INTERPOLATION AND THE LAGRANGE POLYNOMAL

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Abstract

We show that the interpolation polynomial in the lagrange form canbe calculatetod with the some numbers of the arithetic operations. Given a set of *(n+1)* data points and a function *f*, the aim is to determine a polynomial of degree *n* which interpolates *f* at the points inquestion.

Key words: Interpolation, polynomial and laagrange.

Introduction

The problem of determinig a polynomial of degree 1 that passes throwgh the distince points (x_0, y_0) and (x_1, y_1) is the same as appmaximating a function *f* for which $f(x_0) = y_0$ and $f(x_1) = y_1$ by means of a first-decree polynomial interpolating or agreeing with the values of fort he given points [3-5].

Consider the linear polynomial

$$P(x) = \frac{(x - x_1)}{x_0 - x_1} y_0 + \frac{(x - x_0)}{x_1 - x_0} y_1 \tag{1.1}$$

When $x = x_0$.

$$p(x_0) = 1.y_0 + 0.y_1 = y_0 = f(x_0) \tag{1.2}$$

and when $x = x_1$

$$p(x_1) = 0.y_0 + 1.y_1 = y_1 = f(x_1) \tag{1.3}$$

so *p* has the required properties. (See Figure 1.1)

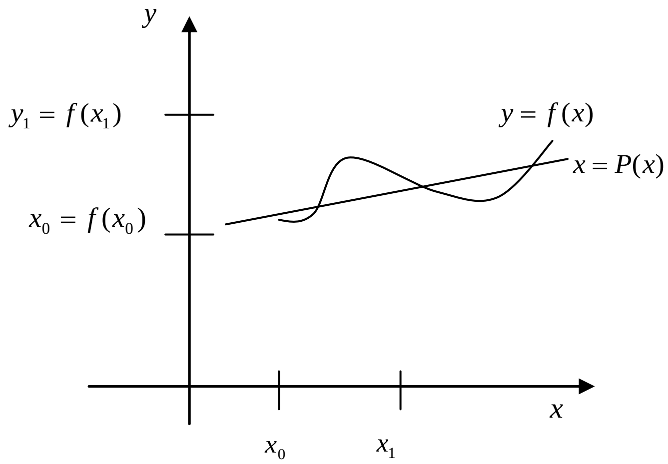


Figure 1.

The tecnique used to construct p is the method of " intepolation" often used in trigonometric or logarithmic tables. What may not be obvious is that *p* is the only polynomial of degree 1 or less with the interpolating property. Has result , however follows.

To generalize the concept of linear interpolation consider the construction of a polynomial of degree at most *n* that passes through the *n+1* points $(x_0, f(x_0)),(x_1, f(x_1)),......,(x_n, f(x_n))$.

The linear polynomial passing through $(x_0, f(x_0))$ and $(x_1. f(x_1))$ is constructed by using the quotients

$$L_0(x) = \frac{(x - x_1)}{(x_0 - x_1)} \text{ and } L_1(x) = \frac{(x - x_0)}{(x_1 - x_0)}$$

When $x = y_0$, $L_0(x_0) = 1$ and $L_1(x_0) = 0$ when $x = x_1$, $L_0(x_1) = 0$ and $L_1(x_1) = 1$

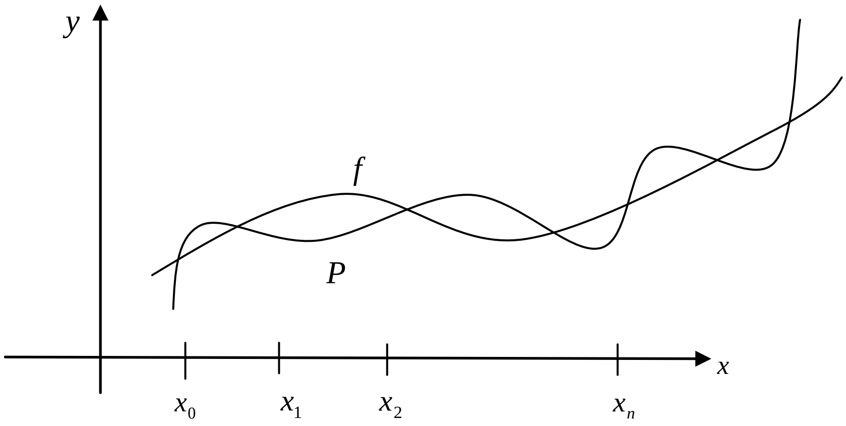


Figure 2.

For the general case,we need to construct, for each *k=0,1,2... n* a quotient $L_{n,k}(x)$ with the property that $L_{n,k}(x_1)=0$ when $i \neq k$ and $L_{n,k}(x)=1$ To safety $L_{nk}(x_1)=0$ for each $i \neq k$ requires that the numerator of L_{nk}... the term

$$(x-x_0)(x-x_1)...(x-x_{k-1})(x-x_{k+1})...x_k \qquad (5)$$

To satisfy $L_{n,k}(x_k)=1$ the denominator of L_{nk} must he equal to when $x=1$

Thus,

$$L_{nk}(x)=\frac{(x-x_0)...(x-x_{k-1})(x-x_{k+1})...(x-x_n)}{(x_k-x_0)...(x_k-x_{k-1})(x_k-x_{k+1})...(x_k-x_n)}=\prod_{\substack{i=0\\i\neq k}}^{n}\frac{(x-x_i)}{x_k-x_i} \qquad (6)$$

A sketch of the graph of a typical L_{nk} (in the case when *n* is even) is known in figure 1.3

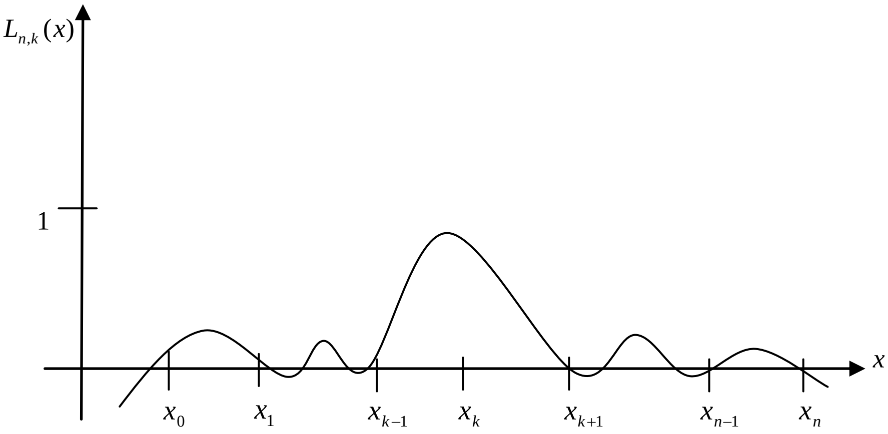


Figure 3.

The interpolating polynomial is easily described now that the form of L_{nk} is known.This polynomial called the *n*th Lagrange interpolation polynomial. Is defined in the following theorem [6], [7], [8].

THEOREM 1. If $x_0\,, x_1\,, \ldots, x_n$ are $(n+1)$ distinct numbers and f is a function whose values are given at these numbers,then there exists a unique polynomial p of degree at most n with the property that

$f(x_k) = P(x_k)$ for each $k = 0,1,\ldots,n$

This polynomial is given by

$$P(x) = f(x_0).L_k(x) + \ldots. + f(x_n)L_k(x) = \sum_{k=0}^{n} f(x_k).L_k(x) \tag{7}$$

Where $$L_{n,k}(x) = \frac{x-x_0}{x_k-x_0}\ldots\frac{x_k-x_{k-1}}{x_k-x_{k-1}}.\frac{x-x_{k+1}}{x_k-x_{k+1}}\ldots\frac{x-x_n}{x_k-x_n} = \prod_{\substack{i=0\\ i\neq k}}^{n} \tag{8}$$

for each $k = 0,1,2,\ldots,n$

EXAMPLE 1.

Using the numbers or nodels, $x_0 = 2, x_1 = 2.5, and\ x_2 = 4$ to find the second interpolating polynomial for $f(x) = 1$ requires that we first determine the coefficent.

Polynomials L_0, L_1, *and* L_2:

$$L_0(x) = \frac{(x-2.5)(x-4)}{(2-2.4)(2-4)} = (6-6.5)x+10$$

$$L_1(x) = \frac{(x-2)(x-4)}{(2.5-2)(2.5-4)} = \frac{(-4x+24)x-32}{3}$$

and $L_2(x) = \dfrac{(x-2)(x-2.5)}{(4-2)(4-2.5)} = \dfrac{(x-4.5)x+5}{3}$

Since $\int(x_0) = \int(2) = 0.5, \int(x_1) = \int(2.5) = 0.4, and \int(x_2) = \int(4) = 0.25$

$$P(x) = \sum_{k=0}^{2} f(x_k)L_k(x)$$

$$= 0.5((x-6.5)x+10) + 0.4\frac{(-4x+24)x-32}{3} + 0.25\frac{(x-4.5)x+5}{3} = (0.05x-0.425)x+1.15$$

An approximation to $f(3) = 3$ is

$$f(3) \approx P(3) = 0.325$$

Taylor polynomial (expanded about $x_0=1$) could be used to reasonably approximate $f(3) = 3$ (See Figure 1.1)

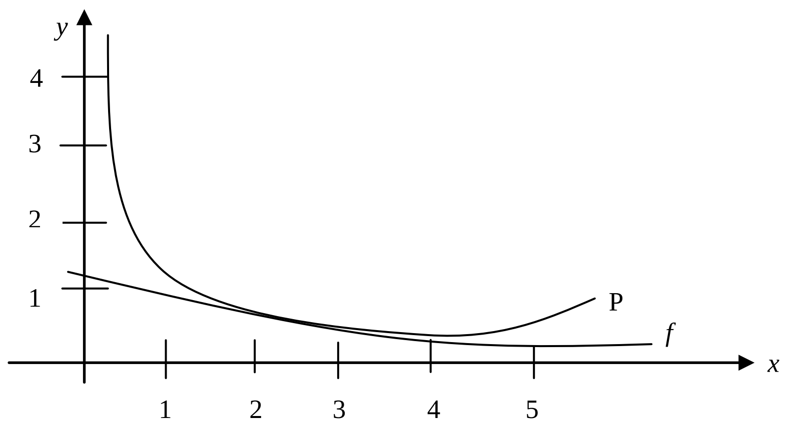


Figure 4.

The next step is to calculate a remainder term or bound for the error involved in approximating a function by an interpolating polynomial. This is done in the following.

THEOREM 2.

If x_0 , $x_1,...,x_n$ are distinct numbers in the interval $[a,b]$ a number $\xi(x)$ in *(a,b)* exists with.

$f(x) = P(x) + \frac{f(n+1)}{(n+1)!}(x-x_0)(x-x_1)...(x-x_a)$ where P is the polynomial given in Eq.(5)

Proof Note first that if $x = x_k$ for $k = 0,1,...,n$ then $f(x_k) = P(x_k)$ and choosing $\xi(x_k)$ arbittarily in *(a,b)* yields Eq.(3.5). If $x \neq x_k$ for any $k = 0,1,...,n$ define the function g for t in $[a,b]$ by

$$g(t) = f(t) - P(t) - [f(x) - P(x)]\frac{(t-x_0)(t-x_1)...(t-x_a)}{(x-x_0)(x-x_1)...(x-x_a)}$$
$$= f(t) - P(t) - [f(x) - P(x)]\prod_{t=0}^{a}\frac{(t-x_1)}{(x-x_1)}$$

Since

$$g(x_k) = f(x_k) - P(x_k) - [f(x) - P(x)]\prod_{t=0}^{n}\frac{(x_k-x_1)}{x-x_1}$$
$$= 0 - [f(x) - P(x].0) = 0$$

Moreover,

$$g(x) = f(x) - P(x) - [f(x) - P(x)]\prod_{t=0}^{n}\frac{(x_k-x_1)}{x-x_1}$$
$$= f(x) - P(x) - [f(x) - P(x)] = 0$$

Thus, $g \in C^{a+1}[a,b]$ and g vanishes at the n+2 distinct number $x, x_1,...,x_a$ By the Generalized Rolle's Theorem, there exists $\xi \equiv \xi(x)$ in (a, b) for which $g^{(a+1)}(\xi) = 0$. Evaluating $g^{(a+1)}$ at ξ gives.

$$0 = g^{(a+1)}(\xi) = f^{(a+1)}(\xi) - P^{(a+1)}(\xi) - [f(x) - P(x)]\frac{d^{a+1}}{d_1^{a+1}}\left(\prod_{i=0}^{a}\frac{t-x_1}{x-x_1}\right)_{t=\xi}$$

Since P is a polynomial of degree at most n, the (n+1)st derivative, $P^{(a+1)}$, is identically zero. Also

$$\prod_{t=0}^{a}\frac{(t-x_1)}{(x-x_1)}=\left(\frac{1}{\prod_{t=0}^{a}(x-x_1)}\right)t^{a+1}+(lower-\deg ree\, terms\ in\ t).$$

and

$$\frac{d^{a+1}}{dt^{a+1}}\prod_{t=0}^{a}\frac{(t-x_1)}{(x-x_1)}=\frac{(n+1)!}{\prod_{t=0}^{a}(x-x_1)}$$

Equation (2.9) now becomes

$$0=f^{(a+1)}(\xi)-0-\left[f(x)-P(x)\right]\frac{(n+1)!}{\prod_{t=0}^{a}(x-x_1)}$$

And, upon solving for *f(x)*.

$$f(x)=P(x)+\frac{f^{(a+1)}(\xi)}{(n+1)!}\prod_{t=0}^{a}(x-x_1)$$

The error Formula in *Theorem 1.2* is an important theoretical restricted because lagrange polynomials are used extensively for deriving numerical differentation and gration methods. Error bounds for these techniques are obtain from the lagrange error Formula. [9-12]

Note that the error form for the Lagrange polynomial is quite similation that for Taylor polynomial. The Taylor polynomial of degree n about x_0 all rates all known information at has an error term of the form

$$\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)^{n+1}$$

The Lagrange polynomial of degree n uses information at the distinct numbers $x_0, x_1, \ldots, x_n$ and in place of $(x-x_0)^n$ its error Formula uses a product the *(n+1)* $(x-x_0),(x-x_1),\ldots,(x-x_n)$

$$\frac{f^{(n+1)}(\xi(x))}{(n+1)!}(x-x_0)(x-x_1)...(x-x_n)$$

Specific use of this error Formula is restricted to those functions) who derivaties known bounds.

EXAMPLE 2. Table 2.1 lists values of a function (The Bessel function of the first kind of zero)various points.The approximations to f (1.5) obtained by various Lagrange polynomials will be compared.[1], [2]

x	$f(x)$
0	0.7651977
3	0.6200860
6	0.4554022
9	0.2818186
2	0.1103623

Table 1.

Since 1.5 is between 1.3 and 1.6 the linear polynomial at 1.5 given by

$$P_1(1.5)=\frac{(1.5-1.6)}{(1.3-1.6)}(0.6200860)+\frac{(1.5-1.3)}{(1.6-1.3)}(0.4554022)=0.5102968$$

Two polynomials of degree two could reasonably be used ,one by letting $x_0=1.3$

$x_1=1.6$ and $x_2=1.9$ which gives

$$P_2(1.5)=\frac{(1.5-1.6)(1.5-1.9)}{(1.3-1.6)(1.3-1.9)}(0.6200860)+\frac{(1.5-1.3)(1.5-1..9)}{(1.6-1.3)(1.6-1.9)}(0.4554022)$$
$$+\frac{(1.5-1.3)(1.5-1.6)}{(1.9-1.3)(1.9-1.6)}(0.2818186)=0.5112857$$

and the other by letting $x_0=1.3$, $x_1=1.6$, $x_2=1.9$ *and* $x_3=1.6$ in which case

$$\bar{P}_2(1.5)=0.5124715$$

In the third-degree case there are also two choices for the polynomial. One is with $x_0 = 1.3$, $x_1 = 1.6$, $x_2 = 1.9$, *and* $x_3 = 2.2$, which gives

$P_3 = (1.5) = 0.5118302$

The other is obtained by letting $x_0 = 1.0$, $x_1 = 1.3$, $x_2 = 1.6$, *and* $x_3 = 1.9$ giving

$\bar{P}_3(1.5) = 0.5118200$

The fourth-degree Lagrange polynomial uses all the entries in the table. With $x_0 = 1.0$, $x_1 = 1.3$, $x_2 = 1.6$, $x_3 = 1.9$ *and* $x_4 = 2.2$ it can be shown that

$P_4(1.5) = 0.5118200$

Since $\bar{P}_3(1.5)$, $P_3(1.5)$ and $P_4(1.5)$ all agree to within 2 x 10^{-5} units,we expect $P_4(1.5)$ to be the most accurate approximation and to be correct to within 2 x 10^{-5} units.

The actual value of $f(1.5)$ is known to be 0.5118277, so the true accuracies of the approximations are as follows:

$|P_1(1.5) - f(1.5)| \approx 1.53$ x 10^{-3}

$|P_2(1.5) - f(1.5)| \approx 5.42$ x 10^{-4}

$|\bar{P}_2(1.5) - f(1.5)| \approx 6.44$ x 10^{-4}

$|P_3(1.5) - f(1.5)| \approx 2.5$ x 10^{-6}

$|\bar{P}_3(1.5) - f(1.5)| \approx 1.50$ x 10^{-5}

$|P_4(1.5) - f(1.5)| \approx 7.7$ x 10^{-6}

P_3 is the most accurate approximation. However, with no knowledge of the actual value of $f(1.5), P_4$ would be accepted as the best approximation. Note that the error or remainder term derived in Theorem 2.3 cannot be applied here, since no knowledge of the fourth derivative of f is available. Unfortunately, this is generally the case.[13]

EXAMPLE 3.

We wish to interpolate $f(x) = x^2$ over the range $1 \le x \le 3$, given these three points:

$$x_0 = 1 \qquad f(x_0) = 1$$
$$x_1 = 2 \qquad f(x_1) = 4$$
$$x_3 = 3 \qquad f(x_3) = 9$$

The interpolating polynomial is:

$$L(x) = 1.\frac{x-2}{1-2} + 4.\frac{x-1}{2-1}.\frac{x-3}{2-3} + 9.\frac{x-1}{3-1}.\frac{x-2}{3-2} = x^2$$

EXAMPLE 4.

We wish to interpolate $f(x) = x^3$ over the range $1 \le x \le 3$ given these 3 points:

$$x_0 = 1 \qquad f(x_0) = 1$$
$$x_1 = 2 \qquad f(x_1) = 8$$
$$x_2 = 3 \qquad f(x_3) = 27$$

The interpolating polynomial is:

$$L(x) = 1.:\frac{x-2}{1-2}.\frac{x-3}{1-2} + 8.\frac{x-1}{2-1}.\frac{x-3}{2-3} + 27.\frac{x-1}{3-1}.\frac{x-2}{3-2} = 6x^2 - 11x + 6$$

EXAMPLE 5.

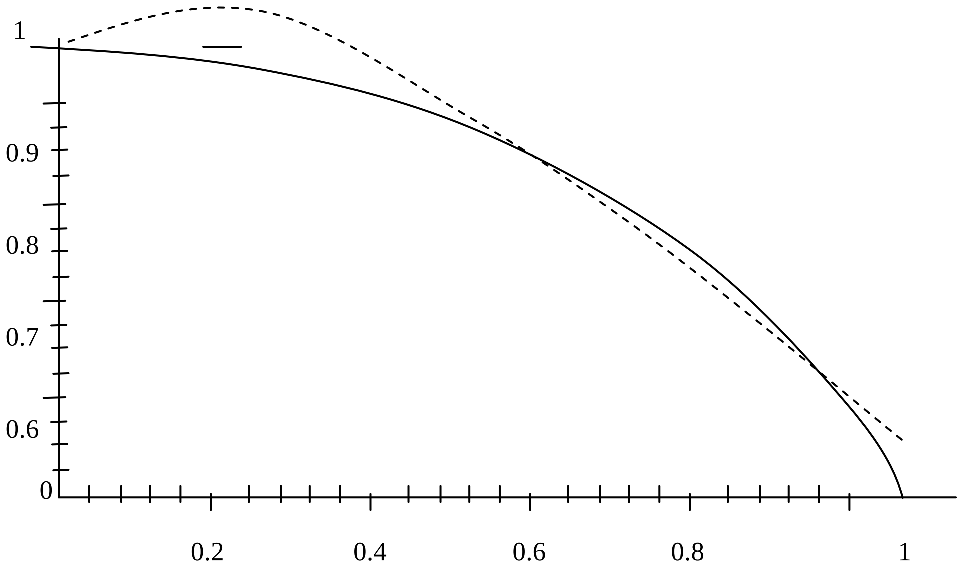
Let $x_0 = 0$, $x_1 = 0.6$, $x_2 = 0.9$, $f(x) = \cos(x^2)$.

a. Find the Lagrange interpolating polynomial $P_2(x)$ *and* $R_2(x)$.

b. Approximate $\int_0^1 \cos(x_2)dx$ *by* $\int_0^1 P_2(x)dx$ *and estimate the approximation error.*

a. $P_2(x) = f(0)\frac{(x-0.6)(x-0.9)}{(-0.6)(-0.9)} + f(0.6)\frac{x(x-0.9)}{0.6(-0.3)} + f(0.9)\frac{x(x-0.6)}{0.9(0.3)}$

$$P_2(x) = \frac{1}{0.54}(x-0.6)(x-0.9) - \frac{\cos(0.36)}{0.18}x(x-0.9) + \frac{\cos(0.81)}{0.27}x(x-0.6)$$



$$- \; y = \cos(x^2), - - - \; y = P_2(x)$$

$f'(x) = -2x\sin(x^2), \quad f''(x) = -2(\sin(x^2)) + 2x^2\cos(x^2)$,

$$f'''(x) = -2(2x\cos(x^2)) + 4x\cos(x^2) - 4x^3\sin(x^2)) = -4(3x\cos(x^2) - 2x^3\sin(x^2))$$

$|R_2(x)| = \dfrac{-4(3x\cos(c) - 2c^3\sin(c))}{6}x(x-0.6)(x-0.9)$ *where c* is in *(0,1).*

b. $\int_0^1 P_2(x)dx = \int_0^1\left(\dfrac{1}{0.54}(x-0.6)(x-0.9)\dfrac{\cos(0.36)}{0.18}x(x-0.9) + \dfrac{\cos(0.81)}{0.27}x(x-0.6)\right)dx$

$$= \frac{1}{0.54}\int_0^1\left((x-0.6)^2 - 0.3(x-0.6)\right)dx - \frac{\cos(0.36)}{0.18}\int_0^1(x^2 - 0.9x)dx + \frac{\cos(0.81)}{0.27}\int_0^1(x^2 - 0.6x)dx$$

$$= \frac{1}{0.54}\left(\frac{1}{3}(x-0.6)^3 - \frac{0.3}{2}(x-0.6)^2\right)\Big|_0^1 - \frac{\cos(0.36)}{0.18}\left(\frac{1}{3}x^3 - \frac{0.9}{2}x^2\right)\Big|_0^1 + \frac{\cos(0.81)}{0.27}\left(\frac{1}{3}x^3 - \frac{0.6}{2}x^2\right)\Big|_0^1$$

$$= \frac{1}{0.54}\left(\frac{1}{3}(0.4)^3 - \frac{0.3}{2}(0.4)^2 + \frac{1}{3}(0.6)^3 + \frac{0.3}{2}(0.6)^2\right) - \frac{\cos(0.36)}{0.18}\left(\frac{1}{3} - \frac{0.9}{2}\right) + \frac{\cos(0.81)}{0.27}\left(\frac{1}{3} - \frac{0.6}{2}\right)$$

$= 0.9201181$

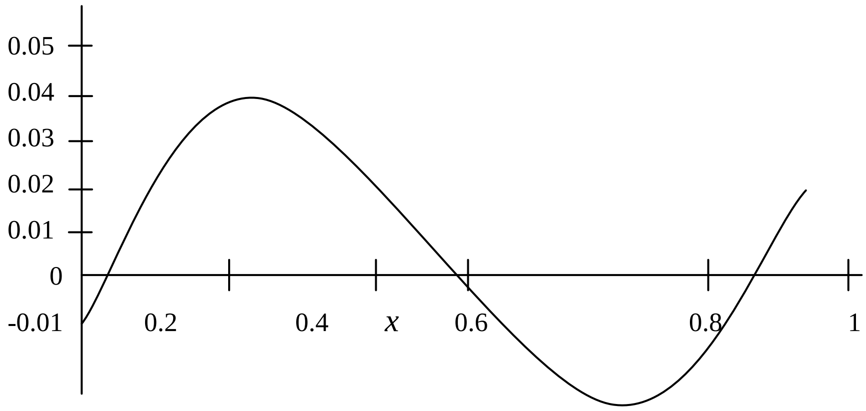
True error

$$\left|\int_0^1 \cos(x^2)dx - \int_0^1 P_2(x)dx\right| = |0.9045242 - 0.9201181| = 0.0155939$$

An approximation error:

$$Error_{approx} = \left|\int_0^1 \cos(x^2)dx - \int_0^1 P_2(x)dx\right| \le \int_0^1 |R_2(x)dx| = \int_0^1 \frac{|f'''(c)|}{3!}|x(x-0.6)(x-0.9)|dx$$

Observe that $|f'''(\Im(x))| \le$ 6.5, and the function $x(x-0.6)(x-0.9)$ changes signs over [0,1]:



$$y = x(x-0.6)(x-0.9), \quad 0 \le x \le 1$$

$$x(x-0.6)(x-0.9) = (x-0.6+0.6)(x-0.6-0.3)$$

$$= \left[(x-0.6)^2 + 0.6(x-0.6)\right](x-0.6-0.3) = (x-0.6)^3 + 0.3(x-0.6)^2 - 0.18(x-0.6)$$

$$x(x-0.6)(x-0.9) = (x-0.9+0.9)(x-0.9+0.3)(x-0.9)$$

$$= \left((x-0.9)^2 + 1.2(x-0.9) + 0.27\right)(x-0.9) = (x-0.9)^3 + 1.2(x-0.9)^2 + 0.27(x-0.9)$$

$$Error_{approx} \le \frac{6.5}{6}\left(\int_0^{0.6} x(x-0.6)(x-0.9)dx - \int_{0.6}^{0.9} x(x-0.6)(x-0.9)dx + \int_{0.9}^{1} x(x-0.6)(x-0.9)dx\right)$$

$$\left[\frac{1}{4}(x-0.6)^4+0.1(x-0.6)^3-0.09(x-0.6)^2\right]\int_0^{0,6} \quad -\left[\frac{1}{4}(x-0.6)^4+0.1(x-0.6)^3-0.09(x-0.6)^2\right]\int_{0,6}^{0,9}$$

$$+\left[\frac{1}{4}(x-0.9)^4+0.4(x-0.9)^3+\frac{0.27}{2}(x-0.9)^2\right]\int_{0,9}^{1}$$

$$+\frac{1}{4}(0.6)^4+0.1(0.6)^3-0.09(0.6)^2-\frac{1}{4}(0.3)^4-0.1(0.3)^3+0.09(0.3)^2+\frac{1}{4}(0.1)^4+0.4(0.1)^3+\frac{0.27}{2}(0.1)^2$$

$$=0.0268979$$

EXAMPLE 6.

$\sin\left(\frac{1}{4}\pi+x\right)=\frac{1}{2}\sqrt{2}(1+x)$ find the approximate formula lets we develop approximate formula and we, this formula to calculate sin43^o

In Taylor formula the next term in the twin terms

$$f(a+h)=f(a)=h\,f''(a)+\frac{h^2}{2}f''(a)+...$$

$$\sin\left(\frac{1}{4}\pi+x\right)=\sin\frac{1}{4}\pi+x\cos\frac{1}{4}\pi=\frac{1}{2}\sqrt{2}+\frac{\sqrt{2}}{2}x=\frac{1}{2}\sqrt{2}(\lambda+x)$$

$$\sin 43^0=\sin\left[\frac{1}{4}\pi+\left(-\frac{\pi}{90^0}\right)\right]=\frac{1}{2}\sqrt{2}(1-0{,}0349x)=\frac{1}{2}\sqrt{2}(\lambda+x)$$

$$\sin 43^0=\sin\left[\frac{1}{4}\pi+\left(-\frac{\pi}{90^0}\right)\right]=\frac{1}{2}\sqrt{2}(1-0{,}0349)=0{,}6824$$

EXAMPLE 7.

Lets calculate $\sin 62^0$ as fifth find the five digits after the comma correctly

Solition: Taylor series according to $(x-\alpha)$moment?

$$\sin x=\sin\alpha+(x-\alpha)\cos\alpha-\frac{(x-\alpha)^2}{2!}\sin\alpha-\frac{(x-\alpha)^3}{3!}\cos\alpha+...$$ dır.

62^0 take the closest and trigonometric functions as take known angle.

$\alpha = 60^0$ take. $x - \alpha = 62^0 - 60^0 = 2^0 = \frac{\pi}{90^0} = 0{,}034907$ and

$$\sin 62^0 = \frac{\sqrt{3}}{2} + \frac{1}{2}(0{,}034907) - \frac{\sqrt{3}}{4}(0{,}034907)^0 - \frac{1}{12}(0{,}034907)^3 + ...$$
$$= 0{,}866025 + 0{,}017454 - 0{,}000528 - 0{,}00004 + ... = 0{,}88295$$

EXAMPLE 8.

$\ln 0{,}97$ find the seven digits after the comma correctly

$$\ln(a - x) = \ln a - \frac{x}{a} - \frac{x^2}{2a^2} - \frac{x^3}{3a^2} - ... + \frac{x^n}{na^n} - ...$$

$a = 1$ *and* $x = 0{,}03$ *take*

$$\ln 0{,}97 = -0{,}03 - \frac{1}{2}(0{,}03)^2 - \frac{1}{3}(0{,}03)^3 - \frac{1}{4}(0{,}03)^4 - \frac{1}{5}(0{,}03)^5 - .... = -0{,}0304592$$

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