DIFFERENCE SCHEMES FOR SINGULARLY PERTURBED BOUNDARY PROBLEMS WHICH APPEARED ON SOLVING ELLIPTIC EQUATIONS OF SPHERE SYMMETRY PROPERTY

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1. INTRODUCTION. Let us consider the following boundary value problem

$$\begin{cases} \left(\frac{\varepsilon}{x}\right)^2 \frac{d}{dx}\left(x^2 \frac{du}{dx}\right) - q(x)u = f(x), \quad x \in (0, \quad 1); & (1) \\ |u(0)| < +\infty, \quad \xi u(1) + \eta\varepsilon(1)u'(1) = \psi. & (2) \end{cases}$$

We assume the functions q, f in (1) to be sufficiently smooth , and additionally the conditions

$$q(x) \geq q_0 > 0 \text{ при } x \in [0,1]; \quad (3)$$

$$\varepsilon \in (0,1]; \ \xi \geq 0; \quad \eta \geq 0; \quad \xi + \eta > 0. \quad (4)$$

are satisfied.

We can obtain problem of type (1), (2) with $\varepsilon = 1/r$ if we shall solve equation of sphere symmetry property $\Delta u - qu = f$ for the sphere of radius equal to *r*. If *r* is big enough, then function *u*(*x*) can form a boundary layer near the point *x*=1. Numerical solving of the similar problems (as called singularly perturbed problems) need to use special difference schemes which guaranteed the uniform convergence of appropriate solution to exact one [1]. There are two fundamental manners to construct the uniformly converging numerical algorithms for singularly perturbed boundary problems. The first of them deals with construction of the “special” difference schemes on the uniform grids and start from A.M.Ilyin’s investigation [2]. The second way is based on using of adapted to properties of solution non-uniform grids, and is connected with N.S.Bakhvalov's name historically [3]. The method [4,5], which permits to associate both of the ways, was used in our paper. Firstly, the method of the discretisation keep particulars of the original differential problem automatically, therefore constructed schemes are those of the special type. Secondly, in the framework of proposed method can be realized the algorithm of grid's adapting. Furthermore, the method permits us to approximate a solution as well as its derivatives at the same time.

It is necessary to remind some properties of the problem (1), (2). In particular, for its solution $u(x)$ the following condition holds (see [6]):

$$u'(0) = 0. \tag{5}$$

Throughout the paper we shall assume that problem (1), (2) has a unique solution from the class $\mathbf{C}^1[0,1] \cap \mathbf{C}^2(0,1)$. Let the operator $\mathbf{L}$ of the problem (1), (2) define by representations

$$\begin{cases} \mathbf{L}v(0) \equiv -\varepsilon v'(0); \\ \mathbf{L}v(x) \equiv -(\varepsilon/x)^2 \left(x^2 v'\right)' + q(x)v, \quad x \in (0,1); \\ \mathbf{L}v(1) \equiv \xi v(1) + \eta\varepsilon v'(1). \end{cases}$$

for functions v from the above described class. Using corresponding methods from [7] we can prove that $\mathbf{L}$ is the operator of monotonic type, therefore follow "theorem of compare" takes place:

Lemma 1. Let us assume that the problem (1), (2) satisfies the conditions (3), (4). Then inequality $|\mathbf{L}u(x)| \le |\mathbf{L}v(x)|$ follows inequality $|u(x)| \le |v(x)|$ for functions $u, v \in \mathbf{C}^1[0,1] \cap \mathbf{C}^2(0,1)$ ($x \in [0,1]$).

Following statement guaranties a uniformly bounded (with respect to ε) solution of the problem (1), (2):

Lemma 2. Let us assume that the problem (1), (2) satisfies the conditions (3), (4). Then its solution can be estimated by

$$|u(x)| \le \max_{0 \le y \le 1} |f(y)| \Big/ q_0 + |\psi| \left(\xi + \eta q_0 \Big/ \left(3 + \sqrt{q_0}\right)\right)^{-1} \tag{6}$$

for any $x \in [0,1]$.

Proof. We denote

$$\theta \equiv \sqrt{q_0}\,, \quad A \equiv \max_{0 \le y \le 1} |f(y)| \Big/ q_0\,, \quad B \equiv |\psi| \left(\xi + \eta\theta^2/(3+\theta)\right)^{-1},$$

$$v_0(x) \equiv \frac{sh(x\theta/\varepsilon)}{x\,sh(\theta/\varepsilon)}$$

and consider "barrier" function

$$v(x) \equiv A + Bv_0(x).$$

It is necessary to verify following representations now

$$\mathbf{L}v(0) = -\varepsilon B v_0'(0) = 0 = \left|\mathbf{L}u(0)\right|;$$

$$\mathbf{L}v(x) = q_0 A + B\left[q(x) - q_0\right]v_0(x) \geq \max_{0\leq y\leq 1}\left|f(y)\right| \geq \left|\mathbf{L}u(x)\right|, \quad x \in (0,1);$$

$$\mathbf{L}v(1) = \xi(A+B) + \eta\theta B\left[cth\left(\theta/\varepsilon\right) - \theta/\varepsilon\right] \geq B\left(\xi + \eta\theta^2/(3+\theta)\right) = \left|\psi\right| = \left|\mathbf{L}u(1)\right|.$$

The last inequality uses estimation

$$cthz - 1/z \geq z/(3+z) \qquad (z > 0),$$

which may by verify easily. Using statement of the *lemma 1* and inequality $v_0(x) \leq 1 \quad (x \in [0,1])$ we complete proof of the lemma.

2. SET OF THE DIFFERENCE SCHEMES for the problem (1),(2). Let *s* and *t* be a constants such that $0 \leq s < t \leq 1$. Let constants $\bar{q}$ and $\bar{f}$ approximate function $q(x)$ and $f(x)$ in interval $[s,t]$; a choice of these constants will be determine later. Multiplying equation (1) to $-x^2 v(x)$, where $v(x)$ is sufficiently smooth testing function, after that integrating a result on $[s,t]$ we obtain:

$$\begin{cases} \left[-\varphi\varepsilon x^2 v + u\varepsilon^2 x^2 v'\right]_t^s + \int_t^s u\left[-\varepsilon^2\left(x^2 v'\right)' + \bar{q}x^2 v\right]dx = -\bar{f}\int_t^s x^2 v dx + \delta(s,t); \\ \delta(s,t) \equiv \int_t^s \left\{\bar{f} - f(x) + \left[\bar{q} - q(x)\right]u(x)\right\}x^2 v dx. \end{cases} \tag{7}$$

Here we denote $\varphi(x) \equiv \varepsilon u'(x)$. We choose testing functions $v^{(0)}(x)$ è $v^{(1)}(x)$ in identity (7) according to

$$-\varepsilon^2\left(x^2 v'\right)' + x^2\bar{q}v = 0, \; x \in (s,t), \tag{8}$$

$$\begin{cases} xv^{(0)}\big|_{x=s} = 1, \qquad xv^{(0)}\big|_{x=t} = 0; \\ xv^{(1)}\big|_{x=s} = 0, \qquad xv^{(1)}\big|_{x=t} = 1. \end{cases} \tag{9}$$

Here $\overline{q} = q^{(0)}$ and $\overline{q} = q^{(1)}$ correspond to functions $v^{(0)}(x)$ and $v^{(1)}(x)$ respectively.

Solution of the problems (8), (9) can be found easily:

$$v^{(0)}(x) = \frac{sh\left((t-x)\sqrt{q^{(0)}}\Big/\varepsilon\right)}{xsh\left((t-s)\sqrt{q^{(0)}}\Big/\varepsilon\right)}; \quad v^{(1)}(x) = \frac{sh\left((x-s)\sqrt{q^{(1)}}\Big/\varepsilon\right)}{xsh\left(\sqrt{q^{(1)}}\,(t-s)/\varepsilon\right)}. \quad (10)$$

Substituting in (7) $\overline{q} = q^{(0)}$, $\bar{f} = f^{(0)}$, $v = v^{(0)}$ we obtain:

$$\varepsilon s\varphi(s) - \varepsilon^2 t\big(u(t)-u(s)\big)/(t-s) + (t-s)q^{(0)}\left[\gamma\left(R^{(0)}\right)tu(t) + \mu\left(R^{(0)}\right)su(s)\right] =$$

$$= -(t-s)f^{(0)}\left[\gamma\left(R^{(0)}\right)t + \mu\left(R^{(0)}\right)s\right] + \delta^{(0)}(s,t). \quad (11)$$

Analogously for $\overline{q} = q^{(1)}$, $\bar{f} = f^{(1)}$, $v = v^{(1)}$ we can obtain

$$-\varepsilon s\varphi(t) + \varepsilon^2 s\big(u(t)-u(s)\big)/(t-s) + (t-s)q^{(1)}\left[\mu\left(R^{(1)}\right)tu(t) + \gamma\left(R^{(1)}\right)su(s)\right] =$$

$$= -(t-s)f^{(1)}\left[\mu\left(R^{(1)}\right)t + \gamma\left(R^{(1)}\right)s\right] + \delta^{(1)}(s,t) \quad (12)$$

In (11) and (12) we denote:

$$R^{(k)} \equiv (t-s)\sqrt{q^{(k)}}\Big/\varepsilon,\ k = 0,1,$$

$$\mu(z) \equiv (zcthz-1)/z^2,\ \gamma(z) \equiv \big(1 - z/shz\big)/z^2. \quad (13)$$

In order to transform the statements described above to the difference schemes for problem (1), (2) let consider some grid on the interval $[0,1]$

$$0 = x_1 < x_2 < \ldots < x_i < \ldots < x_N = 1 \quad (14)$$

and denote:

$$h_i \equiv x_{i+1} - x_i,\ i = 1,2,\ldots,N-1;\ h \equiv \max_{1\le i\le N}\left(h_i\right).$$

Let $v^h \equiv \left\{v_i^h\right\}_{i=1}^N$ denote some mesh function with corresponding norm:

$$\left\|v^h\right\|_{h,\infty} \equiv \max_{1\le i\le N}\left|v_i^h\right|;$$

moreover, we shall denote $(v)^h \equiv \left\{v(x_i) \equiv v_i\right\}_{i=1}^N$ a projection of some continuous function $v(x)$ on the grid (14). Assuming in (11) and (12) $s = x_i$,

$t = x_{i+1}$ and do not taking into account errors of approximation $\delta^{(0)}(x_i, x_{i+1})$ and $\delta^{(1)}(x_i, x_{i+1})$ we obtain discrete problem corresponding to (1), (2):

$$\begin{cases} \varphi_1^h = 0, \\ \varepsilon x_i \varphi_i^h - \varepsilon^2 x_{i+1} Du_i^h + h_i q_i^{(0)} \left[\gamma\left(R_i^{(0)}\right) x_{i+1} u_{i+1}^h + \mu\left(R_i^{(0)}\right) x_i u_i^h\right] = -f_i^{(0)} \sigma_i^{(0)}; \\ -\varepsilon x_{i+1} \varphi_{i+1}^h + \varepsilon^2 x_i Du_i^h + h_i q_i^{(1)} \left[\mu\left(R_i^{(1)}\right) x_{i+1} u_{i+1}^h + \gamma\left(R_i^{(1)}\right) x_i u_i^h\right] = -f_i^{(1)} \sigma_i^{(1)}; \\ \quad i = \overline{1, N-1}; \\ \xi u_N^h + \eta \varphi_N^h = \psi. \end{cases} \tag{15}$$

Here $u^h \equiv \left\{u_i^h\right\}_{i=1}^N$ and $\varphi^h \equiv \left\{\varphi_i^h\right\}_{i=1}^N$ approximate unknown mesh functions $(u)^h$ and $(\varphi)^h$ respectively. We supply constants $q^{(k)}$, $f^{(k)}$, $R^{(k)}$ ($k = 0,1$) by index 'i' and denote

$$Du_i^h \equiv \left(u_{i+1}^h - u_i^h\right)/h_i \; ; \; \sigma_i^{(0)} \equiv h_i\left[\gamma\left(R_i^{(0)}\right)x_{i+1} + \mu\left(R_i^{(0)}\right)x_i\right];$$

$$\sigma_i^{(1)} \equiv h_i\left[\mu\left(R_i^{(1)}\right)x_{i+1} + \gamma\left(R_i^{(1)}\right)x_i\right],\; i = 1,2,\ldots,N-1 .$$

Excluding from the equations (15) values φ_i^h ($i = 1,2,\ldots,N-1$) we can rewrite this problem in the traditional third-points form:

$$\begin{cases} \mathbf{L}^h u_1^h \equiv -\varepsilon^2 x_2 Du_1^h + h_1 q_1^{(0)} \left[\gamma\left(R_1^{(0)}\right) x_2 u_2^h + \mu\left(R_1^{(0)}\right) x_1 u_1^h\right] = -f_1^{(0)} \sigma_1^{(0)}; \\ \mathbf{L}^h u_i^h \equiv -\varepsilon^2 x_{i+1} Du_i^h + \varepsilon^2 x_i Du_{i-1}^h + h_i q_i^{(0)} \left[\gamma\left(R_i^{(0)}\right) x_{i+1} u_{i+1}^h + \mu\left(R_i^{(0)}\right) x_i u_i^h\right] + \\ + h_{i-1} q_{i-1}^{(1)} \left[\mu\left(R_{i-1}^{(1)}\right) x_i u_i^h + \gamma\left(R_{i-1}^{(1)}\right) x_{i-1} u_{i-1}^h\right] = -f_i^{(0)} \sigma_i^{(0)} - f_{i-1}^{(1)} \sigma_{i-1}^{(1)}, \quad i = 2,3,\ldots,N-1; \\ \mathbf{L}^h u_N^h \equiv \eta \varepsilon^2 x_{N-1} Du_{N-1}^h + \eta h_{N-1} q_{N-1}^{(1)} \left[\mu\left(R_{N-1}^{(0)}\right) x_N u_N^h + \gamma\left(R_{N-1}^{(0)}\right) x_{N-1} u_{N-1}^h\right] + \xi \varepsilon u_N^h = \\ = \varepsilon \psi - \eta f_{N-1}^{(1)} \sigma_{N-1}^{(1)}. \end{cases} \tag{16}$$

The first and the latter equations of this system are non-standard approximations of boundary conditions (5) and (2) accordingly. The following statement contains an estimate of convergence of a multitude of the schemes (16):

Theorem 1. Let's assume, that for $i = 1,2,\ldots,N-1$ the inequalities

$$\min\left\{q_i^{(0)}, q_i^{(1)}\right\} \geq \alpha > 0$$

are satisfying with a constant α, independent of ε, N, and the values ε, η satisfy conditions (4). Then the problem (16) has a unique solution $u^h = \left\{u_i^h\right\}_{i=1}^N$. If additionally for $i = 1,2,\ldots, N-1$ and $x \in \left[x_i, x_{i+1}\right]$ the conditions

$$\left|q_i^{(k)} - q(x)\right| + \left|f_i^{(k)} - f(x)\right| \le Ch,\ k = 0,1 \qquad (17)$$

are satisfy with a constant C, independent of ε and N, then a solution u^h of the problem (16) is estimated by

$$\left\|u^h - (u)^h\right\|_{h,\infty} \le Ch, \qquad (18)$$

where C does not depend on ε and N. Thus, the difference scheme (16) converges uniformly in ε with the first rate on any irregular grid.

Proof. Using the appropriate statement from [7] (or discrete principle of maximum from [8]) we can prove, that the operator $\mathbf{L}^h$ of the problem (16) is an operator of a monotone type for any parameters of a grid and number ε. So, for an operator $\mathbf{L}^h$ the discrete variant of a comparison theorem (see *Lemma* 1) is fair. The last statement guarantees a unambiguous resolvability of a problem (16) and used for the proof of an estimate (18).

Comparing equations (16) and equations (11), (12) of the main identity taken for $s = x_i$, $t = x_{i+1}$, we come to a conclusion that grid function $w^h \equiv (u)^h - u^h$ satisfies a system

$$\begin{cases} \mathbf{L}^h w_1 = \delta^{(0)}\left(x_1, x_2\right); \\ \mathbf{L}^h w_i = \delta^{(0)}\left(x_i, x_{i+1}\right) + \delta^{(1)}\left(x_i, x_{i-1}\right), \quad i = 2,3,\ldots,N-1; \\ \mathbf{L}^h w_N = \eta\delta^{(1)}\left(x_{N-1}, x_N\right). \end{cases} \qquad (19)$$

Inequality (6) of a *lemma* 2 and the estimates (17) allow to prove inequalities

$$\left|\delta^{(k)}\left(x_i, x_{i+1}\right)\right| \le Ch\sigma_i^{(k)} \quad (i = 1,2,\ldots,N-1;\quad k = 0,1), \qquad (20)$$

where C does not depend of h and ε. On the other hand, by virtue of definition (16) of operator $\mathbf{L}^h$ we obtain

$$\begin{cases} \mathbf{L}^h (1)_1^h \ge \alpha\sigma_1^{(0)}; \\ \mathbf{L}^h (1)_i^h \ge \alpha\left(\sigma_i^{(0)} + \sigma_{i-1}^{(1)}\right), \quad i = 2,3,\ldots,N-1; \\ \mathbf{L}^h (1)_N^h \ge \eta\alpha\sigma_{N-1}^{(1)}. \end{cases} \qquad (21)$$

Comparing the formulas (19), (20), (21), we have

$$\left|\mathbf{L}^h w_i^h\right| \le \mathbf{L}^h (\alpha^{-1} C h)_i^h,\; i = 1,2,\ldots,N .$$

The last evaluation, because of a comparison theorem, results in an inequality (18), thereby proving the theorem.

After solving a system (16) we can, in case of necessity, to calculate derivatives of the solution, using the formulas (15). For an example we shall consider the following variant of choice of parameters in a multitude (16):

$$\begin{cases} q_i^{(0)} = q(x_i), \quad q_i^{(1)} = q(x_{i+1}); \\ f_i^{(0)} = f(x_i), \quad f_i^{(1)} = f(x_{i+1}), \quad i = 1,2,\ldots,N-1. \end{cases} \tag{22}$$

In this case difference scheme from (16) satisfies the conditions of the *theorem* 1 and looks rather simply ($h_i = h, \quad i = 1,2,\ldots,N-1; \quad \eta = 0, \; \xi = 1$):

$$\begin{cases} h\sqrt{q(0)} \Big/ \left(\varepsilon \cdot sh\left(h\sqrt{q(0)}/\varepsilon\right) - h\sqrt{q(0)}\right)\left(u_1^h - u_2^h\right) + u_1^h = -f(0)/q(0); \\ -1\Big/4sh^2\left(h\sqrt{q(x_i)}/2\varepsilon\right)\left(x_{i+1}u_{i+1}^h - 2x_i u_i^h + x_{i-1}u_{i-1}^h\right) + x_i u_i^h = -x_i f(x_i)/q(x_i); \\ u_N^h = \psi; \qquad\qquad\qquad\qquad i = 2,3,\ldots,N-1. \end{cases}$$

Let's mark, that this scheme is not conservative. The conservatism of the scheme guarantees choice of parameters under the formulas

$$\begin{cases} q_i^{(0)} = q_i^{(1)} = \left(q(x_i) + q(x_{i+1})\right)/2 \equiv q_{i+1/2}, \\ f_i^{(0)} = f_i^{(1)} = \left(f(x_i) + f(x_{i+1})\right)/2 \equiv f_{i+1/2}, \quad i = 1,2,\ldots,N-1. \end{cases} \tag{23}$$

3. NUMERICAL EXAMPLES. Let's present results of numerical experiments permitting to compare new and well-known schemes.

The experiments deals with a calculation of orders of uniform convergence and classical convergence in according with the following algorithm (see also [1,9]). Let $v_\varepsilon(x)$ is solution of an initial differential problem dependent on a parameter $\varepsilon \in (0,1]$ and determined on an interval [0,1]; $v_\varepsilon^h \equiv \left\{v_{\varepsilon,i}^h\right\}_{i=1}^N$ is a mesh function approximating $v_\varepsilon(x_i)$ in the uniform grid $x_i = (i-1)h$ ($i = 1,2,..,N, \quad N = 1/h + 1$) and calculated for $h \in H \equiv \left\{h_0/2^j \,\middle|\, j = 0,1,\ldots,k\right\}$ and $\varepsilon \in E \equiv \left\{\varepsilon_0/2^j \,\middle|\, j = 0,1,\ldots,m\right\}$. Let's designate:

$$\delta(h,\varepsilon) \equiv \left\|(v_\varepsilon)^h - v_\varepsilon^h\right\|_{h,\infty}, \quad \Delta(h) \equiv \Delta(h,v) \equiv \max_{\varepsilon \in E} \delta(h,\varepsilon).$$

The experimental orders of uniform and classical convergence (" p " and " p_0 ") were determined by the formulas

$$p = \frac{1}{\ln 2}\ln\left[\frac{1}{k}\sum_{j=0}^{k-1}\left[\Delta\left(h_0/2^j\right)/\Delta\left(h_0/2^{j+1}\right)\right]\right], \qquad (24)$$

$$p_0 \equiv \frac{1}{\ln 2}\ln\left[\frac{1}{k}\sum_{j=0}^{k-1}\left[\delta\left(h_0/2^j,\varepsilon_0\right)/\delta\left(h_0/2^{j+1},\varepsilon_0\right)\right]\right] \qquad (25)$$

for $h_0 = 1/8, \quad \varepsilon_0 = 1/2, \quad k = 7, \quad m = 8$.

In case of a constant coefficients obtained in section 2 schemes (15) reduce to the exact solution of a problem (1)-(2). Therefore the problem (1)-(2) tests with coefficients:

$$q(x) = q_0 + b_0 x^2,$$
$$f(x) = f_0 + 2a_0\varepsilon^2(10x^2 - 12x + 3) - a_0 q_0 x^2(1-x)^2 - b_0 x^2 u_0$$

for $q_0 = 1, \quad f_0 = 1, \quad a_0 = 10, \quad b_0 = 0.1$ and $\xi = 5, \quad \eta = 0.5, \quad \psi = 1$.

Here $u_0(x)$ is the solution of a problem:

$$u_0(x) = -1 + \left(6/\left(5 + 0.5\varepsilon\left[ctg\left(1/\varepsilon\right)/\varepsilon - 1\right]\right)\right)\cdot\left(sh\left(x/\varepsilon\right)/x sh\left(1/\varepsilon\right)\right) + 10x^2(1-x)^2$$

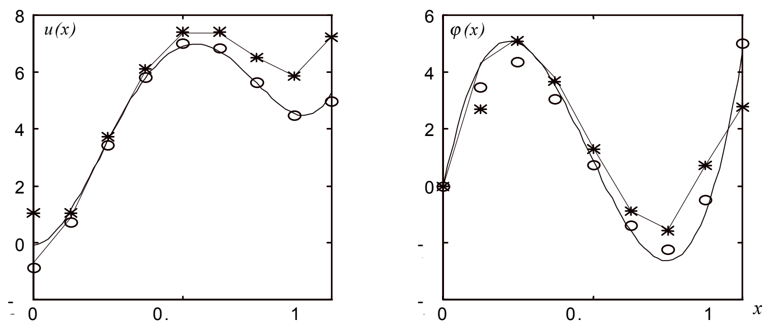
The quantities (24) and (25) were calculated for $v_\varepsilon(x) \equiv u(x)$ and $v_\varepsilon(x) \equiv \varphi(x) \equiv \varepsilon u'(x)$. Table 1 allows us to analyse of experimentally determined orders of convergence's of the difference schemes for the functions $u(x)$ and $\varphi(x)$. Samarskiy's well-known scheme [8] and scheme (16) with parameters (22) and (23) were tested here. Approximate solutions, which were found by means of corresponding difference schemes, were used for calculation of derivatives. In case of the scheme of Samarskiy for calculation of a derivative in internal points of a grid was used the central-difference approximation, the boundary values of a derivative were calculated with use of a directed difference (right point) and under the formula:

$$u'(0) = h\left(q(0)u(0) + f(0)\right)/6.$$

Table 1. The experimental order of convergence

	$u(x)$		$\varphi(x)$	
Scheme	uniform convergence	classical convergence	uniform convergence	classical conver
Samarskiy [8]	0.30	1.10	0.23	0.98
(15), (22)	1.22	2.00	0.84	1.97
(15), (23)	1.06	1.99	0.99	1.98

The analysis of table allows to make a conclusion that the above described experiment confirms the statement of the theorem 1 about uniform convergence's (with the first order) solution of a difference problem (16) to a solution of an initial problem (1), (2). Moreover, by results of this experiment the hypothesis about uniform convergence's (with the first order) streams can be formulated.



Solution of a problem: u(x) and $\varphi = \varepsilon u'(x)$. Number of knots - 9, ε = 1/4, $f_0 = q_0 = b_0 = 1$, $a_0 = 100$, $\xi = \eta = 1$ $\psi = 10$. The continuous line corresponds to a exact solution, '*' - solution on the scheme A, dotted line – on the scheme B and 'o' – on the scheme C.

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