EXISTENCE OF MANY PARAMETRIC SOLUTIONS OF THE SPECIAL SYSTEMS OF PARTIAL INTEGRAL EQUATIONS WITH TWO VARIABLES

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Introduction. Partial Integral Equations arise in the Quantum Theory of Fields [1], in some problems of building Mechanics [2], in some questions of technique, namely, problem of stability of rotor [3] and in the investigations of spectral properties, particularly, bounded states of many particle operators which appear in Quantum Mechanics, Solid Physics [4].

Let us turn our attention to the last studies of E. Culfa on Partial Integral Equations:

Existence and uniqueness of solution of Partial Integral Equations with one-dimensional degenerate kernels are considered and for the solution an exact formula is obtained [5]. Partial Integral Equations with Fredholm's Type kernels are, under some conditions based on the kernels, equivalent to the Fredholm's Second Type Integral Equation [6]. Partial Integral Equations with N-dimensional degenerate kernels are eqivalent to the Fredholm's Second Type Integral Equation with degenerate kernels [7]. A special system of Partial Integral Equations with two variables in common case is investigated, and then by applying new methods, the exact solution formula is we obtained [8].

Let us consider the following special system of Partial Integral Equations with two Variables

$$u(x,y)=f(x,y)+\sum_{i=1}^{m}\int_a^b K_i(x,y)Z_i(t,y)u(t,y)dt+\sum_{j=1}^{n}\int_c^d M_j(x,y)N_j(x,s)u(x,s)ds+$$

$$+\sum_{k=1}^{l}\int_a^b\int_c^d P_k(x,y)Q_k(t,s)u(t,s)dsdt,$$

(1)

where *u(x,y)* is an unknown r dimensional vector, f(x,y) is r dimensional known vector function,

$K_i(x,y), Z_i(x,y), M_j(x,y), N_j(x,y), P_k(x,y)$ and $Q_k(x,y)$ are known continuous r x r dimensional matrix functions on [a,b]x[c,d].

Substitute the followings in the equation (1)

$$y=\frac{\bar{y}-a}{b-a}(d-c)+c,$$

$$s=\frac{\bar{s}-a}{b-a}(d-c)+c,$$

Then the unknown vector function *u(x,y)* becomes

$$\bar{u}(x,\bar{y})=u\left(x,\frac{\bar{y}-a}{b-a}(d-c)+c\right),\ (x,\bar{y})\in[a,b]x[a,b]$$

and we get

$$\bar{u}(x,\bar{y})=f\left(x,\frac{\bar{y}-a}{b-a}(d-c)+c\right)+\sum_{i=1}^{m}\int_a^b K_i\left(x,\frac{\bar{y}-a}{b-a}(d-c)+c\right)Z_i\left(t,\frac{\bar{y}-a}{b-a}(d-c)+c\right)\bar{u}(t,\bar{y})dt+$$

$$+\sum_{j=1}^{n}\int_a^b M_j\left(x,\frac{\bar{y}-a}{b-a}(d-c)+c\right)N_j\left(x,\frac{\bar{s}-a}{b-a}(d-c)+c\right)\frac{d-c}{b-a}\bar{u}(x,\bar{s})d\bar{s}+$$

$$+\sum_{k=1}^{l}\int_a^b\int_a^b\frac{d-c}{b-a}P_k\left(x,\frac{\bar{y}-a}{b-a}(d-c)+c\right)Q_k\left(t,\frac{\bar{s}-a}{b-a}(d-c)+c\right)\bar{u}(t,s)d\bar{s}dt\ .$$

Because of no limitations in general, we can take *c* and *d* as *a* and *b* respectively. It is clear that our problem reduces to finding the solution of system (1). For this aim, we will use the following denotations.

$$\begin{cases} \alpha_i(y) = \int_a^b Z_i(t,y)u(t,y)dt, i = 1,\dots, m, \\ \beta_j(x) = \int_a^b N_j(x,s)u(x,s)ds, j = 1,\dots, n, \\ a_k = \int_a^b \int_a^b Q_k(t,s)u(t,s)dsdt, k = 1,\dots, l, \end{cases}$$

(2)

where $\alpha_i(y)$, $\beta_j(x)$ are r dimensional unknown vector functions and a_k is r dimensional unknown vector.

By putting (2) in the equation (1), we obtain

$$u(x,y) = f(x,y) + \sum_{i=1}^{m} K_i(x,y)\alpha_i(y) + \sum_{j=1}^{n} M_j(x,y)\beta_j(x) + \sum_{k=1}^{l} P_k(x,y)a_k$$

(3)

By substituting (3) in (2), we received

$$\alpha_i(y) = \int_a^b Z_i(t,y)[f(t,y) + \sum_{j=1}^{m} K_j(t,y)\alpha_j(y) + \sum_{j=1}^{n} M_j(t,y)\beta_j(t) + \sum_{j=1}^{l} P_j(t,y)a_j]dt, i = 1,\dots, m,$$

(4)

$$\beta_j(x) = \int_a^b N_j(x,s)[f(x,s) + \sum_{i=1}^{m} K_i(x,s)\alpha_i(s) + \sum_{i=1}^{n} M_i(x,s)\beta_i(x) + \sum_{i=1}^{l} P_i(x,s)a_i]ds, j = 1,\dots, n,$$

(5)

$$a_k=\int_a^b\int_a^b Q_k(t,s)[f(t,s)+\sum_{i=1}^{m}K_i(t,s)\alpha_i(s)+\sum_{i=1}^{n}M_i(t,s)\beta_i(t)+\sum_{i=1}^{l}P_i(t,s)a_i]dsdt, k=1,\dots,l.$$

(6)

Let us denote:

$$f_i(y)=\int_a^b Z_i(t,y)f(t,y)dt,$$

$$A_{ij}(y)=\int_a^b Z_i(t,y)K_j(t,y)dt, i,j=1,\dots,m,$$

$$B_{ik}(y)=\int_a^b Z_i(t,y)P_k(t,y)dt, k=1,\dots,l,$$

$$\bar{f}_p(x)=\int_a^b N_p(x,s)f(x,s)ds, p=1,\dots,n$$

$$\overline{A}_{pq}(x)=\int_a^b N_p(x,s)M_q(x,s)ds, q=1,\dots,n,$$

$$\overline{B}_{pk}(x)=\int_a^b N_p(x,s)P_k(x,s)ds,$$

$$d_k=\int_a^b\int_a^c Q_k(t,s)f(t,s)dsdt,$$

$$D_{ki}(y)=\int_a^b Q_k(t,y)K_i(t,y)dt,$$

$$\overline{D}_{kp}(x)=\int_a^b Q_k(x,s)M_p(x,s)ds,$$

$$C_{kr}=\int_a^b\int_a^b Q_k(t,s)P_r(t,s)dsdt, r=1,\dots,l, \qquad (7)$$

where $f_i(y)$, $\overline{f}_p(x)$ are r dimensional vector functions, d_k is r dimensional vector, $A_{ij}(y)$, $B_{ik}(y)$, $\overline{A}_{pq}(x)$, $\overline{B}_{pk}(x)$, $D_{ki}(y)$, $\overline{D}_{kp}(x)$, are rxr dimensional matrix functions and C_{kr} is r dimensional matrix.

By the help of denotations given in (7) we can rewrite the expressions (4), (5) and (6)

$$\alpha_i(y)=f_i(y)+\sum_{j=1}^{m}A_{ij}(y)\alpha_j(y)+\sum_{k=1}^{l}B_{ik}(y)a_k+\sum_{p=1}^{n}\int_a^b Z_i(t,y)M_p(t,y)\beta_p(t)dt, i=1,...,m,$$

(8)

$$\beta_p(x)=\bar{f}_p(x)+\sum_{q=1}^{n}\overline{A}_{pq}(x)\beta_q(x)+\sum_{k=1}^{l}\overline{B}_{pq}(x)a_k+\sum_{j=1}^{m}\int_a^b N_p(x,s)K_j(x,s)\alpha_j(s)ds, p=1,...,n,$$

(9)

$$a_k=d_k+\sum_{r=1}^{l}C_{kr}a_r+\sum_{j=1}^{m}\int_a^b D_{kj}\alpha_j(s)ds+\sum_{p=1}^{n}\int_a^b \overline{D}_{kp}(t)dt, k=1,...,l.$$

(10)

Under the following conditions:

$$\det\left[I_m - A(y)\right]\neq 0,\quad \forall y\in[a,b],$$
$$\det\left[I_n - \overline{A}(x)\right]\neq 0,\quad \forall x\in[a,b],$$
$$\det\left[I_l - C\right]\neq 0, \qquad (11)$$

where I_m is a mr x mr-identity matrix.

$$A(y)=\begin{pmatrix} A_{11}(y) & A_{12}(y) & \dots & A_{1m}(y) \\ A_{21}(y) & A_{22}(y) & \dots & A_{2m}(y) \\ \dots & \dots & \dots & \dots \\ A_{m1}(y) & A_{m2}(y) & \dots & A_{mm}(y) \end{pmatrix},$$

$$\overline{A}(x)=\begin{pmatrix} \overline{A}_{11}(x) & \overline{A}_{12}(x) & \dots & \overline{A}_{1n}(x) \\ \overline{A}_{21}(x) & \overline{A}_{22}(x) & \dots & \overline{A}_{2n}(x) \\ \dots & \dots & \dots & \dots \\ \overline{A}_{n1}(x) & \overline{A}_{n2}(x) & \dots & \overline{A}_{nn}(x) \end{pmatrix}, \quad C=\begin{pmatrix} C_{11} & C_{12} & \dots & C_{1l} \\ C_{21} & C_{22} & \dots & C_{2l} \\ \dots & \dots & \dots & \dots \\ C_{l1} & C_{l2} & \dots & C_{ll} \end{pmatrix}.$$

From (11), we get

$$\left(I_{m\mathrm{r}}-A(y)\right)^{-1}=\begin{pmatrix} F_{11}(y) & F_{12}(y) & \dots & F_{1m}(y) \\ F_{21}(y) & F_{22}(y) & \dots & \mathrm{F}_{2m}(y) \\ \dots & \dots & \dots & \dots \\ F_{m1}(y) & F_{m2}(y) & \dots & F_{mm}(y) \end{pmatrix},$$

$$\left(I_{m\mathrm{r}}-\overline{A}(x)\right)^{-1}=\begin{pmatrix} \overline{F}_{11}(x) & \overline{F}_{12}(x) & \dots & \overline{F}_{1n}(x) \\ \overline{F}_{21}(x) & \overline{F}_{22}(x) & \dots & \overline{F}_{2n}(x) \\ \dots & \dots & \dots & \dots \\ \overline{F}_{n1}(x) & \overline{F}_{n2}(x) & \dots & \overline{F}_{nn}(x) \end{pmatrix},$$

$$\left(I_{l\mathrm{r}}-C\right)^{-1}=\begin{pmatrix} G_{11} & G_{12} & \dots & G_{1l} \\ G_{21} & G_{22} & \dots & G_{2l} \\ \dots & \dots & \dots & \dots \\ G_{l1} & G_{l2} & \dots & G_{ll} \end{pmatrix},$$

(12)

where $F_{ij}(y)$, $\overline{F}_{kp}(x), G_{k\mathrm{r}}$ are rx r matrixes. If we take into consideration of inverses of the matrixes

$I_{mr} - A(y), I_{nr} - \overline{A}(x), I_{lr} - C, x \in [a,b], y \in [a,b].$

which are given in (12), then we rearrange the systems, then we get the following systems of equations

$$\alpha_i(y) = \sum_{v=1}^{m} F_{iv}(y)[f_v(y) + \sum_{k=1}^{l} B_{vk}(y)a_k + \sum_{p=1}^{n} \int_a^b Z_v(t,y)M_p(t,y)\beta_p(t)dt], i = 1,...,m, \quad (13)$$

$$\beta_p(x) = \sum_{v=1}^{n} \overline{F}_{pv}(x)[\bar{f}_v + \sum_{k=1}^{l} \overline{B}_{vk}(x)a_k + \sum_{j=1}^{m} \int_a^b N_v(x,t)K_j(x,t)\alpha_j(t)dt], p = 1,...,n, \quad (14)$$

$$a_k = \sum_{s=1}^{l} G_{ks}[d_s + \sum_{j=1}^{m} \int_a^b D_{sj}(t)\alpha_j(t)dt + \sum_{q=1}^{n} \overline{D}_{sq}(t)\beta_q(t)dt], k = 1,...,l. \quad (15)$$

Substitute (15) in (13), (14) and interchange x and y. Then we get

$$\alpha_i(x) = \sum_{v=1}^{m} F_{iv}(x)\{f_v(x) + \sum_{k=1}^{l}\sum_{s=1}^{l} B_{vk}(x)G_{ks}[d_s + \sum_{j=1}^{m}\int_a^b D_{sj}(t)\alpha_j(t)dt + \sum_{p=1}^{n}\int_a^b \overline{D}_{sp}(t)\beta_p(t)dt] + \sum_{p=1}^{n}\int_a^b Z_v(t,x)M_p(t,x)\beta_p(t)dt]\}, i = 1,..,m. \quad (16)$$

$$\beta_p(x) = \sum_{v=1}^{n} \overline{F}_{pv}(x)\left\{\bar{f}_v(x) + \sum_{k=1}^{l}\sum_{s=1}^{l} \overline{B}_{vk}(x)G_{ks}\left[d_s + \sum_{j=1}^{m}\int_a^b D_{sj}(t)\alpha_j(t)dt + \sum_{q=1}^{n}\int_a^b \overline{D}_{sq}(t)\beta_q(t)dt\right] + \sum_{j=1}^{m}\int_a^b N_v(x,t)K_j(x,t)\alpha_j(t)dt\right\}, p = 1,...,n. \quad (17)$$

Let us use the following denotations:

$$V(x) = \begin{pmatrix} \alpha(x) \\ \beta(x) \end{pmatrix}, \alpha(x) = \begin{pmatrix} \alpha_1(x) \\ \alpha_2(x) \\ \dots \\ \alpha_m(x) \end{pmatrix}, \beta(x) = \begin{pmatrix} \beta_1(x) \\ \beta_2(x) \\ \dots \\ \beta_n(x) \end{pmatrix},$$

$$K(x,t)=\begin{pmatrix} K_{11}(x,t) & K_{12}(x,t) & \dots & K_{1m}(x,t) & \overline{K}_{11}(x,t) & \overline{K}_{12}(x,t) & \dots & \overline{K}_{1n}(x,t) \\ K_{21}(x,t) & K_{22}(x,t) & \dots & K_{2m(x,t)} & \overline{K}_{21}(x,t) & \overline{K}_{22}(x,t) & \dots & \overline{K}_{2n}(x,t) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ K_{m1}(x,t) & K_{m2}(x,t) & \dots & K_{mm}(x,t) & \overline{K}_{m1}(x,t) & \overline{K}_{m2}(x,t) & \dots & \overline{K}_{mn}(x,t) \\ R_{11}(x,t) & r_{12}(x,t) & \dots & R_{1m}(x,t) & \overline{R}_{11}(x,t) & \overline{R}_{12}(x,t) & \dots & \overline{R}_{1n}(x,t) \\ R_{21}(x,t) & r_{22}(x,t) & \dots & R_{2m}(x,t) & \overline{R}_{11}(x,t) & \overline{R}_{12}(x,t) & \dots & \overline{R}_{2n}(x,t) \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ R_{n1}(x,t) & r_{n2}(x,t) & \dots & R_{nm}(x,t) & \overline{R}_{n1}(x,t) & \overline{R}_{n2}(x,t) & \dots & \overline{R}_{nn}(x,t) \end{pmatrix}$$

$$H(x)=\begin{pmatrix} L(x) \\ \overline{L}(x) \end{pmatrix}, L(x)=\begin{pmatrix} L_1(x) \\ L_2(x) \\ \dots \\ L_m(x) \end{pmatrix}, \overline{L}(x)=\begin{pmatrix} \overline{L}_1(x) \\ \overline{L}_2(x) \\ \dots \\ \overline{L}_m(x) \end{pmatrix}, \quad (18)$$

where,

$$K_{ij}(x,t)=\sum_{\nu=1}^{m}\sum_{k=1}^{m}\sum_{s=1}^{l} B_{\nu k}(x)F_{i\nu}(x)G_{ks}D_{sj}(x),$$

$$\overline{K}_{ip}(x,t)=\sum_{\nu=1}^{m} F_{i\nu}(x)\left[Z_\nu(t,x)M_p(t,x)+\sum_{k=1}^{l}\sum_{s=1}^{l} B_{\nu k}(x)G_{ks}\overline{D}_{sp}(t)\right],$$

$$L_i(x)=\sum_{\nu=1}^{m} F_{i\nu}(x)\left[f_\nu(x)+\sum_{k=1}^{l}\sum_{s=1}^{l} B_{\nu k}(x)G_{ks}\overline{D}_{sp}(t)\right],$$

$$\overline{L}_p(x)=\sum_{\nu=1}^{n} \overline{F}_{i\nu}(x)\left[\overline{f}_\nu(x)+\sum_{k=1}^{l}\sum_{s=1}^{l} \overline{B}_{\nu k}(x)G_{ks}d_s\right], p=1,2,\dots,n$$

$$R_{pq}(x,t)=\sum_{\nu=1}^{n} \overline{F}_{p\nu}(x)\left[\sum_{k=1}^{l}\sum_{s=1}^{l} B_{\nu k}(x)G_{ks}\left(\overline{D}_{sp}(t)+N_\nu(x,t)K_j(x,t)\right)\right],$$

$$\overline{R}_{pq}(x,t)=\sum_{v=1}^{n}\overline{F}_{pv}(x)\sum_{s=1}^{l}\sum_{s=1}^{l}G_{ks}\overline{D}_{sq}(t),$$

(19)

$i,j=1,2,...,m,p,q=1,2,...,n$

Put the denotations (18) and (19) in the vector system (16) and (17), then we get

$$V(x)=\int_a^b K(x,s)V(s)ds+H(x),\ \text{where } x\in[a,b].$$

(20)

Suppose that 1 is eigenvalues of kernel matrix K(x,t) with multiple e and

$\varphi_1(x),\varphi_2(x),...,\ \varphi_e(x)$ are eigenvectors of kernel matrix K(x,t) with respect to the eigenvalue 1. Then, 1 is eigenvalues of kernel matrix K*(x,t) with multiple e, where K* is conjugate to the matrix K. Let

$\varphi_1{}^*(x),\varphi_2{}^*(x),...,\varphi_e{}^*(x)$ be eigenvectors of kernel matrix K*(x,t) with respect to the eigenvalue 1.

Now we introduce the scalar product <u, v> of u = (u_1, u_2,..., u_{m+n}),

v = (v_1, v_2,..., v_{m+n}) as follows <u, v> = $\sum_{i=1}^{m+n} u_i v_i$ and also we assume that the following conditions hold:

$$\int_a^b <\varphi_i{}^*(x),H(x)>dx=0,\ i=1,2,...,e$$

(21)

Then the system (20) has many parametric solutions as follows

$$V(x)=V_0(x)+\sum_{i=1}^{e} C_i\varphi_i(x),$$

(22)

Where $V_0(x)$ is any arbitrary solution of system (20), $C_1, C_2,..., C_e$ are any arbitrary constants.

To solve the system (1), we put firstly the formula (22) in (15) and secondly (15) and (22) into (3) and then we obtained the general solution formula.

By the way, we proved the following Theorem.

Theorem. Let the conditions of (11) and (21) be held. Then the system (1) will have *e* – parametric solutions in space $C_r([a,b]x[a,b])$ and also its general solution can be represented in the forms of (3), (15) and (22).

Remark. If condition (21) is not kept, then the system (1) has no solution.

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