COMPARATION OF TWO METHODS FOR A DIFFERENTIAL EQUATION WITH VARIANT RETARDED ARGUMENT

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Abstract

In this paper, we applied two approximate methods for the solution of a boundary value problem for a differential equation with retarded argument:

$$x''(t) + a(t)x(t - \tau(t)) = f(t)$$
$$x(t) = \varphi(t) \ \ (\lambda_0 \le t \le 0) \ \ x(T) = x_\tau,$$

where $a(t), f(t), \tau(t) \ge 0 \ (0 \le t \le T)$ *and* $\varphi(t) \ (\lambda_0 \le t \le 0)$ *are known as continuous functions.*

Keywords: *ordinary differential equations, boundary value problem, successive approximations method*

GECİKEN DEĞİŞKENLİ DİFERENSİYEL DENKLEM İÇİN İKİ YÖNTEM KARŞILAŞTIRMASI

Özet

Bu makalede, ***geciken*** *değişkenli diferensiyel denklem için konulmuş*

$$x''(t) + a(t)x(t - \tau(t)) = f(t)$$
$$x(t) = \varphi(t) \ \ (\lambda_0 \le t \le 0) \ \ x(T) = x_\tau,$$

sınır değer probleminin yaklaşık çözümünü bulmak için iki yöntem uygulanmış ve bu yöntemler mukayese edilmiştir.

Anahtar Kelimeler: *bayağı diferansiyel denklemler, sınır değer problemi, ardışık yaklaşıklar yöntemi*

1. Introduction

A common method used for the analytical solution of the boundary value problems is the integral equation method [1,2]. With this method, we obtain an integral equation that is equivalent to the boundary value problem and the solution of the integral equation is defined as the solution of the boundary value problem. The equivalent integral equation is usually a Fredholm equation in the classical theory. In this study we obtain a Fredholm-Volterra integral equation different from classical theory for the problem

$$\begin{gathered} x''(t) + a(t)x(t - \tau(t)) = f(t) \\ x(t) = \varphi(t) \ \ (\lambda_0 \le t \le 0) \ \ x(T) = x_\tau, \end{gathered} \qquad (1)$$

where $0 \le t \le T$ and $a(t), f(t), \tau(t) \ge 0 \ (0 \le t \le T)$ and $\varphi(t) \ (\lambda_0 \le t \le 0)$ are known as continuous functions. The Fredholm operator included in the equivalent integral equation is an operator with a degenerated kernel. We applied the modified two sided approximations method and consecutive substitution method for problem (1). Earlier the method two sided approximations method and modified successive approximations method have been investigated in [3].

In this study these methods were applied to the boundary value problem with retarted argument. We investigated the solution for arbitrary continuous function $\tau(t)$.

2. An equivalent integral equation

In problem (1), if we take $\lambda(t)=t-\tau(t)$ then $t_0\in[0,t]$ is a point located at the left side of T such that conditions $\lambda(t_0)=0$ and $\lambda(t)\le 0$ $(0\le t\le t_0)$ are satisfied, where, $\lambda_0=\min_{0\le t\le t_0}\lambda(t)$. We assume that $\lambda(t)$ is a nondecreasing function in the interval $[t_0,t]$ and the equation $\lambda(t)=\sigma$ has differential continuous solution $t=\gamma(\sigma)$ for arbitrary $\sigma\in[0,\lambda(t)]$. It can be seen that if $x^*(t)$ is a solution of the boundary value for problem (1) then $x^*(t)$ is also the solution of the equation

$$x(t)=\hat{h}(t)+\frac{t}{T}\int_0^T (T-s)a(s)x(s-\tau(s))ds-\int_0^t (t-s)a(s)x(s-\tau(s))ds. \qquad (2)$$

Here, $\hat{h}(t)=\varphi(0)-(x_T-\varphi(0))\frac{t}{T}-\frac{t}{T}\int_0^T (T-s)f(s)ds+\int_0^t (t-s)f(s)ds.$

Let $\sigma=s-\tau(s)$. Therefore Eq. (2) can be written as follows:

$$x(t)=h(t)+\frac{t}{T}\int_0^{\lambda(T)}(T-\gamma(\sigma))a(\gamma(\sigma))x(\sigma)\gamma'(\sigma)d\sigma \qquad (3)$$
$$-\int_0^{\lambda(t)}(t-\gamma(\sigma))a(\gamma(\sigma))x(\sigma)\gamma'(\sigma)d\sigma,$$

where

$$h(t)=\hat{h}(t)+\frac{t}{T}\int_{\lambda_0}^{0}(T-\gamma(\sigma))a(\gamma(\sigma))\varphi(\sigma)\gamma'(\sigma)d\sigma \qquad 4)$$
$$-\int_{\lambda_0}^{0}(t-\gamma(\sigma))a(\gamma(\sigma))\varphi(\sigma)\gamma'(\sigma)d\sigma,$$

Let $K_1(\sigma)=(T-\gamma(\sigma))a(\gamma(\sigma))\gamma'(\sigma)$ and $K(t,\sigma)=(t-\gamma(\sigma))a(\gamma(\sigma))\gamma'(\sigma)$.

Therefore, we write

$$x(t)=h(t)+\frac{t}{T}\int_0^{\lambda(T)}K_1(\sigma)x(\sigma)d\sigma-\int_0^{\lambda(t)}K(t,\sigma)x(\sigma)d\sigma \qquad (5)$$

or $x(t)=h(t)+\frac{t}{T}F_\lambda x+V_\lambda x$ ....(6)

where $F_\lambda x\equiv\int_0^{\lambda(T)}K_1(\sigma)x(\sigma)d\sigma$ is the Fredholm operator, $V_\lambda x\equiv-\int_0^{\lambda(t)}K(t,\sigma)x(\sigma)d\sigma$ is the Volterra operator. Eq. (6) is a Fredholm-Volterra integral equation and it is equivalent to problem (1).

3. Modified two sided approximations method

In this section, we will use the modified two sided approach method for the solution of the equation (5). This equation equivalent to the problem (1). Suppose that the sings of the $\gamma'(t)$ and $a(t)$ functions are fixed in the intervals $[0,\lambda(t)]$ and $[0,T]$. Let $a(t)\ge 0$ $(0\le t\le T)$ and $x(t)\ge 0$ $(0\le t\le T)$. Therefore, the Fredholm operator $F_\lambda x$ is non decreasing and the Volterra operator $V_\lambda x$ is non increasing. Let us chose that the values of the first lower approach $\underline{x}_0$ and first upper approach $\bar{x}_0$ and these approach satisfy conditions

$$\underline{x}_0(t)\le h+\frac{t}{T}F_\lambda\underline{x}_0+V_\lambda\bar{x}_0 \qquad \bar{x}_0(t)\ge h+\frac{t}{T}F_\lambda\bar{x}_0+V_\lambda\underline{x}_0 \qquad (7)$$

and $\underline{x}_0 \le \overline{x}_0$ ....(8)

The other approaches are defined by

$$\underline{x}_n(t) = h(t) + \frac{t}{T} F_\lambda \underline{x}_n + V_\lambda \overline{x}_{n-1}$$

$$\overline{x}_n(t) = h(t) + \frac{t}{T} F_\lambda \overline{x}_n + V_\lambda \underline{x}_{n-1} \qquad (9)$$

In order to obtain solution of the problem (1) by using this method we will consider the following theorem.

Theorem 1: Suppose that

$\alpha = 1 - \frac{1}{T} F_\lambda(t) \neq 0$ and $q = \left(\int_0^{\lambda(T)} |K_1(\sigma)|\, d\sigma + 1 \right) \int_0^{\lambda(T)} |K_1(\sigma)|\, d\sigma < 1.$

Therefore the problem (1) has the unique solution and this solution is determined by limit's of the upper and lower approximations are

$$\underline{x}_n = h + \frac{t}{T\alpha} F_\lambda h + V_\lambda \overline{x}_{n-1} + \frac{t}{T\alpha} F_\lambda V_\lambda \overline{x}_{n-1}$$

$$\overline{x}_n = h + \frac{t}{T\alpha} F_\lambda h + V_\lambda \underline{x}_{n-1} + \frac{t}{T\alpha} F_\lambda V_\lambda \underline{x}_{n-1} \quad (n = 1, 2, ...) \cdot$$

The convergent speed is also determined by inequality $\| \overline{x}_n - \underline{x}_n \| \le q^n \| \overline{x}_0 - \underline{x}_0 \|$.

Proof: In order to find approach $\underline{x}_n(t)$ and $\overline{x}_n(t)$ given by (7) we will use the auxiliary equation

$$y(t) = \overline{h}(t) + \frac{t}{T} F_\lambda y \qquad (10)$$

The equation (10) is Fredholm integral equation with degenerated kernel and this equation has a solution. Where $F_\lambda y = c$. By hypothesis we write, $\alpha = 1 - \frac{1}{T} F_\lambda t \neq 0.$

Hence , the solution of the equation (10) is

$$y(t) = \overline{h}(t) + \frac{t}{T\alpha} F_\lambda h \qquad (11)$$

If we chose $\overline{h} = h + V_\lambda \overline{x}_{n-1}$ or $\overline{h} = h + V_\lambda \underline{x}_{n-1}$ then lower and upper approximations are determined by

$$\underline{x}_n = h + \frac{t}{T\alpha} F_\lambda h + V_\lambda \overline{x}_{n-1} + \frac{t}{T\alpha} F_\lambda V_\lambda \overline{x}_{n-1}$$

$$\overline{x}_n = h + \frac{t}{T\alpha} F_\lambda h + V_\lambda \underline{x}_{n-1} + \frac{t}{T\alpha} F_\lambda V_\lambda \underline{x}_{n-1} \quad (n = 1, 2, ...) \qquad (12)$$

Thus $\lim_{n \to \infty} \underline{x}_n(t) = \underline{x}(t)$ $\lim_{n \to \infty} \overline{x}_n(t) = \overline{x}(t)$

Suppose that the first lower and first upper approaches satisfy the conditions (7) and (8). Therefore one can show that approximations $\underline{x}_n(t)$ and $\overline{x}_n(t)$ defined by (12) non decreasing and non increasing respectively and $\underline{x}_n(t)$ not great $\overline{x}_n(t)$. Thus, the limit's of the functions $\underline{x}(t)$ and $\overline{x}(t)$ in (9) is solution of the problem (1) as $n \to \infty$. Now let us calculate $\overline{x}_n(t) - \underline{x}_n(t)$. We write

$$\overline{x}_n(t)-\underline{x}_n(t)=-\frac{t}{T\alpha}F_\lambda V_\lambda(\overline{x}_{n-1}-\underline{x}_{n-1})-V_\lambda(\overline{x}_{n-1}-\underline{x}_{n-1}) \quad \text{.............................(13)}$$

from (12). Hence , we obtain

$$|\overline{x}_n(t)-\underline{x}_n(t)|\leq\left[\frac{1}{|\alpha|}\left(\int_0^{\lambda(T)}|K_1(\sigma)|\,d\sigma+1\right)\int_0^{\lambda(T)}|K_1(\sigma)|\,d\sigma\right]\|\overline{x}_{n-1}(t)-\underline{x}_{n-1}(t)\|.$$

By the hypothesis we write $q=\frac{1}{|\alpha|}\left(\int_0^{\lambda(T)}|K_1(\sigma)|\,d\sigma+1\right)\int_0^{\lambda(T)}|K_1(\sigma)|\,d\sigma<1.$

Therefore $\|\overline{x}_n(t)-\underline{x}_n(t)\|\leq q^n\|\overline{x}_0(t)-\underline{x}_0(t)\|$.

4. The consecutive substitution method

If we substitute right side of equation (6) instead of x in the operator $V_\lambda x$ to equation (6) then we obtain

$$x(t)=h(t)+V_\lambda h+\left[\frac{t}{T}+V_\lambda\frac{t}{T}\right]F_\lambda x+V_\lambda^2 x \quad \text{...(14)}$$

If we rewrite the right side of equation (6) instead of x in the operator $V_\lambda^2 x$ to equation (14) then we have

$$x(t)=h(t)+V_\lambda h+V^2{}_\lambda h+\left[\frac{t}{T}+V_\lambda\frac{t}{T}+V^2{}_\lambda\frac{t}{T}\right]F_\lambda x+V_\lambda^3 x \quad \text{.................................(15)}$$

When this operation is applied of n time we have

$$x=\sum_{i=0}^{n}V_\lambda^i h+\sum_{i=0}^{n}V_\lambda^i\frac{t}{T}F_\lambda x+V_\lambda^{n+1}x$$

If we choose $h_n(t)=\sum_{i=0}^{n}V_\lambda^i h,\quad a_n(t)=\sum_{i=0}^{n}V_\lambda^i\frac{t}{T}$ then it becomes

$$x=h_n(t)+a_n(t)F_\lambda x+V_\lambda^{n+1}x \quad \text{..(16)}$$

Now we can proof that the formula it is true

$$|V_\lambda^n x|\leq\frac{[K\lambda(T)]^n}{n!}\quad(n\in N) \quad \text{..(17)}$$

for the operator $V_\lambda x\equiv-\int_0^{\lambda(t)}K(t,s)x(s)ds\qquad(0\leq t\leq T)$ (18)

As a result of this we neglect the operator $|V_\lambda^{n+1}x|$ in equation (16) for n 's which are big enough. Thus the consecutive approximations are formed by taking the Volterra operator into consideration.

$$x_n(t)=h_n(t)+a_n(t)F_\lambda x_n \quad \text{..(19)}$$

Theorem 2: Let $\tau(t)\geq 0, a(t), f(t)\quad(0\leq t\leq T)$ be the functions given in problem (1) and $\alpha_n=1-\int_0^{\lambda(T)}K_1(s)\,h_n(s)\,ds\neq 0$ also $\lim_{n\to\infty}A_n\frac{[K\lambda(T)]^n}{n!}=0$ such that

$$A_n=1+\frac{\|a_n\|}{|\alpha_n|}\int_0^{\lambda(T)}|K_1(s)|\,ds$$

Then the limit of the approximations $x_n(t) = h_n(t) + \frac{a_n(t)}{\alpha_n}\int_0^{\lambda(T)} K_1(s)\, h_n(s)\, ds$ converges to the solution of problem (1) and the speed of the convergence is determined by

$$| x_n(t) - x(t) | \le A_n \frac{[K\lambda(T)]^n}{n!}$$

Proof: Eq. (19) is the Fredholm integral equation with a degenerated kernel. The solution of

$$x_n(t) = h_n(t) + \int_0^{\lambda(T)} a_n(t)\, K_1(s)\, x_n(s)\, ds \quad \text{......(20)}$$

is the same as the solution of eq. (16) and problem (1). Now, let us find the solution of equation (20). So, we use auxiliary equation, $y(t) = h_n(t) + a_n(t) F_\lambda y$ where $F_\lambda y$ is shown as $F_\lambda y = c_n$. Thus $y(t)$ is like that

$$y(t) = h_n(t) + a_n(t)\, c_n \quad \text{......(21)}$$

Therefore, $c_n = \int_0^{\lambda(T)} K_1(s)\, h_n(s)\, ds + c_n \int_0^{\lambda(T)} K_1(s)\, a_n(s)\, ds$

When $\alpha_n = 1 - \int_0^{\lambda(T)} K_1(s)\, h_n(s)\, ds \neq 0$ is given c_n is found as follows

$$c_n = \frac{1}{\alpha_n}\int_0^{\lambda(T)} K_1(s)\, h_n(s)\, ds \quad \text{......(22)}$$

If we use equation (22) in (21), for $n = 1,2,...$ then

$$x_n(t) = h_n(t) + \frac{a_n(t)}{\alpha_n}\int_0^{\lambda(T)} K_1(s)\, h_n(s)\, ds \quad \text{......(23)}$$

This operation is the approximate solution of problem(1) that is, the limit of $x_n(t)$ converges to the solution of problem (1).

Now let us determine the error of the approximate solution of eq. (23). Using (16) and (19), we reach $x - x_n = a_n(t)\, F_\lambda (x - x_n) + V_\lambda^{n+1} x$

If it is accepted that ε is $\varepsilon = x - x_n$, then we obtain the Fredholm integral equation with a degenerated kernel

$$\varepsilon = a_n(t)\, F_\lambda \varepsilon + V_\lambda^{n+1} x \quad \text{......(24)}$$

It was proven that solution of equation (24) is being found by using the following formula: $\varepsilon = V_\lambda^{n+1} x + \frac{a_n(t)}{\alpha_n}\int_0^{\lambda(T)} K_1(s)\, V_\lambda^{n+1} x(s) ds$

Thus, we write $| \varepsilon | \le [1 + \frac{\| a_n \|}{| \alpha_n |}\int_0^{\lambda(T)} | K_1(s) | ds] \| V_\lambda^{n+1} x(s) \|$

Then, by the hypothesis, $A_n = 1 + \frac{\| a_n \|}{| \alpha_n |}\int_0^{\lambda(T)} | K_1(s) | \, ds$ and we have

$$| x_n(t) - x(t) | \le A_n \frac{[K\lambda(T)]^n}{n!} \| x \| \quad \text{......(25)}$$

Example 1. Let us consider the boundary value problem:

$$x''(t) + tx(t - \frac{1}{2}\sqrt{t}) = t^2 - \frac{1}{2}t^{\frac{3}{2}} \quad 0 \le t \le 1), \text{ ....}(26)$$

$$x(t) = 0 \quad -1/16 \le t \le 0), \quad x(1) = 1$$

This equation can be written as the Fredholm-Volterra integral equation

$$x(t) = \frac{409}{420}t + \frac{1}{12}t^4 - \frac{2}{35}t^{\frac{7}{2}} + \frac{t}{16}\int_0^{1/2}\left[3 + 4\sigma - 16\sigma^2 + \frac{3 + 28\sigma - 80\sigma^2}{\sqrt{1+16\sigma}}\right]x(\sigma)\,d\sigma \quad \text{....(27)}$$

$$-\frac{1}{16}\int_0^{t-\sqrt{t}/2}\left[(4t-1) + (16t-12)\sigma - 16\sigma^2 + \frac{(4t-1) + (48t-20)\sigma - 80\sigma^2}{\sqrt{1+16\sigma}}\right]x(\sigma)d\sigma.$$

Let $h(t) = \frac{409}{420}t + \frac{1}{12}t^4 - \frac{2}{35}t^{\frac{7}{2}}$, $K_1(\sigma) = \frac{1}{16}\left[3 + 4\sigma - 16\sigma^2 + \frac{3 + 28\sigma - 80\sigma^2}{\sqrt{1+16\sigma}}\right]$,

$$K(t,\sigma) = \frac{1}{16}\left[(4t-1) + (16t-12)\sigma - 16\sigma^2 + \frac{(4t-1) + (48t-20)\sigma - 80\sigma^2}{\sqrt{1+16\sigma}}\right]$$

and $F_\lambda x \equiv \int_0^{1/2} K_1(\sigma)x(\sigma)\,d\sigma,$ $\quad V_\lambda\, x \equiv -\int_0^{t-\sqrt{t}/2} K(t,\sigma)x(\sigma)d\sigma.$

Therefore, the integral equation (20) can be written as

$$x(t) = h(t) + tF_\lambda x + V_\lambda x \text{ ....}(28)$$

and this equation is equivalent to problem (26). Some values of the solution of this equation are obtained by using the method of modified two sided approximations and the method of consecutive approximations of order two which are given in Table 1, where the first approximation is $x_0(t) = (409/420)t$.

Table 1. Values at some point in the interval [0,1]

t_i	$x(t_i)$	$x_2^{(1)}(t_i)$	$x_2^{(2)}(t_i)$	$\varepsilon_1(t_i)$	$\varepsilon_2(t_i)$
0.00	0.00	0.0000000	0.0000000	0.0000000	0.0000000
0.25	0.25	0.2501598	0.2501078	0.0001598	0.0001078
0.50	0.50	0.5001671	0.5000684	0.0001671	0.0000684
1.00	1.00	0.9999999	0.9999046	-0.0000001	-0.0000954

$x_2^{(1)}$: The modified two sided approach method; $x_2^{(2)}$: The consecutive substitution method. ε_1 and ε_2 are speed of the modified two sided approach method and the consecutive substitution method, respectively.

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