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Asymptotic stability for mixed fractional delay differential equations

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Abstract

This paper is concerned with the stability analysis of nonlinear mixed fractional delay differential equations using Krasnoselskii's fixed point theorem in a weighted Banach space.

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1. Introduction

Fractional differential equations have become an important field of applied mathematics and modeling of many physical phenomena associated to very rapid and very short changes, for more details we refer to the books ([7, 11, 15, 16, 20, 21]) and the references therein. In particular, initial value problems and boundary value problems related to the qualitative theory of the existence, uniqueness and stability of solutions for fractional differential equations have been mainly discussed by a lot of authors especially in last three decades or so. Beside, several methods have been employed to prove the existence, uniqueness and stability of solutions for fractional boundary value problems among the spectra theory, the critical point theory, method of upper and lower solutions and the fixed point theorems [1, 2, 3, 4, 5, 6, 9, 12, 13, 14], among others.

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Agarwal et al. [4], investigated an existence result for the following Caputo fractional order functional differential equations with delay using the Krasnoselskii's fixed point theorem

$$\begin{cases} {}^{C}D^{\alpha}[u(t) - g(t, u(t-r))] = f(t, u(t-r)), \ t \geq t_0, \\ u(t) = \Phi(t), \ t \in [t_0 - r, t_0], \ 0 < \alpha \leq 1. \end{cases}$$

In [5], Bashir et al. studied the qualitative theory of the following boundary value problem with delay

$$\begin{cases} D^{\alpha}\left(D^{\beta}u(t) - g(t, u(t-r))\right) = f(t, u(t-r)), \quad t \in [1, b], \\ u(t) = \phi(t), \quad t \in [1-r, 1], \\ D^{\beta}u(1) = \eta \in \mathbb{R}, \end{cases}$$

where D^{α} and D^{β} are the Caputo-Hadamard fractional derivatives, $0 < \alpha, \beta < 1$.

Ge and Kou [9], by utilizing the Krasnoselskii's fixed point theorem, discussed the stability and asymptotic stability of the zero solution to the following Caputo type fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}u(t) = f(t, u(t)), \ t \geq 0, \ 1 < \alpha \leq 2, \\ u(0) = u_0, \ u'(0) = u_1. \end{cases}$$

Furthermore, in [6], Boulares et al. discussed the stability and asymptotic stability of the zero solution of the following boundary value problem with delay

$$\begin{cases} {}^{C}D^{\alpha}u(t) = f(t, u(t), (t - r(t))) + {}^{C}D^{\alpha-1}g(t, u(t - r(t))), \ t \geq 0, \\ u(t) = \phi(t), \ t \in [m_0, 0], \ u'(0) = u_1, \ 1 < \alpha < 2. \end{cases}$$

where $m_0 = \inf_{t \geq 0}\{t - r(t)\}$. Motivated by the works mentioned above and the papers [1, 2, 3, 8, 10, 12, 13, 14, 17, 18, 19, 23, 24] and the references therein, We aim to enrich the field of differential equations by talking about the analysis of qualitative theory of the subjects of the stability and asymptotic stability of the zero solution to the following initial value problem of mixed Riemann-Liouville-Caputo fractional differential equations with delay on unbounded interval

$$\begin{cases} {}^{RL}D^{\alpha}[{}^{C}D^{\beta}u(t) - g(t, u(t-r))] = f(t, u(t-r)), \ t \geq 0, \\ u(t) = \Phi(t), \ t \in [-r, 0], \\ \lim\limits_{t \to 0} t^{1-\alpha}\, {}^{C}D^{\beta}u(t) = 0, \ u'(0) = u_0 \in \mathbb{R}, \end{cases} \tag{1}$$

where ${}^{RL}D^{\alpha}$ and ${}^{C}D^{\beta}$ are the left Riemann Liouville and left Caputo fractional derivatives respectively, $0 < \alpha \leq 1$, $1 < \beta \leq 2$, $r > 0$, $f, g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ are continuous functions with $f(t, 0) = g(t, 0) = 0$ and we denote the solution of (1) by $u(t, \Phi, u_0)$.

To this aim, we start by transformation (1) into fixed point problem using some mathematical skills of fractional integral and derivative, then we use Krasnoselskii's fixed point theorem in appropriate weighted Banach space.

2. preliminaries

In this section, we introduce some notations, definitions, and preliminary concepts that which we need in later and can be found in [16, 17, 21, 22] as well as we present the equivalent integral equation of (1).

Let C_{λ} be the class of all continuous functions defined on $[-r, +\infty)$ with the norm

$$\|u\|_{\lambda} = \sup_{t \geq -r}\left\{e^{-\lambda t}|u(t)|\right\},$$

for all positive real number $\lambda > 1$. Also $C_r = C([-r, 0])$ is endowed with norm

$$\|\Phi\|_C := \sup\{|\Phi(t)| : t \in [-r, 0]\}.$$

We present in the following some basic concepts of fractional calculus.

Definition 2.1 ([16, 21])**.** *The Riemann-Liouville fractional integral of the function u of order $\alpha > 0$ is defined by*

$$I^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds,$$

where Γ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt$.

Definition 2.2 ([16, 21])**.** *The Riemann-Liouville fractional derivative of the function u of order $\alpha \in (n-1, n]$ is defined by*

$${}^{RL}D^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s)\, ds.$$

Definition 2.3 ([16, 21])**.** *The Caputo fractional derivative of the function u of order $\alpha \in (n-1, n]$ is defined by*

$${}^{C}D^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)\, ds.$$

Let $\alpha > 0$ be a real number, we have two following lemmas.

Lemma 2.4. *The unique solution of linear fractional differential equation*

$${}^{RL}D^{\alpha} u(t) = 0,$$

is given by

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + ... + c_n t^{\alpha-n},\ c_i \in \mathbb{R},\ i = 1, 2, ..., n,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of α.

Lemma 2.5. *The unique solution of linear fractional differential equation*

$${}^{C}D^{\alpha} u(t) = 0,$$

is given by

$$u(t) = c_1 + c_2 t + ... + c_n t^{n-1},\ c_i \in \mathbb{R},\ i = 1, 2, ..., n,$$

where $n = [\alpha] + 1$.

Lemma 2.6. *Problem (1) is equivalent to the following Caputo type fractional differential equation with delay*

$$\begin{cases} {}^{C}D^{\beta} u(t) = I^{\alpha} f(t, u(t-r)) + g(t, u(t-r)),\ t \geq 0, \\ u(t) = \Phi(t),\ t \in [-r, 0], \\ u'(0) = u_0. \end{cases} \tag{2}$$

Proof. Using Lemma 2.4, equation one of (1) can be written as

$${}^{C}D^{\beta} u(t) = I^{\alpha} f(t, u(t-r)) + g(t, u(t-r)) + c_0 t^{\alpha-1},$$

using condition $\lim\limits_{t \to 0} t^{1-\alpha}\, {}^{C}D^{\beta} u(t) = 0$, we get $c_0 = 0$. Then we obtain the desired result. □

Lemma 2.7. *Let f and g are continuous functions. Then $u \in C([-r,+\infty))$ is a solution of the problem (2) if and only if u is a solution of the delay Cauchy type problem*

$$\begin{cases} u'(t) = I^{\alpha+\beta-1} f(t, u(t-r)) + I^{\beta-1} g(t, u(t-r)) + u_0, \ t \geq 0, \\ u(t) = \Phi(t), \ t \in [-r, 0]. \end{cases} \tag{3}$$

Proof. Let $u \in C([-r,+\infty))$ be a solution of the problem (2), for any $t \geq 0$, we have

$${}^{C}D^{\beta} u(t) = \left({}^{C}D^{\beta-1} D^{1} u\right)(t) = I^{\alpha} f(t, u(t-r)) + g(t, u(t-r)).$$

According to Lemma 2.5 and the condition $u'(0) = u_0$, one gets

$$u'(t) = I^{\beta-1}\left[I^{\alpha} f(t, u(t-r)) + g(t, u(t-r))\right] + u_0,$$

which means that u is a solution of the problem (3).

Conversely, let u be a solution of the problem (3). Then, for any $t \geq 0$, it is easy to see that

$$\begin{aligned} &{}^{C}D^{\beta} u(t) \\ &={}^{C} D^{\beta-1} u'(t) \\ &={}^{C} D^{\beta-1}\left(I^{\alpha+\beta-1} f(t, u(t-r)) + I^{\beta-1} g(t, u(t-r))\right) +{}^{C} D^{\beta-1} u_0 \\ &= I^{\alpha} f(t, u(t-r)) + g(t, u(t-r)). \end{aligned}$$

Besides, we have $u'(0) = u_0$. □

Lemma 2.8. *Let $k \in \mathbb{R}^*$ satisfies that $|k| \leq \frac{\lambda-1}{2}$, clearly $\lambda + k > 0$. Then (3) is equivalent to the following Volterra type integral equation*

$$\begin{cases} u(t) = \Phi(0) e^{-kt} + \frac{1-e^{-kt}}{k} u_0 + k \int_0^t e^{-k(t-s)} u(s) ds \\ +\frac{1}{\Gamma(\alpha+\beta-1)} \int_0^t \int_\tau^t e^{-k(t-s)} (s-\tau)^{\alpha+\beta-2} ds f(\tau, u(\tau-r)) d\tau \\ +\frac{1}{\Gamma(\beta-1)} \int_0^t \int_\tau^t e^{-k(t-s)} (s-\tau)^{\beta-2} ds g(\tau, u(\tau-r)) d\tau, \ t \geq 0, \\ u(t) = \Phi(t), \ t \in [-r, 0]. \end{cases} \tag{4}$$

Proof. Let k defined above. It is clear that (3) can be written as follow

$$\begin{cases} u'(t) + ku(t) = ku(t) + \frac{1}{\Gamma(\alpha+\beta-1)} \int_0^t (t-s)^{\alpha+\beta-2} f(s, u(s-r)) ds \\ +\frac{1}{\Gamma(\beta-1)} \int_0^t (t-s)^{\beta-2} g(s, u(s-r)) ds + u_0, \\ u(t) = \Phi(t), \ t \in [-r, 0]. \end{cases}$$

By the variation of constants formula, we obtain (4).

Conversely, it is clear that

$$\left(e^{kt} u(t)\right)' = \left(u'(t) + ku(t)\right) e^{kt},$$

using this fact, we get

$$\begin{aligned}
&\left(u'(t)+ku(t)\right)e^{kt}\\
&=\Bigg[\Phi(0)+k\int_0^t e^{ks}u(s)ds\\
&+\frac{1}{\Gamma(\alpha+\beta-1)}\int_0^t\int_\tau^t e^{ks}(s-\tau)^{\alpha-2}dsf(\tau,u(\tau-r))d\tau\\
&+\frac{1}{\Gamma(\beta-1)}\int_0^t\int_\tau^t e^{ks}(s-\tau)^{\beta-2}dsg(\tau,u(\tau-r))d\tau+\frac{e^{kt}-1}{k}u_0\Bigg]'\\
&=e^{kt}u_0+ke^{kt}u(t)+\Bigg[\int_0^t e^{k\tau}I^{\alpha+\beta-1}f(\tau,u(\tau-r))d\tau\\
&+\int_0^t e^{k\tau}I^{\beta-1}g(\tau,u(\tau))d\tau\Bigg]'\\
&=e^{kt}\left(u_0+I^{\alpha+\beta-1}f(t,u(t-r))+I^{\beta-1}g(t,u(t-r))+ku(t)\right).
\end{aligned}$$

This means that

$$u'(t)=I^{\alpha+\beta-1}f(t,u(t-r))+I^{\beta-1}g(t,u(t-r))+u_0.$$

On the other hand, if (4) holds we have $u(0)=\Phi(0)$.

From the argument above, we get that the system (3) can be equivalently written as (4). □

Definition 2.9. *The trivial solution $u=0$ of (1) is said to be*

(i) stable in Banach space C_λ, if for every $\epsilon>0$, there exists a $\delta=\delta(\epsilon)>0$ such that $|\Phi(t)|+|u_0|\leq\delta$ implies that the solution $u(t)=u(t,\Phi,u_0)$ exists for all $t\in[-r,+\infty)$ and satisfies $\|x\|_\lambda\leq\epsilon$.

(ii) asymptotically stable, if it is stable in C_λ and there exists a number $\sigma>0$ such that $|\Phi(t)|+|u_0|\leq\sigma$ implies $\lim_{t\to\infty}\|u(t)\|_\lambda=0$.

Our main results based on the Krasnoselskii fixed point theorem.

Lemma 2.10 (Krasnoselskii fixed point theorem [22])**.** *If $\mathcal{M}$ is a nonempty bounded, closed and convex subset of a Banach space E, $\mathcal{A}$ and $\mathcal{B}$ two operators defined on $\mathcal{M}$ with values in E such as*

(i) $\mathcal{A}u+\mathcal{B}v\in\mathcal{M}$, for all $u,v\in\mathcal{M}$,

(ii) $\mathcal{A}$ is continuous and compact,

(iii) $\mathcal{B}$ is a contraction.

Then there exists $w\in\mathcal{M}$ such as: $w=\mathcal{A}w+\mathcal{B}w$.

In order to prove (ii), we need to the following modified compactness criterion.

Lemma 2.11 ([17])**.** *Let $\mathcal{M}$ be a subset of the Banach space C_λ. Then $\mathcal{M}$ is relatively compact in C_λ if the following conditions are satisfied*

i) $\left\{e^{-\lambda t}u(t):u\in\mathcal{M}\right\}$ is uniformly bounded,

ii) $\left\{e^{-\lambda t}u(t):u\in\mathcal{M}\right\}$ is equicontinuous on any compact interval of $\mathbb{R}$,

iii) $\left\{e^{-\lambda t}u(t):u\in\mathcal{M}\right\}$ is equiconvergent at infinity, i.e. for any given $\epsilon>0$, there exists a $T_0>0$ such that for all $u\in\mathcal{M}$ and $t_1,t_2>T_0$, it holds

$$\left|e^{-\lambda t_2}u(t_2)-e^{-\lambda t_1}u(t_1)\right|<\epsilon.$$

3. Main results

This section devoted to presenting and proving our main results. Before this end, we introduce the following hypotheses.

$(H1)$ $f, g : I \times C_r \to \mathbb{R}$ are continuous functions.

$(H2)$ There exists a constant $l > 0$ and a bounded continuous function $\eta > 0$ so that if $|u|, |v| \leq l$ then

$$|g(t, u) - g(t, v)| \leq \eta(t) |u - v|, \text{ for all } t \in \mathbb{R}_+.$$

$(H3)$ There exist a constant $\gamma > 0$ and two continuous functions $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$, $\Psi : (0, \gamma] \to \mathbb{R}_+$ such that

$$\left| f(t, e^{\lambda(t-r)} u) \right| \leq e^{\lambda t} \zeta(t) \Psi(|u|),$$

holds for all $t \geq 0$, $0 < |u| \leq \gamma$, where Ψ is nondecreasing function and $\zeta \in L^1([0, \infty))$.

Theorem 3.1. *Assume that $(H1) - (H3)$ hold. Then the trivial solution $u = 0$ of (1) is stable in C_λ, provided that there exist constants $M_1, M_2 > 0$ such that*

$$\Psi(z) \sup_{t \geq 0} \int_0^t e^{-\lambda(t-\tau)} \mathcal{K}(t - \tau) \zeta(\tau) d\tau \leq z M_2, \tag{5}$$

for all $z \in (0, \gamma]$, and

$$\sup_{t \geq 0} \int_0^t e^{-\lambda(t-\tau)} \mathcal{H}(t - \tau) \eta(\tau) d\tau \leq M_1 < 1 - \frac{|k|}{\lambda + k} - M_2 < 1, \tag{6}$$

where

$$\mathcal{K}(t - \tau) = \frac{1}{\Gamma(\alpha + \beta - 1)} \int_\tau^t e^{-k(t-s)} (s - \tau)^{\alpha+\beta-2} ds, \ \text{if } t - \tau \geq 0,$$

and

$$\mathcal{H}(t - \tau) = \frac{1}{\Gamma(\beta - 1)} \int_\tau^t e^{-k(t-s)} (s - \tau)^{\beta-2} ds, \ \text{if } t - \tau \geq 0.$$

Proof. In the proof of this theorem, we use the fact that $e^{-\lambda t} = e^{-\lambda(t-\tau)} e^{-\lambda\tau}$ for all $t \geq \tau$. For any given $\epsilon > 0$, we first prove the existence of $\delta > 0$ such that

$$|\phi(t)| + |u_0| < \delta \text{ implies } \|u\| \leq \epsilon.$$

Let $0 < \delta \leq \frac{|k|}{|k|+2} \left[\left(1 - M_1 - M_2 - \frac{|k|}{\lambda+k} \right) \epsilon \right]$. Consider the nonempty closed convex subset $B_\epsilon = \{u \in C_\lambda([-r, +\infty), \mathbb{R}) : e^{-\lambda t} |u(t)| \leq \epsilon$ for $t \geq 0$ and $u(t) = \phi(t)$ if $t \in [-r, 0]\}$ for any $\epsilon > 0$. We define two mapping $\mathcal{A}, \mathcal{B} : B_\epsilon \to C_\lambda([-r, +\infty], \mathbb{R})$ by

$$(\mathcal{A}u)(t) = \begin{cases} 0 \text{ if } t \in [-r, 0], \\ k \int_0^t e^{-k(t-s)} u(s) ds + \int_0^t \mathcal{K}(t - \tau) f(\tau, u(\tau - r)) d\tau \text{ if } t \in I, \end{cases} \tag{7}$$

and

$$(\mathcal{B}u)(t) = \begin{cases} \Phi(t) \text{ if } t \in [-r, 0], \\ \Phi(0) e^{-kt} + \frac{1-e^{-kt}}{k} u_0 + \int_0^t \mathcal{H}(t - \tau) g(\tau, u(\tau - r)) d\tau \text{ if } t \in I. \end{cases} \tag{8}$$

Clearly, for $u \in B_\epsilon$, both $\mathcal{A}u$ and $\mathcal{B}u$ are continuous functions on $[-r, +\infty)$. Also, for $u \in B_\epsilon$, for any $t \geq 0$, we have

$$\begin{aligned}
& e^{-\lambda t} \left| (\mathcal{A}u)(t) \right| \\
& \leq |k| e^{-\lambda t} \int_0^t e^{-k(t-s)} |u(s)| \, ds + \int_0^t e^{-\lambda t} \mathcal{K}(t-\tau) \left| f(\tau, u(\tau - r)) \right| d\tau \\
& \leq |k| \int_0^t e^{-\lambda(t-s)} e^{-k(t-s)} \left| e^{-\lambda s} u(s) \right| ds \\
& + \int_0^t e^{-\lambda t} \mathcal{K}(t-\tau) \zeta(\tau) \Psi(|u(\tau - r)|) \, d\tau \\
& \leq |k| \, \|u\|_\lambda \int_0^\infty e^{-(k+\lambda)s} ds \\
& + \int_0^t e^{-\lambda(t-\tau)} \mathcal{K}(t-\tau) \zeta(\tau) \Psi\left(e^{-\lambda(\tau - r)} |u(\tau - r)| \right) d\tau \\
& \leq \left(\frac{2|k|}{\lambda + k} + M_2 \right) \epsilon < \infty,
\end{aligned} \tag{9}$$

and

$$\begin{aligned}
& e^{-\lambda t} \left| (\mathcal{B}u)(t) \right| \\
& \leq |\Phi(0)| e^{-(\lambda + k)t} + \frac{e^{-\lambda t} + e^{-(\lambda + k)t}}{|k|} |u_0| \\
& + \int_0^t e^{-\lambda t} \mathcal{H}(t-\tau) g(\tau, u(\tau - r)) d\tau \\
& \leq |\Phi(0)| + 2\frac{|u_0|}{|k|} + \int_0^t e^{-\lambda(t-\tau)} \mathcal{H}(t-\tau) e^{-\lambda\tau} g(\tau, u(\tau - r)) d\tau \\
& \leq |\Phi(0)| + 2\frac{|u_0|}{|k|} + \int_0^t e^{-\lambda(t-\tau)} \mathcal{H}(t-\tau) \eta(\tau) \left| e^{-\lambda\tau} u(\tau) \right| d\tau \\
& \leq |\Phi(0)| + 2\frac{|u_0|}{|k|} + \left\{ \int_0^t e^{-\lambda(t-\tau)} \mathcal{H}(t-\tau) \eta(\tau) \, d\tau \right\} \|u\|_\lambda \\
& \leq |\Phi(0)| + 2\frac{|u_0|}{|k|} + M_1 \epsilon < \infty.
\end{aligned} \tag{10}$$

Then $\mathcal{A}B_\epsilon \subset C_\lambda$ and $\mathcal{B}B_\epsilon \subset C_\lambda$. Now we shall to prove that there exists at least one fixed point of the operator $\mathcal{A} + \mathcal{B}$. To this end, we divide the proof into three claims.

Claim 1. We show that $\mathcal{A}u + \mathcal{B}v \in B_\epsilon$ for all $u, v \in B_\epsilon$, we combine (9) and (10) to get

$$\|\mathcal{A}u + \mathcal{B}v\|_\lambda \leq \frac{|k| + 2}{|k|} \delta + \left(M_1 + M_2 + \frac{2|k|}{\lambda + k} \right) \epsilon \leq \epsilon, \tag{11}$$

this means that $\mathcal{A}u + \mathcal{B}v \in B_\epsilon$, for all $u, v \in B_\epsilon$.

Claim 2. Obviously, $\mathcal{A}$ is continuous operator, it remains to prove that $\mathcal{A}B_\epsilon$ is relatively compact in C_λ. In fact, from (11), we get that $\left\{ e^{-\lambda t} u(t) : u \in B_\epsilon \right\}$ is uniformly bounded in C_λ. Moreover, a classical theorem states the fact that the convolution of an L^1-function with a function tending to zero, does also tend to zero. Then we conclude that for $t \geq \tau$, we have

$$\begin{aligned}
& \lim_{t \to \infty} e^{-\lambda(t-\tau)} \mathcal{K}(t-\tau) \\
& = \lim_{t \to \infty} \frac{1}{\Gamma(\alpha + \beta - 1)} \int_\tau^t \left[e^{-\lambda(t-s)} e^{-k(t-s)} \right] \left[e^{-\lambda(s-\tau)} (s-\tau)^{\alpha + \beta - 2} \right] ds \\
& = \lim_{t \to \infty} \frac{1}{\Gamma(\alpha + \beta - 1)} \int_0^{t-\tau} \left[e^{-(\lambda + k)(t - \tau - s)} \right] \left[e^{-\lambda s} s^{\alpha + \beta - 2} \right] ds = 0.
\end{aligned} \tag{12}$$

Together with the continuity of functions $\mathcal{K}$ and $t \longmapsto e^{-\lambda t}$, we get that there exists a constant $M_3 > 0$ such that

$$e^{-\lambda(t-\tau)}\mathcal{K}(t-\tau) \leq M_3.$$

Also, for any fixed $T_0 \geq 0$ and any $t_1, t_2 \in [0, T_0]$, $t_1 < t_2$, we have

$$\begin{aligned}
&\left|e^{-\lambda t_2}(\mathcal{A}u)(t_2) - e^{-\lambda t_1}(\mathcal{A}u)(t_1)\right| \\
&= \left|k\int_0^{t_2} e^{-\lambda t_2}e^{-k(t_2-s)}u(s)ds - k\int_0^{t_1} e^{-\lambda t_1}e^{-k(t_1-s)}u(s)ds\right. \\
&+ \int_0^{t_2} e^{-\lambda t_2}\mathcal{K}(t_2-\tau) f(\tau, u(\tau-r))\,d\tau \\
&\left. - \int_0^{t_1} e^{-\lambda t_1}\mathcal{K}(t_1-\tau) f(\tau, u(\tau-r))\,d\tau\right| \\
&\leq |k|\int_0^{t_1} \left|e^{-\lambda t_2}e^{-k(t_2-s)} - e^{-\lambda t_1}e^{-k(t_1-s)}\right| |u(s)|\,ds \\
&+ \int_0^{t_1} \left|e^{-\lambda t_2}\mathcal{K}(t_2-\tau) - e^{-\lambda t_1}\mathcal{K}(t_1-\tau)\right| |f(\tau, u(\tau-r))|\,d\tau \\
&+ |k|\int_{t_1}^{t_2} e^{-\lambda t_2}e^{-k(t_2-s)}|u(s)|\,ds + \int_{t_1}^{t_2} e^{-\lambda t_2}\mathcal{K}(t_2-\tau)|f(\tau, u(\tau-r))|\,d\tau.
\end{aligned}$$

Thus

$$\begin{aligned}
&\left|e^{-\lambda t_2}(\mathcal{A}u)(t_2) - e^{-\lambda t_1}(\mathcal{A}u)(t_1)\right| \\
&\leq |k|\int_0^{t_1} \left|e^{-(\lambda+k)(t_2-s)} - e^{-(\lambda+k)(t_1-s)}\right| \left|e^{-\lambda s}u(s)\right| ds \\
&+ \int_0^{t_1} \left|e^{-\lambda t_2}\mathcal{K}(t_2-\tau) - e^{-\lambda t_1}\mathcal{K}(t_1-\tau)\right| \zeta(\tau)\,\Psi(|u(\tau-r)|)\,d\tau \\
&+ \int_{t_1}^{t_2} e^{-\lambda(t_2-\tau)}\mathcal{K}(t_2-\tau)\,\zeta(\tau)\,e^{-\lambda\tau}\Psi(|u(\tau-r)|)\,d\tau \\
&+ |k|\int_{t_1}^{t_2} e^{-(\lambda+k)(t_2-s)}\left|e^{-\lambda s}u(s)\right| ds \\
&\leq \left\{|k|\int_0^{t_1} \left|e^{-(\lambda+k)(t_2-s)} - e^{-(\lambda+k)(t_1-s)}\right| ds + |k|\int_{t_1}^{t_2} e^{-(\lambda+k)(t_2-s)}ds\right\}\epsilon \\
&+ \left\{\int_0^{t_1} \left|e^{-\lambda t_2}\mathcal{K}(t_2-\tau) - e^{-\lambda t_1}\mathcal{K}(t_1-\tau)\right| \zeta(\tau)\,d\tau\right. \\
&\left. + \|\zeta\|_\infty M_3(t_2-t_1)\right\}\Psi(\epsilon) \\
&\to 0 \text{ as } t_2 \to t_1,
\end{aligned}$$

this means that $\left\{e^{-\lambda t}u(t) : u \in B_\epsilon\right\}$ is equicontinuous on any compact interval of $\mathbb{R}_+$, it remains to show that the set $\left\{e^{-\lambda t}u(t) : u \in B_\epsilon\right\}$ is equiconvergent at infinity. In fact, for any $\epsilon_1 > 0$ such that $\epsilon \leq \frac{\lambda+k}{6|k|}\epsilon_1$, there exists a $L > 0$ satisfies

$$M_3\int_L^\infty \zeta(\tau)\,d\tau \leq \frac{\epsilon_1}{6}.$$

According to (12), we get that

$$\begin{aligned}
&\lim_{t\to\infty}\sup_{\tau\in[0,L]} e^{-\lambda(t-\tau)}\mathcal{K}(t-\tau) \\
&\leq \max\left\{\lim_{t\to\infty} e^{-\lambda(t-L)}\mathcal{K}(t-L), \lim_{t\to\infty} e^{-\lambda t}\mathcal{K}(t)\right\} = 0.
\end{aligned}$$

Then, there exists $T > L$ such that for $t_1, t_2 \geq T$, we obtain

$$\begin{aligned}
&\sup_{\tau\in[0,L]} \left| e^{-\lambda t_2}\mathcal{K}\left(t_2-\tau\right)e^{\lambda\tau} - e^{-\lambda t_1}\mathcal{K}\left(t_1-\tau\right)e^{\lambda\tau}\right| \\
&\leq \sup_{\tau\in[0,L]} \left| e^{-\lambda(t_2-\tau)}\mathcal{K}\left(t_2-\tau\right)\right| + \sup_{\tau\in[0,L]} \left| e^{-\lambda(t_1-\tau)}\mathcal{K}\left(t_1-\tau\right)\right| \\
&\leq \frac{\epsilon_1}{6}\left(\Psi\left(\epsilon\right)\int_0^\infty \zeta\left(\tau\right)d\tau\right)^{-1}.
\end{aligned}$$

Furthermore, for $t \geq s$, we have

$$\lim_{t\to\infty} e^{-(\lambda+k)(t-s)} = 0,$$

then for $t_1, t_2 \geq T$, on gets

$$\begin{aligned}
&\sup_{s\in[0,L]} \left| e^{-(\lambda+k)(t_2-s)} - e^{-(\lambda+k)(t_1-s)}\right| \\
&\leq \sup_{s\in[0,L]} \left| e^{-(\lambda+k)(t_2-s)}\right| + \sup_{s\in[0,L]} \left| e^{-(\lambda+k)(t_1-s)}\right| \leq \frac{\epsilon_1}{6}\left(\epsilon\left|k\right|\right)^{-1}.
\end{aligned}$$

Therefore, for $t_1, t_2 \geq T$, we have

$$\begin{aligned}
&\left| e^{-\lambda t_2}\left(\mathcal{A}u\right)\left(t_2\right) - e^{-\lambda t_1}\left(\mathcal{A}u\right)\left(t_1\right)\right| \\
&= \Bigg| k\int_0^{t_2} e^{-\lambda t_2}e^{-k(t_2-s)}u(s)ds - k\int_0^{t_1} e^{-\lambda t_1}e^{-k(t_1-s)}u(s)ds \\
&+ \int_0^{t_2} e^{-\lambda t_2}\mathcal{K}\left(t_2-\tau\right)f\left(\tau, u(\tau-r)\right)d\tau \\
&- \int_0^{t_1} e^{-\lambda t_1}\mathcal{K}\left(t_1-\tau\right)f\left(\tau, u(\tau-r)\right)d\tau \Bigg| \\
&\leq \epsilon\left|k\right|\int_0^L \left| e^{-(\lambda+k)(t_2-s)} - e^{-(\lambda+k)(t_1-s)}\right| ds + 2\epsilon\left|k\right|\int_0^\infty e^{-(\lambda+k)s}ds \\
&+ \Psi\left(\epsilon\right)M_3\int_L^{t_2}\zeta\left(\tau\right)d\tau + \Psi\left(\epsilon\right)M_3\int_L^{t_1}\zeta\left(\tau\right)d\tau \\
&+ \Psi\left(\epsilon\right)\int_0^L \left| e^{-\lambda t_2}\mathcal{K}\left(t_2-\tau\right)e^{\lambda\tau} - e^{-\lambda t_1}\mathcal{K}\left(t_1-\tau\right)e^{\lambda\tau}\right| \zeta\left(\tau\right)d\tau \\
&\leq \frac{\epsilon_1}{6} + \frac{2\epsilon\left|k\right|}{\lambda+k} + 2\Psi\left(\epsilon\right)M_3\int_L^\infty \zeta\left(\tau\right)d\tau + \frac{\epsilon_1}{6} \leq \epsilon_1,
\end{aligned}$$

this achieves the proof.

Claim 3. We show that $\mathcal{B} : B_\epsilon \to C_\lambda$ is a contraction mapping. In fact, for any $u, v \in B_\epsilon$, from $(H2)$, we have

$$\begin{aligned}
&\sup_{t\geq 0} e^{-\lambda t}\left|\left(\mathcal{B}u\right)(t) - \left(\mathcal{B}v\right)(t)\right| \\
&= \sup_{t\geq 0}\left\{\int_0^t e^{-\lambda t}\mathcal{H}\left(t-\tau\right)\left|g(\tau, u(\tau-r)) - g\left(\tau, v(\tau-r)\right)\right| d\tau\right\} \\
&\leq \sup_{t\geq 0}\left\{\int_0^t e^{-\lambda t}\mathcal{H}\left(t-\tau\right)\eta\left(\tau\right)\left|u\left(\tau\right) - v\left(\tau\right)\right| d\tau\right\} \\
&\leq \sup_{t\geq 0}\left\{\int_0^t e^{-\lambda(t-\tau)}\mathcal{H}\left(t-\tau\right)\eta\left(\tau\right)\left[e^{-\lambda\tau}\left|u(\tau) - v\left(\tau\right)\right|\right] d\tau\right\} \\
&\leq \left\{\sup_{t\geq 0}\int_0^t e^{-\lambda(t-\tau)}\mathcal{H}\left(t-\tau\right)\eta\left(\tau\right)d\tau\right\}\left\|u-v\right\|_\lambda \leq M_1\left\|u-v\right\|_\lambda,
\end{aligned}$$

from (7) $\mathcal{A}$ is a contraction mapping.

By 2.10, there exists at least one fixed point of the operator $\mathcal{A}+\mathcal{B}$.

Finally, for any $\epsilon_2>0$, if $0<\delta_1\leq\frac{|k|}{|k|+2}\left[\left(1-M_1-M_2-\frac{|k|}{\lambda+k}\right)\epsilon_2\right]$, then $|\phi(t)|+|u_0|<\delta_1$ implies that

$$\begin{aligned}
&e^{-\lambda t}\left|u\left(t\right)\right|\\
&\leq|k|\,e^{-\lambda t}\int_0^t e^{-k(t-s)}\left|u(s)\right|ds+\int_0^t e^{-\lambda t}\mathcal{K}\left(t-\tau\right)\left|f(\tau,u(\tau-r))\right|d\tau\\
&+\left|\Phi\left(0\right)\right|e^{-(\lambda+k)t}+\frac{e^{-\lambda t}+e^{-(\lambda+k)t}}{|k|}\left|u_0\right|\\
&+\int_0^t e^{-\lambda t}\mathcal{H}\left(t-\tau\right)g(\tau,u(\tau-r))d\tau\\
&\leq|k|\int_0^t e^{-\lambda(t-s)}e^{-k(t-s)}\left|e^{-\lambda s}u(s)\right|ds\\
&+\int_0^t e^{-\lambda t}\mathcal{K}\left(t-\tau\right)\zeta\left(\tau\right)\Psi\left(\left|u(\tau-r)\right|\right)d\tau\\
&+\left|\Phi\left(0\right)\right|+2\frac{|u_0|}{|k|}+\int_0^t e^{-\lambda(t-\tau)}\mathcal{H}\left(t-\tau\right)\eta\left(\tau\right)\left|e^{-\lambda\tau}u\left(\tau\right)\right|d\tau\\
&\leq\left(M_1+M_2+\frac{|k|}{\lambda+k}\right)\|u\|_\lambda+\frac{|k|+2}{|k|}\delta_1,
\end{aligned}$$

this means that

$$\|u\|_\lambda\left(1-\left(M_1+M_2+\frac{|k|}{\lambda+k}\right)\right)\leq\frac{|k|+2}{|k|}\delta_1,$$

so,

$$\|u\|_\lambda\leq\frac{|k|+2}{|k|\left(1-M_1-M_2-\frac{|k|}{\lambda+k}\right)}\delta_1\leq\epsilon_2.$$

Thus, we know that the trivial solution of (1) is stable in Banach space C_λ. □

Theorem 3.2. *Suppose that all conditions of Theorem 3.1 are satisfied, and for any $R>0$, there exist functions $\varphi_R,\psi_R\in L^1\left([0,+\infty)\right)$, $\varphi_R(t),\psi_R\left(t\right)>0$ such that $|u|\leq R$ implies*

$$e^{-\lambda t}\left|f(t,u)\right|\leq\varphi_R(t),\ \ e^{-\lambda t}\left|g(t,u)\right|\leq\psi_R(t)\ \ a.e.\ t\in[0,+\infty)\,. \tag{13}$$

Then the trivial solution of (1) is asymptotically stable.

Proof. First, according to the Theorem 3.1, the trivial solution of (1) is stable in the Banach space C_λ. Next, we shall prove that the trivial solution $u=0$ of (1) is attractive. To this purpose, we define the subset of B_R

$$B_R^*=\left\{u\in B_R,\ \lim_{t\to\infty}e^{-\lambda t}u(t)=0\right\},\ \text{for any } R>0.$$

It is a nought to prove that $\mathcal{A}u+\mathcal{B}v\in B_R^*$ for any $u,v\in B_R^*$, i.e.

$$e^{-\lambda t}\left[\left(\mathcal{A}u\right)(t)+\left(\mathcal{B}v\right)(t)\right]\to0\text{ as }t\to\infty,$$

where

$$\begin{aligned}
&\left(\mathcal{A}u\right)(t)+\left(\mathcal{B}v\right)(t)\\
&=k\int_0^t e^{-k(t-s)}u(s)ds+\int_0^t\mathcal{K}\left(t-\tau\right)f(\tau,u(\tau-r))d\tau\\
&+\Phi\left(0\right)e^{-kt}+\frac{1-e^{-kt}}{k}u_0+\int_0^t\mathcal{H}\left(t-\tau\right)g(\tau,v(\tau-r))d\tau.
\end{aligned}$$

In fact, for $u, v \in B_r^*$, based on the fact that used in the proof of Theorem 3.1 (Claim 2), it follows that

$$e^{-\lambda t} k \int_0^t e^{-k(t-s)} u(s) ds = k \int_0^t e^{-(\lambda+k)(t-s)} \left(e^{-\lambda s} u(s)\right) ds \to 0,$$

and

$$e^{-\lambda(t-\tau)} \mathcal{K}(t-\tau) \to 0,$$

as $t \to \infty$. Together with the hypothesis $\varphi_R, \psi_R \in L^1([0, +\infty))$ and using same way of (12) on the function $\mathcal{H}$, we obtain

$$\begin{aligned} & e^{-\lambda t} \int_0^t \mathcal{K}(t-\tau) \left| f(\tau, u(\tau - r)) \right| d\tau \\ & \leq \int_0^t \left[e^{-\lambda(t-\tau)} \mathcal{K}(t-\tau) \right] \varphi_R(\tau) d\tau \to 0, \end{aligned}$$

and

$$\begin{aligned} & e^{-\lambda t} \int_0^t \mathcal{H}(t-\tau) \left| g(\tau, u(\tau - r)) \right| d\tau \\ & \leq \int_0^t \left[e^{-\lambda(t-\tau)} \mathcal{H}(t-\tau) \right] \psi_R(\tau) d\tau \to 0, \end{aligned}$$

as $t \to \infty$. Furthermore, it is easy to see that

$$\Phi(0) e^{-kt} + \frac{1 - e^{-kt}}{k} u_0 \in B_R^*.$$

Thus, the trivial solution of (1) is asymptotically stable. □

Example 3.3. *Let us consider the following initial value problem of mixed Riemann-Liouville-Caputo fractional differential equations with delay on unbounded interval*

$$\begin{cases} {}^{RL}D^{1/2}[{}^{C}D^{3/2}u(t) - \frac{1}{4}e^{-4t}\sin(u(t-2))] = \frac{1}{5}e^{-4t}u^2(t-2), \ t \geq 0, \\ u(t) = \sin(t), \ t \in [-2, 0], \\ \lim\limits_{t \to 0} t^{1-\alpha}\, {}^{C}D^{3/2}u(t) = 0, \ u'(0) = 1. \end{cases} \tag{14}$$

Then $\alpha = 1/2$, $\beta = 3/2$, $r = 2$, $u_0 = 1$, $\Phi(t) = \sin(t)$, $g(t, u(t-2)) = \frac{1}{4}e^{-4t}\sin(u(t-2))$, $f(t, u(t-2)) = \frac{1}{5}e^{-4t}u^2(t-2)$, $\lambda = 3$, $K = 2$, $\gamma = 1$. Doing straightforward computations, we obtain $\eta(t) = \frac{1}{4}e^{-4t}$, $\zeta(t) = \frac{1}{5}e^{-t}$, $\Psi(|u|) = u^2$, $M_2 = \frac{\sqrt{3}}{5}$, $M_1 = \frac{3}{16}$. Then the trivial solution of (14) is stable in C_3 follows from Theorem 3.1.

Moreover, let $\varphi_R(t) = \frac{1}{5}e^{-t}R^2$ and $\psi_R(t) = \frac{1}{4}e^{-t}|\sin(R)|$, then $\varphi_R, \psi_R \in L^1([0, +\infty))$. Hence, by Theorem 3.2, we get that the trivial solution of (14) is asymptotically stable.

4. Conclusion

In this paper, by utilizing the Krasnoselskii fixed point theorem in a weighted Banach space, we investigate the stability and asymptotic stability of the trivial solution for nonlinear fractional differential equations with the left Riemann Liouville and left Caputo fractional derivatives of orders $\alpha \in (0, 1]$ and $\beta \in (1, 2]$ respectively. We establish the equivalence between the fractional differential equation and the integral equation on an infinite interval. Two main theorems are obtained. We also put an example to illustrate our results. However, we still have works to improve our constraint conditions for they are a little complicated in reality.

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