\ &= \{(g_1,g_2) \in G_1 \times G_2 : (g_1N_1) \times (g_2N_2) \subseteq A_1 \times A_2\}\\ &= \{g_1 \in G_1 : g_1N_1 \subseteq A_1\} \times \{g_2 \in G_2 : g_2N_2 \subseteq A_2\}\\ &= \underline{N_1}(A_1) \times \underline{N_2}(A_2).\end{aligned}$$

The proposition above shows that the lower approximation of the direct products of two sets is equal to the direct products of their lower approximations.

Example 13 *From Example 10, we obtain* $\underline{(N_1)}(A_1) = \{e, (123), (132)\}$ *and* $\underline{(N_2)}(A_2) = \{0,2\}$. *Then* $\underline{(N_1 \times N_2)}(A_1 \times A_2) = \underline{(N_1)}(A_1) \times \underline{(N_2)}(A_2)$.

Proposition 14 *Let N_1 and N_2 be normal subgroups of groups G_1 and G_2, respectively. Let A_1 and A_2 be non-empty subsets of G_1 and G_2, respectively. Then*

$$\overline{(N_1 \times N_2)}(A_1 \times A_2) = \overline{(N_1)}(A_1) \times \overline{(N_2)}(A_2).$$

Proof. *For any* $(g_1, g_2) \in \overline{(N_1 \times N_2)}(A_1 \times A_2)$, *we have*

$$\begin{aligned}
\overline{(N_1 \times N_2)}(A_1 \times A_2) &= \{(g_1, g_2) \in G_1 \times G_2 : (g_1, g_2)(N_1 \times N_2) \cap (A_1 \times A_2) \neq \emptyset\} \\
&= \{(g_1, g_2) \in G_1 \times G_2 : (g_1N_1 \times g_2N_2) \cap (A_1 \times A_2) \neq \emptyset\} \\
&= \{(g_1, g_2) \in G_1 \times G_2 : (g_1N_1 \cap A_1) \times (g_2N_2 \cap A_2) \neq \emptyset\} \\
&= \{g_1 \in G_1 : g_1N_1 \cap A_1 \neq \emptyset\} \times \{g_2 \in G_2 : g_2N_2 \cap A_2 \neq \emptyset\} \\
&= \overline{(N_1)}(A_1) \times \overline{(N_2)}(A_2).
\end{aligned}$$

The above proposition shows that the upper approximation of the direct products of two sets is equal to the direct products of their upper approximations.

4. DIRECT PRODUCT OF ROUGH SUBGROUPS

Now, we will present the direct product of rough subgroups in a group according to the direct product of normal subgroups. For this purpose, we give some of their properties.

Definition 15 *Let N_1 and N_2 be normal subgroups of groups G_1 and G_2, respectively. Let A_1 and A_2 be non-empty subsets of G_1 and G_2, respectively. If $\overline{(N_1 \times N_2)}(A_1 \times A_2)$ is a subgroup (normal subgroup) of $G_1 \times G_2$, then $A_1 \times A_2$ is called a upper rough subgroup (normal subgroup) of $G_1 \times G_2$. Similarly, if $\underline{(N_1 \times N_2)}(A_1 \times A_2)$ is a subgroup (normal subgroup) of $G_1 \times G_2$, then $A_1 \times A_2$ is called a lower rough subgroup (normal subgroup) of $G_1 \times G_2$.*

Proposition 16 *Let N_1 and N_2 be normal subgroups of groups G_1 and G_2, respectively. Let A_1 and A_2 be subgroups of G_1 and G_2, respectively. If $N_1 \times N_2 \subseteq A_1 \times A_2$, then $A_1 \times A_2$ is a lower rough subgroup of $G_1 \times G_2$.*

Proof. *Let (e_1, e_2) be the identity of $G_1 \times G_2$. Since $(e_1, e_2)(N_1 \times N_2) = N_1 \times N_2 \subseteq A_1 \times A_2$, then $(e_1, e_2) \in (N_1 \times N_2)(A_1 \times A_2)$. Let (a_1, a_2) and (b_1, b_2) be any elements of $\underline{(N_1 \times N_2)}(A_1 \times A_2)$. Then $(a_1, a_2)(N_1 \times N_2) \subseteq A_1 \times A_2$ and $(b_1, b_2)(N_1 \times N_2) \subseteq A_1 \times A_2$. It is well known that $N_1 \times N_2$ is a normal subgroup and $A_1 \times A_2$ is a subgroup of $G_1 \times G_2$. Then, we have*

$$\begin{aligned}
(a_1b_1, a_2b_2)(N_1 \times N_2) &= (a_1, a_2)(b_1, b_2)(N_1 \times N_2) \\
&= ((a_1, a_2)(N_1 \times N_2)(b_1, b_2)(N_1 \times N_2)) \subseteq (A_1 \times A_2)(A_1 \times A_2) \subseteq A_1 \times A_2.
\end{aligned}$$

This implies that

$$(a_1b_1, a_2b_2) \in (N_1 \times N_2)(A_1 \times A_2).$$

Let (a_1, a_2) be an arbitrary element of $\underline{(N_1 \times N_2)}(A_1 \times A_2)$. Therefore

$$(a_1, a_2) = (a_1e_1, a_2e_2) = (a_1, a_2)(e_1, e_2) \in (a_1, a_2)(N_1 \times N_2) \subseteq A_1 \times A_2.$$

Since $A_1 \times A_2$ is a subgroup of $G_1 \times G_2$, we have $({a_1}^{-1}, {a_2}^{-1}) \in A_1 \times A_2$.

Thus we have

$$({a_1}^{-1}, {a_2}^{-1})(N_1 \times N_2) \subseteq (A_1 \times A_2)(A_1 \times A_2) \subseteq A_1 \times A_2.$$

This implies that $({a_1}^{-1}, {a_2}^{-1}) \in \underline{(N_1 \times N_2)}(A_1 \times A_2)$.
Hence $\underline{(N_1 \times N_2)}(A_1 \times A_2)$ *is a subgroup of* $G_1 \times G_2$.

Proposition 17 *Let N_1 and N_2 be normal subgroups of groups G_1 and G_2, respectively. Let A_1 and A_2 be normal subgroups of G_1 and G_2, respectively. If $N_1 \times N_2 \subseteq A_1 \times A_2$, then $A_1 \times A_2$ is a lower normal subgroup of $G_1 \times G_2$.*

Proof. B*y Proposition 16, we showed that $A_1 \times A_2$ was a lower rough subgroup $G_1 \times G_2$. It is enough to show that $\underline{(N_1 \times N_2)}(A_1 \times A_2)$ is normal subgroup of $G_1 \times G_2$. Let (a_1, a_1) and (g_1, g_2) be arbitrary elemens of $\underline{(N_1 \times N_2)}(A_1 \times A_2)$ and $G_1 \times G_2$, respectively. Therefore, $(a_1, a_2)(N_1 \times N_2) \subseteq A_1 \times A_2$. It is well known that $N_1 \times N_2$ and $A_1 \times A_2$ are normal subgroups of $G_1 \times G_2$. Therefore, we have*

$$\begin{aligned}(g_1 a_1 {g_1}^{-1}, g_2 a_2 {g_2}^{-1})(N_1 \times N_2) &= (g_1, g_2)(a_1, a_2)({g_1}^{-1}, {g_2}^{-1})(N_1 \times N_2)\\ &= (g_1, g_2)(a_1, a_2)(g_1, g_2)^{-1}(N_1 \times N_2)\\ &= (g_1, g_2)((a_1, a_2)(N_1 \times N_2))(g_1, g_2)^{-1}\\ &\subseteq (g_1, g_2)(A_1 \times A_2)(g_1, g_2)^{-1} \subseteq A_1 \times A_2.\end{aligned}$$

This implies that $(g_1 a_1 {g_1}^{-1}, g_2 a_2 {g_2}^{-1}) \in \underline{(N_1 \times N_2)}(A_1 \times A_2)$.
Hence $\underline{(N_1 \times N_2)}(A_1 \times A_2)$ *is a normal subgroup of* $G_1 \times G_2$.

Proposition 18 *Let N_1 and N_2 be normal subgroups of groups G_1 and G_2, respectively. Let A_1 and A_2 be subgroups of G_1 and G_2, respectively. Then $A_1 \times A_2$ is a upper rough subgroup of $G_1 \times G_2$.*

Proof. *Let (e_1, e_2) be the identity of $G_1 \times G_2$. Since $N_1 \times N_2$ and $A_1 \times A_2$ are subgroups $G_1 \times G_2$, we have $(e_1, e_2) \in A_1 \times A_2$ and $(e_1, e_2) \in (e_1, e_2)(N_1 \times N_2)$. Thus*

$$(e_1, e_2) \in (e_1, e_2)(N_1 \times N_2) \cap A_1 \times A_2.$$

This implies that $(e_1, e_2) \in \overline{(N_1 \times N_2)}(A_1 \times A_2)$.

Let (a_1, a_2) and (b_1, b_2) be arbitrary elements of $\overline{(N_1 \times N_2)}(A_1 \times A_2)$. Therefore, there are the elements (g_1, g_2) and (h_1, h_2) in $G_1 \times G_2$ such that

$$(g_1, g_2) \in (a_1, a_2)(N_1 \times N_2) \cap (A_1 \times A_2)$$

and

$$(h_1, h_2) \in (b_1, b_2)(N_1 \times N_2) \cap (A_1 \times A_2).$$

Thus

$$(g_1, g_2) \in (a_1, a_2)(N_1 \times N_2), (h_1, h_2) \in (b_1, b_2)(N_1 \times N_2),$$

$$(g_1, g_2) \in A_1 \times A_2 \text{ and } (h_1, h_2) \in A_1 \times A_2.$$

Then

$$(g_1, g_2)(h_1, h_2) = (g_1 h_1, g_2 h_2) \in A_1 \times A_2$$

and

$$(g_1, g_2)(h_1, h_2) \in ((a_1, a_2)(N_1 \times N_2))((b_1, b_2)(N_1 \times N_2)).$$

Since $N_1 \times N_2$ *is a normal subgroup of* $G_1 \times G_2$ *and* $(g_1, g_2)(h_1, h_2) = (g_1h_1, g_2h_2)$*, then*

$$(g_1h_1, g_2h_2) \in ((a_1, a_2)(b_1, b_2))(N_1 \times N_2) = (a_1b_1, a_2b_2)(N_1 \times N_2).$$

Thus $(g_1h_1, g_2h_2) \in (a_1b_1, a_2b_2)(N_1 \times N_2) \cap (A_1 \times A_2)$*. This implies that*

$$(a_1b_1, a_2b_2) \in \overline{(N_1 \times N_2)}(A_1 \times A_2).$$

Let (a_1, a_2) *be any element of* $\overline{(N_1 \times N_2)}(A_1 \times A_2)$*. Then*

$$(g_1, g_2) \in (a_1, a_2)(N_1 \times N_2) \cap (A_1 \times A_2)$$

for some $(g_1, g_2) \in G_1 \times G_2$*, that is,* $(g_1, g_2) \in (a_1, a_2)(N_1 \times N_2), (g_1, g_2) \in A_1 \times A_2$*. Thus* $(g_1, g_2)^{-1} = (g_1^{-1}, g_2^{-1}) \in A_1 \times A_2$ *and* $(g_1, g_2) = (a_1, a_2)(n_1, n_2)$ *for some* $(n_1, n_2) \in N_1 \times N_2$*, and so* $(n_1, n_2)^{-1} = (n_1^{-1}, n_2^{-1}) \in N_1 \times N_2$*. Since* $N_1 \times N_2$ *is a normal subgroup of* $G_1 \times G_2$*, we have*

$$\begin{aligned}(g_1, g_2)^{-1} &= ((a_1, a_2)(n_1, n_2))^{-1} \\ &= (n_1, n_2)^{-1}(a_1, a_2)^{-1} \in (N_1 \times N_2)(a_1^{-1}, a_2^{-1}) \\ &= (a_1^{-1}, a_2^{-1})(N_1 \times N_2).\end{aligned}$$

Thus

$$(g_1, g_2)^{-1} \in (a_1, a_2)^{-1}(N_1 \times N_2) \cap (A_1 \times A_2),$$

and so

$$(a_1, a_2)^{-1} \in \overline{(N_1 \times N_2)}(A_1 \times A_2).$$

Hence $\overline{(N_1 \times N_2)}(A_1 \times A_2)$ *is a subgroup of* $G_1 \times G_2$*.*

Proposition 19 *Let* N_1 *and* N_2 *be normal subgroups of groups* G_1 *and* G_2*, respectively. Let* A_1 *and* A_2 *be normal subgroups of* G_1 *and* G_2*, respectively. Then* $A_1 \times A_2$ *is an upper rough normal subgroup of* $G_1 \times G_2$*.*

Proof. *It is enough to show that* $\overline{(N_1 \times N_2)}(A_1 \times A_2)$ *is normal subgroup of* $G_1 \times G_2$*. Let* (a_1, a_2) *and* (g_1, g_2) *be an arbitrary element of* $\overline{(N_1 \times N_2)}(A_1 \times A_2)$ *and* $G_1 \times G_2$*, respectively. Therefore, there is an element* (h_1, h_2) *in* $G_1 \times G_2$ *such that* $(h_1, h_2) \in (a_1, a_2)(N_1 \times N_2) \cap (A_1 \times A_2)$*.* Then $(h_1, h_2) \in (a_1, a_2)(N_1 \times N_2)$ and $(h_1, h_2) \in (A_1 \times A_2)$. Since $N_1 \times N_2$ is normal,

$$\begin{aligned}(g_1h_1g_1^{-1}, g_2h_2g_2^{-1}) &= (g_1, g_2)(h_1, h_2)(g_1, g_2)^{-1} \in (g_1, g_2)((a_1, a_2)(N_1 \times N_2))(g_1, g_2)^{-1} \\ &= ((g_1, g_2)(a_1, a_2))((N_1 \times N_2)(g_1, g_2)^{-1}) \\ &= ((g_1, g_2)(a_1, a_2))((g_1, g_2)^{-1}(N_1 \times N_2)) \\ &= ((g_1, g_2)(a_1, a_2)(g_1, g_2)^{-1})(N_1 \times N_2) \\ &= (g_1a_1g_1^{-1}, g_2a_2g_2^{-1})(N_1 \times N_2).\end{aligned}$$

Since $(A_1 \times A_2)$ is normal,

$$(g_1h_1g_1^{-1}, g_2h_2g_2^{-1}) = (g_1, g_2)(h_1, h_2)(g_1, g_2)^{-1} \in (g_1, g_2)(A_1 \times A_2)(g_1, g_2)^{-1} \subseteq A_1 \times A_2.$$

Thus

$$(g_1h_1g_1^{-1}, g_2h_2g_2^{-1}) \in (g_1a_1g_1^{-1}, g_2a_2g_2^{-1})(N_1 \times N_2) \cap (A_1 \times A_2),$$

and so

$$({g_1 a_1 g_1}^{-1}, {g_2 a_2 g_2}^{-1}) \in \overline{(N_1 \times N_2)}(A_1 \times A_2).$$

Hence $\overline{(N_1 \times N_2)}(A_1 \times A_2)$ is a normal subgroup of $G_1 \times G_2$.

The following propositions give the characterization of the upper and lower approximations.

Proposition 20 *Let N_1 and N_2 be normal subgroups of groups G_1 and G_2, respectively. Let A_1 and A_2 be normal subgroups of G_1 and G_2, respectively. If $N_1 \times N_2 \subseteq A_1 \times A_2$, then*

$$\underline{(N_1 \times N_2)}(A_1 \times A_2) = A_1 \times A_2.$$

Proof. *This can be easily got from Proposition 12 and Theorem 4.*

Proposition 21 *Let N_1 and N_2 be normal subgroups of groups G_1 and G_2, respectively. Let A_1 and A_2 be non-empty subsets of G_1 and G_2, respectively. Then*

$$\overline{(N_1 \times N_2)}(A_1 \times A_2) = A_1 N_1 \times A_2 N_2.$$

Proof. *This can be easily got from Proposition 14 and Theorem 3.*

HOMOMORPHIC IMAGE OF DIRECT PRODUCTS OF ROUGH SUBGROUPS

This last section is reserved into properties of the direct products of the rough approximations of a subgroup in a group under homomorphisms between the two groups.

Let $\varphi_i: G_i \to H_i$ be group homomorphisms (i=1,2). Define σ by

$$\sigma: G_1 \times G_2 \to H_1 \times H_2$$
$$(g_1, g_2) \mapsto (\varphi_1(g_1), \varphi_2(g_2)).$$

Then σ is a homomorphism and $Ker\sigma = Ker\varphi_1 \times Ker\varphi_2$, $Im\sigma = Im\varphi_1 \times Im\varphi_2$. If $\varphi_i: G_i \to H_i$ is an isomorphism (i=1,2), then $\sigma: G_1 \times G_2 \to H_1 \times H_2$ is an isomorphism.

Proposition 22 *Let N_1 and N_2 be normal subgroups of groups G_1 and G_2, respectively. Let A_1 and A_2 be normal subgroups of G_1 and G_2, respectively. Then*
(1)$G_1 \times G_2/\overline{(N_1 \times N_2)}(A_1 \times A_2) \cong G_1/\overline{N_1}(A_1) \times G_2/\overline{N_2}(A_2)$,

(2)$G_1 \times G_2/\underline{(N_1 \times N_2)}(A_1 \times A_2) \cong G_1/\underline{N_1}(A_1) \times G_2/\underline{N_2}(A_2)$.

Proof. *(1) By the Proposition 14,we get $\overline{(N_1 \times N_2)}(A_1 \times A_2) = \overline{(N_1)}(A_1) \times \overline{(N_2)}(A_2)$. Since A_1 and A_2 are normal subgroups of G_1 and G_2, respectively, by Theorem 3 it follows that $\overline{(N_1)}(A_1)$ and $\overline{(N_2)}(A_2)$ are normal subgroups of G_1 and G_2, respectively. Then*

$$G_1 \times G_2/\overline{(N_1 \times N_2)}(A_1 \times A_2) = G_1 \times G_2/(\overline{(N_1)}(A_1) \times \overline{(N_2)}(A_2)).$$

By Theorem 8, we get

$$G_1 \times G_2/\overline{(N_1 \times N_2)}(A_1 \times A_2) \cong G_1/\overline{N_1}(A_1) \times G_2/\overline{N_2}(A_2),$$

(2) By the Proposition 12, we get $\underline{(N_1 \times N_2)}(A_1 \times A_2) = \underline{N_1}(A_1) \times \underline{N_2}(A_2)$. *Since* A_1 *and* A_2 *are normal subgroups of* G_1 *and* G_2*, respectively, by Theorem 4 it follows that* $\underline{N_1}(A_1)$ *and* $\underline{N_2}(A_2)$ *are normal subgroups of* G_1 *and* G_2*, respectively. Then*

$$G_1 \times G_2/\underline{(N_1 \times N_2)}(A_1 \times A_2) = G_1 \times G_2/(\underline{N_1}(A_1) \times \underline{N_2}(A_2)).$$

By the Theorem 8, we get

$$G_1 \times G_2/\underline{(N_1 \times N_2)}(A_1 \times A_2) \cong G_1/\underline{N_1}(A_1) \times G_2/\underline{N_2}(A_2).$$

Proposition 22 *Let* $\varphi_i: G_i \rightarrow H_i$ *be group epimorphisms (i=1,2). Let* N_1 *and* N_2 *be normal subgroups and* A_1 *and* A_2 *be subgroups of* G_1 *and* G_2*, respectively. Then* $\sigma: G_1 \times G_2 \rightarrow H_1 \times H_2$ *is an epimorphism and*

$$(1)\ \sigma(\underline{(N_1 \times N_2)}(A_1 \times A_2)) = \underline{\sigma(N_1 \times N_2)}(\sigma(A_1 \times A_2)),$$

$$(2)\ \sigma(\overline{(N_1 \times N_2)}(A_1 \times A_2)) = \overline{\sigma(N_1 \times N_2)}(\sigma(A_1 \times A_2)).$$

Proof. *Let* $\varphi_i: G_i \rightarrow H_i$ *be group epimorphism (i=1,2). Define* σ *by*

$$\sigma : G_1 \times G_2 \rightarrow H_1 \times H_2, \quad (g_1, g_2) \mapsto (\varphi_1(g_1), \varphi_2(g_2)).$$

Then σ *is an epimorphism.*

(1) Since $\sigma: G_1 \times G_2 \rightarrow H_1 \times H_2$ *is an epimorphism,* N_1 *and* N_2 *are normal subgroups and* A_1 *and* A_2 *are subgroups of* G_1 *and* G_2*, respectively, we obtain* $\varphi_1(N_1)$ *and* $\varphi_2(N_2)$ *are normal subgroups and* $\varphi_1(A_1)$ *and* $\varphi_2(A_2)$ *are subgroups of* H_1 *and* H_2*, respectively. Thus* $\varphi_1(N_1) \times \varphi_2(N_2)$ *is a normal subgroup and* $\varphi_1(A_1) \times \varphi_2(A_2)$ *is a subgroup of* $H_1 \times H_2$*, respectively. Then*

$$\begin{aligned}
\sigma(\underline{(N_1 \times N_2)}(A_1 \times A_2)) &= \sigma((\underline{N_1}(A_1) \times \underline{N_2}(A_2)), \textit{by Proposition } 12 \textit{ on } G_1 \times G_2, \\
&= \varphi_1(\underline{N_1}(A_1)) \times \varphi_2(\underline{N_2}(A_2)) \\
&= \underline{\varphi_1(N_1)}(\varphi_1(A_1)) \times \underline{\varphi_2(N_2)}(\varphi_2(A_2)), \quad \textit{by Theorem } 5\ (1), \\
&= \underline{(\varphi_1(N_1) \times \varphi_2(N_2))}(\varphi_1(A_1) \times \varphi_2(A_2)), \textit{by Proposition } 12 \textit{ on } H_1 \times H_2, \\
&= \underline{\sigma(N_1 \times N_2)}(\sigma(A_1 \times A_2)).
\end{aligned}$$

(2) Similar to the proof of (1).

Proposition 23 *Let* $\varphi_i: G_i \rightarrow H_i$ *be group epimorphisms (i=1,2). Let* N_1' *and* N_2' *be normal subgroups and* A_1' *and* A_2' *be subgroups of* H_1 *and* H_2*, respectively. Then* $\sigma: G_1 \times G_2 \rightarrow H_1 \times H_2$ *is an epimorphism and*

$$(1)\ \sigma^{-1}(\underline{(N_1' \times N_2')}(A_1' \times A_2')) = \underline{\sigma^{-1}(N_1' \times N_2')}(\sigma^{-1}(A_1' \times A_2')),$$

$$(2)\ \sigma^{-1}(\overline{(N_1' \times N_2')}(A_1' \times A_2')) = \overline{\sigma^{-1}(N_1' \times N_2')}(\sigma^{-1}(A_1' \times A_2')).$$

Proof. *Similar to the proof of Proposition 22.*

Proposition 24 *Let* $\varphi_i: G_i \rightarrow H_i$ *be group epimorphisms (i=1,2). Let* N_i*,* A_i *be normal subgroups of* G_i *(i=1,2). Then*

$(1)\ \textit{If } ker\varphi_i \subseteq N_i \subseteq A_i, \textit{then}$

$$G_1/\underline{N_1}(A_1) \times G_2/\underline{N_2}(A_2) \cong H_1/\underline{\varphi_1(N_1)}(\varphi_1(A_1)) \times H_2/\underline{\varphi_2(N_2)}(\varphi_2(A_2)),$$

$(2) If\ ker\varphi_i \subseteq N_i,\ then$

$$G_1/\overline{N_1}(A_1) \times G_2/\overline{N_2}(A_2) \cong H_1/\overline{\varphi_1(N_1)}(\varphi_1(A_1)) \times H_2/\overline{\varphi_2(N_2)}(\varphi_2(A_2)).$$

Proof. *(1) By the Theorem 7 (1) we get*

$$G_1/\underline{N_1}(A_1) \cong H_1/\underline{\varphi_1(N_1)}(\varphi_1(A_1))\ and\ G_2/\underline{N_2}(A_2) \cong H_2/\underline{\varphi_2(N_2)}(\varphi_2(A_2)).$$

Hence

$$G_1/\underline{N_1}(A_1) \times G_2/\underline{N_2}(A_2) \cong H_1/\underline{\varphi_1(N_1)}(\varphi_1(A_1)) \times H_2/\underline{\varphi_2(N_2)}(\varphi_2(A_2)).$$

(2) By the Theorem 7 (2) we get

$$G_1/\overline{N_1}(A_1) \cong H_1/\overline{\varphi_1(N_1)}(\varphi_1(A_1))\ and\ G_2/\overline{N_2}(A_2) \cong H_2/\overline{\varphi_2(N_2)}(\varphi_2(A_2)).$$

Hence

$$G_1/\overline{N_1}(A_1) \times G_2/\overline{N_2}(A_2) \cong H_1/\overline{\varphi_1(N_1)}(\varphi_1(A_1)) \times H_2/\overline{\varphi_2(N_2)}(\varphi_2(A_2)).$$

4. CONCLUSION

In this study, we showed that it was possible to apply the rough sets theory to the area of direct products of groups. Then, the concepts of direct product of rough approximations and subgroups in a group were first defined and some of their basic properties were proven. We also showed that for two sets the rough approximations of their direct product were equal to the direct product of their rough approximations. Furthermore, we discussed the structure of direct product of rough approximations of subgroups under homomorphisms between two groups.

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