REGULARIZATION AND UNIQUENESS OF SOLUTIONS OF NONLINEAR VOLTERRA OPERATOR EQUATIONS OF THE THIRD KIND

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Let H be a Hilbert space, $<\cdot,\cdot>$ be a scalar product in H, $\|\cdot\|_H$ be a norm in H. We denote by $C([t_o, T]; H)$ the Banach space of continuous functions defined in $[t_o, T]$, $t_o < T$ attaining the values in H, with the norm

$$\| u(t) \|_c = \sup_{t\in[t_o,T]} \| u(t) \|_H .$$

We denote by $C_\varphi^\gamma([t_o, T]; H)$, $0 < \gamma \le 1$ the linear space of all functions $u(t)$ defined in $[t_o, T]$, attaining the values in H such that

$$\| u(t) - u(s) \|_H \le c\ |\varphi(t) - \varphi(s)|^\gamma.$$

Here $c \ge 0$ is a constant depending on $u(t)$, but independent of t and s; $\varphi(t)$ is a real measurable nondecreasing function in $[t_o, T]$. We denote by $L(H)$ the space of linear bounded operators acting from H into H and by $\|\cdot\|$ the norm in $L(H)$.

We consider the operator equation

$$A(t)\, u(t) + \int_{t_o}^{t} K(t, s, u(s))ds = f(t),\ t \in [t_o, T], \qquad (1)$$

where for the fixed

$$t \in [t_o, T],\ (t, s) \in \overline{G} = \{(t, s) \mid t_o \le s \le t \le T$$

the operator $A(t) \in L(H)$ and the operator $K(t, s, .)$ acts from H into H (the integral is taken in the Bochner sense).

Integral and operator equations of the first kind and the third kind arise both from theory and practice. In particular, different inverse problems for differential equations and the problems of integral geometry are reduced to the Volterra equations of the first kind and the third kind. The different problems Volterra equations of the first kind and the third kind were studied in works [1-12].

We shall be further that

$$K(t, s, u) = Q(t, s)u + K_1(t, s, u), \qquad (2)$$

where $Q(t,s) \in L(H)$ for $(t, s) \in \overline{G}$, $k_1(t, s, 0) = 0$ is a trivial element in H for $(t, s) \in \overline{G}$.

We shall be interested in the following conditions.

1. For each $t \in (t_o, T]$, we have

$$\| Q(t, s)\| \in L^{q_1}(t_o, t),\ q_1 \ge 1,\ \| Q(t, t)\| \in L^1(t_o, T),$$

$$\| A(t)\| \in C[t_o, T].$$

For each $u \in H$, we have

$$\langle \frac{1}{2}[A(t) + A^{*}(t)]u, \mathrm{u}\rangle \geq a(t) \| u \|_{H}^{2}, \quad \| A(t) \| \leq \alpha\, a(t),$$

$$\langle \frac{1}{2}[Q(t,t) + Q^{*}(t,t)]u, \mathrm{u}\rangle \geq \lambda(\mathrm{t}) \| u \|_{H}^{2},$$

$$\langle \frac{1}{2}[A^{*}(t)Q(t,t) + Q^{*}(t,t)A(t)]u, \mathrm{u}\rangle \geq \alpha_{o}\, a(\mathrm{t})\, \lambda(t) \| u \|_{H}^{2},$$

where $A^{*}(t)$ is conjugate to the operator $A(t)$ for $t \in (t_o, T]$, $a(t) \geq 0$ and $\lambda(t) \geq 0$ for

$t \in [t_o, T]$, $a(t)$ is the nondecreasing continuous function in $[t_o, T]$ and $a(t_o) = 0$, α and α_o are the positive contants and $\lambda(t) \in L^{q_1}(t_o, T)$, $q_1 \geq 1$.

2. For $\tau > \eta$, for each (τ, s), $(\eta, s) \in G = \{(t, s) | t_o < s < t < T\}$, (s, u_1), $(s, u_2) \in [t_o, T] \times H$, and

(τ, s, u_1), (τ, s, u_2), (η, s, u_1), $(\eta, s, u_2) \in G \times H$

the following estimates hold

$$\| Q(\tau, s) - Q(\eta, s) \| \leq l(s)[\int_{\eta}^{\tau} \lambda(s) ds + a(\tau)],$$

$$\| K_1(s, s, u_1) - K_1(s, s, u_2) \|_H \leq l_1(s) \| u_1 - u_2 \|_H,$$

$$\| K_2(\tau, s, u_1) - K_1(\eta, s, u_1) - K_2(\tau, s, u_2) + K_1(\eta, s, u_1) \|$$

$$\leq l_2(s)\left[\int \lambda(\,.\,)\ \ +\ .\ (\tau)\right] \| u_1 - u_2 \| ,$$

where $l(t)$, $l_1(t)$ and $l_2(t)$ are nonnegative functions from $L^{q_1}(t_o, T)$, $q_1 \geq 1$, $a(t)$ and $\lambda(t)$ are defined as in the condition 1.

We shall need, further, the following lemmas.

LEMMA 1. Let condition 1 hold. Then the following statements hold.

1) For each $\varepsilon > 0$, for each $t \in [t_o, T]$ the operator $\varepsilon I + A(t)$ has the inverse operator

$(\varepsilon I + A(t)^{-1} \in L(H)$ such that

$$\| (\varepsilon I + A(t))^{-1} \| \leq 1 / (\varepsilon + a(t)),$$

where I is a unit operator in H.

2) For each $\varepsilon > 0$, for each $u \in H$ the following estimate hold

$$\langle -[Q(t,t)\,(\varepsilon I + A(t))^{-1} + ((\varepsilon I + A(t))^{-1})^{*} Q^{*}(t, t)]u, u\rangle \leq$$

$$\leq - \frac{2\theta\lambda(t)}{\theta_1^2(\varepsilon + a(t))} \| u \|_H^2 , \quad t \in [t_o, T] ,$$

where $\theta = min\,\{1 ; \alpha_o\}$, $\theta_1 = max\,\{1; \alpha\}$.

PROOF. 1) For each $u \in H$, we have

$$\| (\varepsilon I + A(t))\, u \|_H^2 = \varepsilon^2 \| u \|_H^2 + 2\varepsilon \left\langle \frac{1}{2}[A(t) + A^*(t)]\, u, u \right\rangle +$$

$$+ \| A(t)\, u \|_H^2 , \quad t \in [t_o, T] .$$

From the condition 1, we obtain

$$\| A(t)\, u \|_H \geq a(t) \| u \|_H , \; t \in [t_o, T] .$$

Taking into account the condition 1 and the last inequality from the last equality, we have

$$\| u \|_H \leq (\varepsilon + a(t))^{-1} \| (\varepsilon I + A(t)\, u \|_H, \; t \in [t_o, T] \qquad (3)$$

Introducing the designation

$$v = (\varepsilon I + A(t))\, u ,$$

from the inequality (3), we obtain

$$\| (\varepsilon I + A(t))^{-} 1 v \|_H \leq (\varepsilon + a(t))^{-1} \| v \|_H .$$

Hence, we have the required estimate.

2) For each $u \in H$ and for $v = (\varepsilon I + A(t))^{-1} u,$ we have

$$2\langle\frac{1}{2}[Q(t,t)\,(\varepsilon I + A(t))^{-1} + (\varepsilon I + A(t))^{-1})\,Q^*(t,t)]\,u,\mathrm{u}\rangle =$$

$$= 2\varepsilon\,\langle\frac{1}{2}[Q(t,t) + Q^*(t,t)]\,\mathrm{v},\mathrm{v}\rangle +$$

$$+2\langle\frac{1}{2}[A^*(t)\;Q(t,t) + Q^*(t,t)A(t)]\,\mathrm{v},\mathrm{v}\rangle. \qquad (4)$$

Taking into account the condition 1, from the inequality

$$\|\,u\|_H \le \|(\varepsilon I + A(t))\|\,\|(\varepsilon I + A(t))^{-1}\,u\,\|_{\mathrm{H}},$$

we obtain

$$\|\,(\varepsilon I + A(t))^{-1}\,u\,\|_{\mathrm{H}} \ge (\varepsilon + \alpha\,a(t))^{-1}\|\,u\,\|_{\mathrm{H}},\quad t\in\,[t_o,\,T]\,.$$

Taking into account the condition 1 and the last inequality, from (4) , we have

$$\langle[Q(t,t)\,(\varepsilon I + A(t))^{-1} + (\,(\varepsilon I + A(t))^{-1})^*\,Q^*(t,t)]\,u,\mathrm{u}\rangle \ge$$

$$\frac{2\lambda(t)(\varepsilon+\alpha_o a(t))}{(\varepsilon+\alpha a(t))^2}\,\|\,u\,\|_H^2 \ge \frac{2\theta\lambda(t)}{\theta_1^2(\varepsilon+a(t))}\,\|\,u\,\|_H^2\;.$$

Hence, we obtain the required estimate. The lemma 1 is proved.

LEMMA 2. Let for each $t\in [t_o, T]$ $B(t), C(t) \in L(H)$ and $\| B(t)\|\in C[t_o, T]$ and $\| C(t)\|\in L[t_o, T]$. Then for each $f(t)\in C([t_o, T]; H)$ the following operator equation

$$u(t) = \int_{t_o}^{t} B(t)C(s)u(s)ds + f(t)$$

has a unique solution $u(t)\in C([t_o, T]; H)$. This solution is defined by the formula

$$u(t) = f(t) + \int_{t_o}^{t} R(t,s) f(s)ds, \quad t\in [t_o, T] ,$$

where $R(t,s)$ is the resolvent of the operator kernel $B(t)\, C(s)$ and $R(t,s)$ is defined by the formula

$$R(t, s) = B(t)\, X(t,s)C(s). \tag{5}$$

Here $R(t,s)$ is the solution of the following operator equation

$$R(t,s) = \int_{s}^{t} B(t)C(\tau)R(\tau, s)d\tau + B(t)C(s), \ (t,s) \in G, \tag{6}$$

$X(t,s)$ is the evolutionary operator of the operator equation

$$\frac{d}{dt}v(t) = C(t)B(t)v(t), \ t\in [t_o, T],$$

that is

$$\frac{d}{dt}X(t,s) = C(t)\,B(t)\,X(t,s), \ (t,s) \in G,$$

$$X(s, s) = I, \ s \in [t_o, T].$$

PROOF. Substituting (5) into (6), we have

$$\int_s^t B(t)C(\tau)R(\tau,s)d\tau = \int_s^t B(t)\,C(\tau)B(\tau)X(\tau, s)C(s)\,d\tau =$$

$$= B(t)[\int_s^t C(\tau)B(\tau)X(\tau, s)\,d\tau]\,C(s) =$$

$$= B(t)[\,X(t,s) - I]C(s) = R(t, s) - B(t)C(s), \ (t,s) \in G.$$

The lemma 2 is proved.

LEMMA 3. Let the condition 1 holds and

$$\| Q(t, t) \| \le N_o\,\lambda(t), \ t\in [t_o, T]\ .$$

Then for each $\varepsilon > 0$ the operator kernel $[-(\varepsilon I + A(t))^{-1}Q(s, s)]$ has the resolvents

$R(t, s, \varepsilon) = -(\varepsilon I + A(t))^{-1} X(t, s, \varepsilon)\, Q(s, s),$ (7)

where $X(t, s, \varepsilon)$ is the evolutionary operator of the operator equation

$$\frac{dv(t,\varepsilon)}{dt} = -Q(t, t)\,(\varepsilon I + A(t))^{-1}\, v(t, \varepsilon),$$

that is

$$\frac{d}{dt} X(t, s, \varepsilon) = -Q(t, t)\,(\varepsilon I + A(t))^{-1}\, X(t, s, \varepsilon),$$

$$X(s, s) = I, \quad s \in [t_o, T],$$

$$\frac{d}{ds} X(t, s, \varepsilon) = X(t, s, \varepsilon)\, Q(s, s)\,(\varepsilon I + A(s))^{-1}. \qquad (8)$$

For $R(t, s, \varepsilon)$ and $X(t, s, \varepsilon)$ the following estimates holds

$$\| R(t, s, \varepsilon) \| \le \frac{N_o \lambda(s)}{\varepsilon + a(t)} \exp\left\{ -\frac{\theta}{\theta_1^2} \int_s^t \frac{\lambda(\tau)}{\varepsilon + a(\tau)} d\tau \right\}, \ (t, s) \in \overline{G}. \quad (9)$$

$$\| X(t, s, \varepsilon) \| \le \exp\left[-\frac{\theta}{\theta_1^2} \int_s^t \frac{\lambda(\tau)}{\varepsilon + a(\tau)} d\tau \right], \quad (t, \tau) \in \overline{G}, \ \ \varepsilon > 0. \quad (10)$$

PROOF. Taking into account the lemma 2, we obtain the formula (7).

Let $u \in H$ be arbitrary. Then, taking into account the lemma 1, we have

$$\frac{d}{dt} \| X(t, s, \varepsilon) u \|_H^2 = 2Re \langle \frac{d}{dt} X(t, s, \varepsilon) u, X(t, s, \varepsilon) u \rangle =$$

$$= 2Re \langle - Q(t, t) (\varepsilon I + A(t))^{-1} X(t, s, \varepsilon) u, X(t, s, \varepsilon) u \rangle$$

$$= \langle - [Q(t, t) (\varepsilon I + A(t))^{-1} + ((\varepsilon I + A(t))^{-1})^* Q^*(t,t)] \times$$

$$X(t, s, \varepsilon) u, X(t, s, \varepsilon) u \rangle \leq \frac{-2\theta\lambda(t)}{\theta_1^2(\varepsilon + a(t))} \| X(t, s, \varepsilon) u \|_H^2 .$$

From here, integrating from s to t and taking into account that $X(s, s, \varepsilon) = I$ is a unit operator in H, we obtain

$$\frac{\| X(t,s,\varepsilon)u \|_H}{\| u \|_H} \leq exp \left[-\frac{\theta}{\theta_1^2} \int_s^t \frac{\lambda(\tau)d\tau}{\varepsilon + a(\tau)} \right], \ (t, \tau) \in G, \ \varepsilon > 0.$$

Hence, we have the estimate (10).

Taking into account the lemma 1, the estimate (10) and the condition of the lemma 3, we obtain the estimate (9). The lemma 3 is proved.

Parallel with equation (1) we shall consider the operator equation of the second kind

$$(\varepsilon I + A(t))\, v(t, \varepsilon) + \int_{t_o}^{t} K(t,s, v(s, \varepsilon))\, ds = f(t) + \varepsilon\, u(t_o),\ t \in [t_o, T], \quad (11)$$

where $\varepsilon > 0$ is a small parameter.

The solution of operator equation (11) we shall seek as follows

$$v(t, \varepsilon) = u(t) + \xi(t, \varepsilon), \quad (12)$$

where $u(t)$ solves equation (1). Substituting (12) into (11), and using the operator $(\varepsilon I + A(t))^{-1}$, we have

$$\xi(t, \varepsilon) = -\int_{t_o}^{t} (\varepsilon I + A(t))^{-1} Q(s, s)\, \xi(s, \varepsilon) ds -$$

$$-\int_{t_o}^{t} (\varepsilon I + A(t))^{-1} [\, Q(t, s) - Q(s, s)]\, \xi(s, \varepsilon) ds -$$

$$-\int_{t_o}^{t} (\varepsilon I + A(t))^{-1} [K_1(t, s, u(s) + \xi(s, \varepsilon)) -$$

$$-K_1(t, s, u(s))] ds + \varepsilon\, (\varepsilon I + A(t))^{-1} [u(t_o) - u(t)].$$

Hence, using the resolvents of the operator kernel $[-(\varepsilon I + A(t))^{-1} Q(s, s)]$ and the Dirichlet formula, we obtain

$$\xi(t,\varepsilon) = \int_{t_o}^{t} H(t,s,\varepsilon)\,\xi(t,\varepsilon)ds + \int_{t_o}^{t} F(t,s,\xi(t,\varepsilon),\varepsilon)ds + \psi(t,\varepsilon). \quad (13)$$

Here

$$H(t,s,\varepsilon) = -\,(\varepsilon I + A(t))^{-1}[\,Q(t,s) - Q(s,s)] -$$

$$-\int_{s}^{t} R(t,\tau,\varepsilon)\,(\varepsilon I + A(\tau))^{-1}[Q(\tau,s) - Q(s,s)]\,d\tau, \quad (14)$$

$$F(t,s,\xi(s,\varepsilon),\varepsilon) = -(\varepsilon I + A(t))^{-1}[K_1(t,s,u(s)) + \xi(s,\varepsilon)) -$$

$$-\,K_1(t,s,u(s))] - \int_{s}^{t} R(t,\tau,\varepsilon)\,(\varepsilon I + A(\tau))^{-1}[K_1(\tau,s,u(s) +$$

$$+\,\xi(s,\varepsilon)) - K_1(\tau,s,u(s))]d\tau, \quad (15)$$

$$\psi(t,\varepsilon) = \varepsilon\,(\varepsilon I + A(t))^{-1}\,[u(t) - u(t_o)] +$$

$$+\varepsilon \int_{t_o}^{t} R(t,\tau,\varepsilon)\,(\varepsilon I + A(\tau))^{-1}\,[u(t_o) - u(\tau)]d\tau, \quad (16)$$

where the resolvent $R(t,\tau,\varepsilon)$ is defined by the formula (7).

LEMMA 4. Let the conditions 1, 2 hold. Then

$$\| H(t, s, \varepsilon) \| \le c_o\, l(s), \tag{17}$$

$$\| F(t, s, \xi(s, \varepsilon), \varepsilon)\|_H \le$$

$$\le \left\{ l_1(s) \frac{1}{\varepsilon + a(t)} \exp\left[-\frac{\theta}{\theta_1^2} \int_s^t \frac{\lambda(\tau)}{\varepsilon + a(\tau)} d\tau \right] + c_1 l_2(s) \right\} ||\xi(s, \varepsilon)||_H\,, \tag{18}$$

where $H(t, s, \varepsilon)$ and $F(t, s, \xi(s, \varepsilon), \varepsilon)$ are defined as in (14) and (15),

$$c_o = \frac{\theta_1^2}{\theta e} + 1 + c_1\,, \qquad c_1 = N_o \left[\left(\frac{\theta_1^2}{\theta} \right)^2 + \frac{\theta_1^2}{\theta} \right].$$

PROOF. Taking into account the formulas (7) and (8), from (14) and (15), it follows that

$$H(t, s, \varepsilon) = - (\varepsilon I + A(t))^{-1} X(t, s, \varepsilon)[\, Q(t, s) - Q(s, s)] +$$

$$+ \int_s^t R(t, \tau, \varepsilon)\, (\varepsilon I + A(\tau))^{-1} [Q(t, s) - Q(\tau, s)]\, d\tau, \tag{19}$$

$$F(t, s, \xi(s, \varepsilon), \varepsilon) = -(\varepsilon I + A(t))^{-1} X(t, s, \varepsilon)\,[K_1(t, s, u(s)+$$

$$+\xi(s, \varepsilon)) - K_1(t, s, u(s))] + \int_s^t R(t, \tau, \varepsilon)\,(\varepsilon I + A(\tau))^{-1}[K_1(t, s, u(s) +$$

$$+\xi(s, \varepsilon)) - K_1(t, s, u(s)) - K_1(\tau, s, u(s)+ \xi(s, \varepsilon)) + K_1(\tau, s, u(s))]d\tau . \quad (20)$$

By the condition 2, the lemma 1 and the estimate (10), the fist term of (19) has the following estimate

$$\| (\varepsilon I + A(t)^{-1} X(t, s, \varepsilon)[\, Q(t, s) - Q(s, s)] \| \le$$

$$\le l(s) \left[\int_s^t \frac{\lambda(\tau)}{\varepsilon + a(\tau)} d\tau + \frac{a(t)}{\varepsilon + a(t)} \right] exp \left[-\frac{\theta}{\theta_1^2} \int_s^t \frac{\lambda(\tau)}{\varepsilon + a(\tau)} d\tau \right] \times$$

$$l(s) \left[\sup_{\tau \ge 0} \left(\tau e^{-\frac{\theta}{\theta_1^2}\tau} \right) + 1 \right] = l(s) \left(\frac{\theta_1^2}{\theta e} + 1 \right). \quad (21)$$

ing into account the condition 2, the lemmas 1 and 3, the second term of (19) may be estimated as follows

$$\| \int_s^t R(t, \tau, \varepsilon)\,(\varepsilon I + A(\tau)^{-1}[Q(t, s) - Q(\tau, s)]\, d\tau \| \le$$

$$\leq \int_s^t N_o \; exp\left[-\frac{\theta}{\theta_1^2}\int_s^t \frac{\lambda(\tau)}{\varepsilon + a(\tau)}d\tau\right]\frac{\lambda(\tau)}{\varepsilon + a(\tau)}\, l(s)\left[\int_\tau^t \frac{\lambda(\tau)}{\varepsilon + a(\tau)} + \frac{a(t)}{\varepsilon + a(t)}\right]d\tau$$

$\leq$

$$\leq N_o\, l(s)\left[\int_0^\infty e^{-\frac{\theta}{\theta_1^2}\nu}\nu d\nu + \int_0^\infty e^{-\frac{\theta}{\theta_1^2}\nu} d\nu\right] = N_o\, l(s)\left[\left(\frac{\theta_1^2}{\theta}\right)^2 + \frac{\theta_1^2}{\theta}\right]. \qquad (22)$$

From the estimates (21) and (22) , we obtain the estimate (17).

Now, we estimate $F(t, s, \xi(s, \varepsilon), \varepsilon)$. By the condition 2, the lemma 1 and the estimate (10) , the first term of (20) has the following estimate

$$\| (\varepsilon I + A(t))^{-1} X(t, s, \varepsilon)[K_1(t, s, u(s) + \xi(s, \varepsilon)) - K_1(t, s, u(s))] \|_H \leq$$

$$\leq \frac{l_1(s)\, \|\xi(s,\varepsilon)\|_H}{\varepsilon + a(t)}\, exp\left[-\frac{\theta}{\theta_1^2}\int_s^t \frac{\lambda(\tau)}{\varepsilon + a(\tau)}d\tau\right]. \qquad (23)$$

Taking into account the condition 2, the lemmas 1 and 3, the second term of (20) may be estimated as follows

$$\| \int_s^t R(t, \tau, \varepsilon)\, (\varepsilon I + A(\tau))^{-1}\, [K_1(t, s, u(s) + \xi(s, \varepsilon)) - K_1(t, s, u(s)) -$$

$$-K_1(\tau, s, u(s) + \xi(s, \varepsilon)) + K_1(\tau, s, u(s))]d\tau \|_H \leq$$

$$\le N_o \int_s^t exp\left[-\frac{\theta}{\theta_1^2}\int_s^t \frac{\lambda(\tau)}{\varepsilon + a(\tau)}d\tau\right]\frac{\lambda(\tau)}{\varepsilon + a(\tau)}\, l_2(s)\times$$

$$\left[\int_\tau^t \frac{\lambda(\tau)}{\varepsilon + a(\tau)}d\tau + \frac{a(t)}{\varepsilon + a(t)}\right] \|\xi(s, \varepsilon)\|_H \, d\tau \le$$

$$\le N_o\, l_2(s)\left[\left(\frac{\theta_1^2}{\theta}\right)^2 + \frac{\theta_1^2}{\theta}\right] \|\xi(s, \varepsilon)\|_H . \tag{24}$$

From the estimates (23) and (24) , we have the estimate (18) . The lemma 4 is proved.

LEMMA 5. Let the condition 1 holds and

$\| Q(t, t) \| \le N_0\, \lambda(t)$ for all $t \in [t_o, T]$,

$\varphi(t) = \int_{t_o}^{t} \lambda(\tau)\, d\tau + a(t)$, for $t \in [t_o, T]$,

$\psi(t, s)$ is defined by the formula (16). Then the following statements hold.

1) If $u(t) \in C_\varphi^\gamma([t_o, T]; H)$, $0 < \gamma \le 1$ then

$$\|\psi(t, s)\|_c \le c_2\, \varepsilon^\gamma , \tag{25}$$

where

$$c_2 = \sup_{\nu \ge 0}\left[\exp(-\frac{\theta}{\theta_1^2}\nu)(\nu+1)^{\gamma}\right] + N_o M_{\gamma} \int_0^{\infty} e^{-\frac{\theta}{\theta_1^2}\nu} (\nu+1)^{\gamma} d\nu,$$

$$M_{\gamma} = \sup_{t,s\in[t_o,T]} \| u(t) - u(s)\|_H / |\varphi(t) - \varphi(s)|^{\gamma}$$

2) If $\varphi(t)$ is the increasing function on $[t_o, T]$ and $u(t) \in C[t_o, T; H]$, then

$$\| \psi(t, s) \|_c \le \left(1 + \frac{N_o\theta_1^2}{\theta}\right)\left[\omega_{\bar{u}}(\varepsilon^{\beta}) + \frac{2\theta_1^2}{\theta} e^{\frac{\theta}{\theta_1^2}} \| u(t) \|_c \, \varepsilon^{1-\beta}\right], \qquad (26)$$

where

$$\omega_{\bar{u}}(\delta) = \max_{\substack{|x-y|\le\delta \\ x,y\in[0,\varphi(T)]}} \| u(\varphi^{-1}(x)) - u(\varphi^{-1}(y))\|_H .$$

PROOF. Taking into account the formulas (7) and (8), from (16), it follows that

$$\psi(t, s) = -\varepsilon\, (\varepsilon I + A(t))^{-1} X(t, t_o, \varepsilon)\, [u(t) - u(t_o)] +$$

$$+ \varepsilon \int_{t_o}^{t} R(t, \tau, \varepsilon)\, (\varepsilon I + A(\tau))^{-1} [u(t) - u(\tau)] d\tau. \qquad (27)$$

In case 1), taking into account the estimates (9) and (10), from (27), we obtain

$$\|\psi(t,s)\|_H \le \varepsilon^{\gamma}\left(\frac{\varepsilon}{\varepsilon+a(t)}\right)^{1-\gamma} exp\left[-\frac{\theta}{\theta_1^2}\int_{\tau}^{t}\frac{\lambda(\tau)}{\varepsilon+a(\tau)}d\tau\right]\times$$

$$\left[\int_{t_o}^{t}\frac{\lambda(\tau)}{\varepsilon+a(\tau)}d\tau+\frac{a(t)}{\varepsilon+a(t)}\right]^{\gamma} + \varepsilon^{\gamma}N_o\int_{t_o}^{t}\left(\frac{\varepsilon}{\varepsilon+a(t)}\right)^{1-\gamma}\times$$

$$exp\left[-\frac{\theta}{\theta_1^2}\int_{\tau}^{t}\frac{\lambda(\tau)}{\varepsilon+a(\tau)}d\tau\right]\frac{\lambda(\tau)}{\varepsilon+a(\tau)}.$$

$$M_{\gamma}\left|\int_{\tau}^{t}\frac{\lambda(\tau)}{\varepsilon+a(\tau)}d\tau+\frac{at)-a(\tau)}{\varepsilon+a(t)}\right|^{\gamma}d\tau \le$$

$$\varepsilon^{\gamma}\left\{\sup_{\nu\ge 0}\left[\exp\left(-\frac{\theta}{\theta_1^2}\nu\right)(\nu+1)^{\gamma}\right]+N_oM_{\gamma}\int_0^{\infty}e^{-\frac{\theta}{\theta_1^2}\nu}(\nu+1)^{\gamma}d\nu\right\}.$$

The estimate (25) is proved.

2) Let $t_o \le t \le \varphi^{-1}(\varepsilon^{\beta})$, $0 < \beta < 1$. Then, taking into account the estimates (9) and (10) , from (27) , we have

$$\| \psi (t, s) \|_H \le \omega_{\bar{u}}(\varepsilon^{\beta}) \exp\left[-\frac{\theta}{\theta_1^2} \int_{t_o}^{t} \frac{\lambda(\tau)}{\varepsilon + a(\tau)} d\tau \right] +$$

$$+N_o\, \omega_{\bar{u}}(\varepsilon^{\beta}) \int_{t_o}^{t} \exp\left[-\frac{\theta}{\theta_1^{\,2}} \int_{\tau}^{t} \frac{\lambda(\tau)}{\varepsilon + a(\tau)} d\tau \right] \frac{\lambda(\tau)}{\varepsilon + a(\tau)} d\tau \le$$

$$\le \left(1 + \frac{N_o \theta_1^2}{\theta} \right) \omega_{\bar{u}}(\varepsilon^{\beta}) . \qquad (28)$$

If $\varphi^{-1}(\varepsilon^{\beta}) \le t \le T$, then taking into account the estimates (9) and (10), we obtain

$$\| \varepsilon \ (\varepsilon I + A(t))^{-1} X (t, t_o , \varepsilon) \ [u(t) - u(t_o)] \|_H \le$$

$$\le 2 \| u(t) \|_c \ \frac{\varepsilon}{\varepsilon + a(t)} \ e^{\frac{\theta}{\theta_1^2}(1 - \frac{a(t)}{\varepsilon + a(t)})} \ exp \left(- \frac{\theta}{(\varepsilon + a(t))\theta_1^2} \int_{t_o}^{t} K(s,s) ds \right) \le$$

$$\le 2 \varepsilon^{1-\beta} \ \frac{\varepsilon^{\beta}}{\varepsilon + a(t)} \ \| u (t) \|_c \ e^{\frac{\theta}{\theta_1^2}} e^{-\frac{\theta \varphi(\varepsilon^{\beta})}{\theta_1^2 (\varepsilon + a(t))}} \le$$

$$\le 2 \varepsilon^{1-\beta} \ e^{\frac{\theta}{\theta_1^2}} \ \| u(t) \|_c \ \sup_{\tau \ge 0} \left(\tau e^{-\frac{\theta}{\theta_1^2}\tau} \right) = 2 \varepsilon^{1-\beta} \ \frac{\theta_1^2}{\theta} \ e^{\frac{\theta}{\theta_1^2}} \ \| u(t) \|_c , \qquad (29)$$

$$\|\varepsilon \int_{t_o}^{\varphi^{-1}(\varphi(t)-\varepsilon^{\beta})} R(t,\ \tau,\ \varepsilon)\ (\varepsilon I + A(\tau))^{-1}[u(t) -$$

$$- u(\tau)]d\tau + \varepsilon \int_{\varphi^{-1}(\varphi(t)-\varepsilon^{\beta})}^{t} R(t,\ \tau,\ \varepsilon)\ (\varepsilon I + A(\tau))^{-1}[u(t) - u(\tau)]d\tau\|_H \leq$$

$$\leq 2N_o\,\|\,u(t)\|_c \int_{t_o}^{\varphi^{-1}(\varphi(t)-\varepsilon^{\beta})} \frac{\varepsilon}{\varepsilon + a(t)}\, exp\left[-\frac{\theta}{\theta_1^2}\int_{\tau}^{t}\frac{\lambda(\tau)}{\varepsilon + a(\tau)}d\tau\right]$$

$$\frac{\lambda(\tau)}{\varepsilon + a(\tau)}\,d\tau + N_o\,\omega_{\bar{u}}(\varepsilon^{\beta}) \int_{\varphi^{-1}(\varphi(t)-\varepsilon^{\beta})}^{t} \frac{\varepsilon}{\varepsilon + a(t)}$$

$$exp\left[-\frac{\theta}{\theta_1^2}\int_{\tau}^{t}\frac{\lambda(\tau)}{\varepsilon + a(\tau)}d\tau\right]\frac{\lambda(\tau)}{\varepsilon + a(\tau)}\,d\tau =$$

$$= 2N_o\,\|\,u(t)\|_c\ \frac{\theta_1^2\varepsilon}{c(\varepsilon + a(t))\theta}\ exp\left[-\frac{\theta}{\theta_1^2}\int_{\tau}^{t}\frac{\lambda(\tau)}{\varepsilon + a(\tau)}\,d\tau\right]\Big|_{\tau=t_o}^{\tau=\varphi^{-1}(\varphi(t)-\varepsilon^{\beta})} +$$

$$+ N_o\omega_{\bar{u}}(\varepsilon^{\beta})\frac{\theta_1^2\varepsilon}{\theta(\varepsilon + a(t))}\exp\left[\frac{\theta}{\theta_1^2}\int_{\tau}^{t}\frac{\lambda(\tau)}{\varepsilon + a(\tau)}d\tau\right]\Big|_{\tau=\varphi^{-1}(\varphi(t)-\varepsilon^{\beta})}^{\tau=t} \leq$$

$$\le 2N_o \| u(t)\|_c \frac{\theta_1^2 \varepsilon}{c(\varepsilon + a(t))\theta} \; exp\left[-\frac{\theta}{\theta_1^2} \int_\tau^t \frac{\lambda(\tau)}{\varepsilon + a(\tau)} d\tau \right] \times$$

$$exp\left[\frac{\theta}{\theta_1^2} (1 - \frac{a(t)}{\varepsilon + a(t)} + \frac{a(\varphi^{-1}(\varphi(t) - \varepsilon^\beta))}{\varepsilon + a(t)} \right] + \frac{N_o \theta_1^2}{\theta} \omega_{\bar{u}} (\varepsilon^\beta) =$$

$$= \| u(t)\|_c \frac{2N_o \theta_1^2}{\theta} e^{\frac{\theta}{\theta_1^2}} \varepsilon^{1-\beta} \frac{\varepsilon^\beta}{\varepsilon + a(t)} \times$$

$$exp\left[\frac{\theta}{\theta_1^2} \frac{(\varphi(t) - \varphi(\varphi^{-1}(\varphi(t) - \varepsilon^\beta)))}{(\varepsilon + a(t))} \right] +$$

$$+ \frac{N_o \theta_1^2}{\theta} \omega_{\bar{u}} (\varepsilon^\beta) \le \frac{2N_o \theta_1^2}{\theta} e^{\frac{\theta}{\theta_1^2}} \varepsilon^{1-\beta} \times$$

$$\sup_{\tau \ge 0} \left(\tau e^{-\frac{\theta}{\theta_1^2}\tau} \right) \| u(t)\|_c + \frac{N_o \theta_1^2}{\theta} \omega_{\bar{u}} (\varepsilon^\beta) =$$

$$= \frac{N_o \theta_1^2}{\theta} \left[\frac{2\theta_1^2}{\theta} e^{\frac{\theta}{\theta_1^2}} \varepsilon^{1-\beta} \| u(t) \|_c + \omega_{\bar{u}} (\varepsilon^\beta) \right]. \tag{30}$$

Taking into account the estimates (28) , (29) and (30), we obtain from (27) the estimate (26) . The lemma 5 is proved.

THEOREM. Let the conditions 1, 2 hold and $\| Q(t, t) \| \le N_0 \lambda(t)$ and $l_1(t) \le \beta \lambda(t)$ for almost all $t \in [t_o, T]$, $0 < \beta < \theta/\theta_1^2$,

$$\varphi(t) = \int_{t_o}^{t} \lambda(\tau)d\tau + a(t), \; t \in [t_o, T].$$

Then the following statements hold.

1) If equation (1) has a solution

$$u(t) \in C_\varphi^\gamma([t_o, T]; H), \; 0 < \gamma \le 1,$$

then the solution $v(t, s)$ of equation (13) converges in the norm $C([t_o, T]); H)$ to $u(t)$ for

$\varepsilon \to 0$ and the estimate

$$\| v(t, \varepsilon) - u(t) \|_c \le M_2 c_2 \varepsilon^\gamma, \qquad (31)$$

holds. Here

$$M_2 = c_3 \exp\{c_3 \int_{t_o}^{T} [c_o l(s) + c_2 l_2(s)]ds\}, \; c_3 = 1 + \frac{\beta\theta_1^2}{\theta - \beta\theta_1^2},$$

c_o and c_2 are defined as in lemmas 4 and 5.

2) If $\varphi(t)$ is the increasing function on $[t_o, T]$ and equation (1) has a solution

$$u(t) \in C([t_o, T]; H]),$$

then the solution $v(t, s)$ of equation (13) converges in the norm $C([t_o, T]; H)$ to $u(t)$ for $\varepsilon \to 0$ and the estimate

$$||v(t, \varepsilon) - u(t)||_c \leq M_3 \left[\omega_{\bar{u}}(\varepsilon^{\beta}) + \frac{2\theta_1^2}{\theta} e^{\frac{\theta}{\theta_1^2}} \| u(t) \|_c \, \varepsilon^{1-\beta} \right], \qquad (32)$$

holds. Here

$$M_3 = M_2 \left(1 + \frac{N_o \theta_1^2}{\theta}\right),$$

$w_{\bar{u}}(\delta)$ is defined as in lemma 5.

PROOF. Solution of equation (11) we shall seek in form (12). By the lemma 4 and the condition, from (13), we have

$$|| \xi(t, \varepsilon)||_H \leq \int_{t_o}^{t} \frac{\beta\lambda(s)}{\varepsilon + a(s)} exp \left[-\frac{\theta}{\theta_1^2} \int_s^t \frac{\lambda(\varepsilon)}{\varepsilon + a(\tau)} d\tau \right] || \xi(t, \varepsilon)||_H \, ds$$

$$+ \int_{t_o}^{t} [c_o l(s) + c_1 l_2(s)] \, || \xi(t, \varepsilon)||_H \, ds + || \psi(t, \varepsilon)||_H .$$

Hence, using the resolvents of the kernel

$$\frac{\beta\lambda(s)}{\varepsilon + a(s)} exp \left[-\frac{\theta}{\theta_1^2} \int_s^t \frac{\lambda(\tau)}{\varepsilon + a(\tau)} d\tau \right],$$

we obtain

$$|| \xi(t, \varepsilon)||_H \le \int_{t_o}^{t} [c_o l(s) + c_1 l_2(s)] \, || \xi(t, \varepsilon)||_H \, ds + \int_{t_o}^{t} \frac{\beta\lambda(s)}{\varepsilon + a(s)} \times$$

$$exp \left[-(\frac{\theta}{\theta_1^2} - \beta) \int_s^t \frac{\lambda(\tau)}{\varepsilon + a(\tau)} d\tau \right] \times$$

$$\times \left\{ \int_{t_o}^{s} [c_o l(\tau) + c_1 l_2(\tau)] \| \xi(\tau, \varepsilon) \|_H \, d\tau + \| \psi(s, \varepsilon) \|_H \right\} ds + || \psi(t, \varepsilon)||_H .$$

From the Dirichlet formula and the last inequality, it follows that

$$|| \xi(t, \varepsilon)||_H \le c_3 \int_{t_o}^{t} [c_o l(s) + c_2 l_2(s)] \, || \xi(s, \varepsilon)||_H \, ds + c_3 \, || \psi(t, \varepsilon)||_c ,$$

where $c_3 = 1 + \dfrac{\beta\theta_1^2}{\theta - \beta\theta_1^2}$.

Hence, taking into account the lemma 5 and applying the Gronuall-Bellman inequality, we obtain estimates (31) and (32) . The theorem is proved.

COROLLARY. Let the conditions 1, 2 hold and $|| Q(t, t) || \le N_0 \, \lambda(t)$ and

$l_1(t) \leq \beta\, \lambda(t)$ for almost all $t \in [t_o, T]$, $0 < \beta < \alpha_1$, $\alpha_1 = min\{1, \theta/\theta_1^2\}$,

$$\varphi(t) = \int_{t_o}^{t} \lambda(\tau)\, d\tau + a(t), \qquad t \in [t_o, T].$$

Then the following statements hold.

1) If there exists a number $\delta \in (t_o, T]$ such that $\varphi(t)$ is the increasing function on $[t_o, \delta]$, then the solution of (1) is unique in $C_\varphi^\gamma([t_o, T]; H)$, $0 < \gamma \leq 1$.

2) If $\varphi(t)$ is the increasing function on $[t_o, T]$, then the solution of (1) in unique in

$C([t_o, T]) ; H)$.

PROOF. Let $u_1(t)$ and $u_2(t)$ be two solutions of equation (1) from $C([t_o, T]) ; H)$ or $C_\varphi^\gamma([t_o, T]; H)$, $0 < \gamma \leq 1$. Then, from (1), we have

$$A(t)[u_1(t_o) - u_2(t_o)] + \int_{t_o}^{t} Q(s, s)[u_1(t_o) - u_2(t_o)]\, ds =$$

$$= A(t)\{[u_2(t) - u_2(t_o)] - [u_1(t) - u_1(t_o)]\} +$$

$$+ \int_{t_o}^{t} Q(s, s)\{[u_2(s) - u_2(t_o)] - [u_1(s) - u_1(t_o)]\} -$$
$$- \int_{t_o}^{t} [Q(t, s) - Q(s, s)][u_1(s) - u_2(s)]ds -$$
$$- \int_{t_o}^{t} [K_1(s, s, u_1(s)) - K_1(s, s, u_2(s))]\, ds - \int_{t_o}^{t} [K_1(t, s, u_1(s)) -$$

$$- K_1(s, s, u_1(s)) + K_1(s, s, u_2(s)) - K_1(t, s, u_2(s))]ds .$$

Hence, taking into account the conditions, we obtain

$$\varphi(t)\, \| u_1(t_o) - u_2(t_o)\|_H^2 \leq \varphi(t)\, \| u_1(t_o) - u_2(t_o)\|_H \{[\omega_1(t - t_o) +$$

$$+ \omega_2(t - t_o)] (\alpha + N_o + \beta) + [\int_{t_o}^{t} l(s) + l_2(s)) ds] [\, || u_1(t) ||_c +$$

$$+ || u_2(t) ||_c + \beta \; || u_1(t_o) - u_2(t_o)||_H \},$$

where

$$\omega_i(\delta) = \max_{|t-s|\le\delta} || u_i(t) - u_i(s)||_H, \quad i = 1, 2.$$

This estimate yield $u_1(t_o) = u_2(t_o)$. Further , from the estimate (32) and that the equations (11) are Volterra equations, it follows that $u_1(t) = u_2(t)$ for $t \in [t_o, T]$. The corollary is proved.

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