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On Various g-Topology in Statistical Metric Spaces

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Abstract

The purpose of this paper is to analyze the significance of new g-topologies defined in statistical metric spaces and we prove various properties for the neighbourhoods defined by Thorp in statistical metric spaces. Also, we give a partial answer to the questions, namely "What are the necessary and sufficient conditions that the g-topology of $typeV$ to be of $typeV_D$?, the g-topology of $typeV_\alpha$ to be the g-topology of $typeV_D$? and the g-topology of $typeV_\alpha$ to be a topology?" raised by Thorp in 1962. Finally, we discuss the relations between λ_Ω-open sets in generalized metric spaces and various g-topology neighbourhoods defined in statistical metric spaces. Also, we prove weakly complete metric space is equivalent to a complete metric space if Ω satisfies the $\mathscr{V}$-property.

1. Introduction

Fréchet introduced the notion of an abstract metric space in the year 1906 [1] from which the concept of "distance" appears. The notion of distance is defined in terms of functions, points and sets. Indeed, in many situations, it is appropriate to look upon the distance concept as a statistical rather than a determinate one. More precisely, instead of associating a number to the distance d(p, q) with every pair of points p, q, one should associate a distribution function F_{pq} and for any positive number x, interpret $F_{pq}(x)$ as the probability that the distance from p to q be less than x.

Using this idea, Menger [3] defined a statistical metric space using the probability function in the year 1942. In 1943, shortly after the appearance of Menger's article, Wald [10] published an article in which he criticized Menger's generalized triangle inequality. In 1951, Menger [5] continued his study of statistical metric spaces and in [4], he studied the behaviour of probabilistic theory.

In 1960, Schweizer et. al gave some properties of neighbourhoods defined by Thorp [7]. Thorp introduced some g-topologies in a statistical metric space [9] and he studied the properties of t-function in [8]. Further, Thorp proved some results using g-topologies defined in a statistical metric space [9]. Finally, he raised some questions about the relationship between various g-topologies defined in [9].

A *statistical metric space* (SM space) [9] is an ordered pair (S,F) where S is a non-null set and F is a mapping from $S\times S$ into the set of distribution functions (that is, real-valued functions of a real variable which are everywhere defined, non decreasing, left-continuous and have infimum 0 and supremum 1).

The distribution function $F(p,q)$ associated with a pair of points p and q in S is denoted by F_{pq}. Moreover, $F_{pq}(x)$ represents the probability that the "distance" between p and q is less than x.

The functions F_{pq} are assumed to satisfy the following:
(SM-I) $F_{pq}(x)=1$ for all $x>0$ if and only if $p=q$.
(SM-II) $F_{pq}(0)=0$.
(SM-III) $F_{pq}=F_{qp}$.
(SM-IV) If $F_{pq}(x)=1$ and $F_{qr}(y)=1$, then $F_{pr}(x+y)=1$.

We often find it convenient to work with the tails of the distribution functions rather than with these distribution functions themselves. Then the *tail* [9] of F_{pq}, denoted by G_{pq}, is defined by $G_{pq}(x)=1-F_{pq}(x)$ for all $x\in\mathbb{R}$.

Let (S,F) be a statistical metric space. Then the menger inequality is,
(SM-IVm) $F_{pr}(x+y)\geq T(F_{pq}(x),F_{qr}(y))$ holds for all points $p,q,r\in S$ and for all numbers $x,y\geq 0$ where T is a 2-place function on the

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unit square satisfying:
(T-I) $0 \leq T(a,b) \leq 1$ for all $a,b,c \in [0,1]$.
(T-II) $T(c,d) \geq T(a,b)$ if $c \geq a, d \geq b$ (monotonicity) for all $a,b,c,d \in [0,1]$.
(T-III) $T(a,b) = T(b,a)$ (commutativity) for all $a,b \in [0,1]$.
(T-IV) $T(1,1) = 1$.
(T-V) $T(a,1) > 0$ for all $a > 0$.

2. Preliminaries

In this section, we recall some basic definitions in [9] and give some examples for these definitions in a statistical metric space.

Let (S,F) be a statistical metric space, $p \in S$ and u,v be positive numbers. Then $N_p(u,v) = \{q \in S \mid F_{pq}(u) > 1-v\} = \{q \in S \mid G_{pq}(u) < v\}$ [9] is called the (u,v)-*sphere* with center p.

The following Example 2.1 shows that the existence of (u,v)-sphere in a statistical metric space.

Example 2.1. Consider the SM space (S,F) where S denotes the possible outcomes of getting a tail when a coin is tossed once. Then $S = \{0,1\}$. Here $F_{pq}(u)$ is the probability that the "distance" between p and q is less than u where $u > 0$ and $p,q \in S$. Fix $p = 0$. Then

$$N_p(u,v) = \begin{cases} S & if \quad 0<u<1, v>1, \\ S & if \quad u>1, v>0, \\ \{0\} & if \quad 0<u<1, 0<v<1, \\ S & if \quad u>1, v>1. \end{cases}.$$

Fix $p = 1$. Then

$$N_p(u,v) = \begin{cases} S & if \quad 0<u<1, v>1, \\ S & if \quad u>1, v>0, \\ \{1\} & if \quad 0<u<1, 0<v<1, \\ S & if \quad u>1, v>1. \end{cases}.$$

For fixed positive numbers u and v, define $U(u,v)$ [9] by $U(u,v) = \{(p,q) \in S \times S \mid G_{pq}(u) < v\}$.

Example 2.2. Consider the SM space (S,F) where S denotes the possible outcomes of rolling a dice. Then $S = \{1,2,3,4,5,6\}$ and the distribution function $F_{pq}(x)$ is the probability that the "distance" between p and q is less than u where $u > 0$ and $p,q \in S$. Then $U(u,v) = \{(p,q) \in S \times S : p = q\}$ for $0 < u < 1, 0 < v < 1$. For $1 < u < 2, 0 < v < 1$, $U(u,v) = \{(p,q) \in S \times S : d(p,q) \leq 1\}$. For $2 < u < 3, 0 < v < 1$, $U(u,v) = \{(p,q) \in S \times S : d(p,q) \leq 2\}$. For $3 < u < 4, 0 < v < 1$, $U(u,v) = \{(p,q) \in S \times S : d(p,q) \leq 3\}$. For $4 < u < 5, 0 < v < 1$, $U(u,v) = \{(p,q) \in S \times S : d(p,q) \leq 4\}$. For $u > 5, 0 < v < 1$, $U(u,v) = \{(p,q) \in S \times S : d(p,q) \leq 5\} = S \times S$. Now $0 < u < 1, v > 1$. Then $U(u,v) = S \times S$. Also, $U(u,v) = S \times S$, for $u > 1, v > 1$.

For any set Z of ordered pairs of positive numbers, $\mathscr{N}(Z) = \{N_p(u,v) \mid (u,v) \in Z, p \in S\}$ and $\mathscr{U}(Z) = \{U(u,v) \mid (u,v) \in Z\}$.

A non-null collection $\{N_p\}$ of subsets $\mathscr{N}(Z)$ in a set S associated with a point $p \in S$ is a family of neighbourhoods for p if each N_p contains p. Let the family of neighbourhoods be associated with each point p of a set S. The set S and the collection of neighbourhoods is called the *g-topological space of typeV* [9].

Using the following conditions, Thorp [9] introduced new g-topologies in a statistical metric space (S,F).

N_0. *typeV*.
N_1. For each point p and each neighbourhood U_p of p, there is a neighbourhood W_p of p such that for each point q of W_p, there is a neighbourhood U_q of q contained in U_p.
N_2. For each point p and each pair of neighbourhoods U_p and W_p of p, there is a neighbourhood of p contained in the intersection of U_p and W_p.

The following are various g-topologies in a statistical metric space (S,F) defined by Thorp [9].
(a) If the conditions N_0 and N_2 are satisfied, then the collection of neighbourhoods on S is called the *g-topology of typeV*$_D$.
(b) The collection of neighbourhoods on S is called the *g-topology of typeV*$_\alpha$ if the conditions N_0 and N_1 are satisfied.
(c) A g-topology is a *topology* if the conditions N_0, N_1 and N_2 are satisfied.

Let S be a set and P be a partially ordered $(<)$ set with least element 0. A *generalized écart* [9] (g-écart for short) is a mapping G from $S \times S$ into P. If a g-écart G satisfies $G(p,p) = 0$ and the set S consists of more than one point, the g-écart g-topology for S is the g-topology determined from G, and its partially ordered range set P, as follows. For each $f > 0$ in P and each $p \in S$, the f-sphere for p is $N_p(f) = \{q \in S \mid G(p,q) < f\}$. Then for each $p \in S$, the collection of f-spheres, $\mathscr{N}_p(P) = \{N_p(f) \mid f > 0 \text{ in } P\}$ is a family of neighbourhoods for p.

The g-écart associated with a statistical metric space (S,F) is the mapping G defined by $G(p,q) = G_{pq}$ [9].

Example 2.3. Let $S = \mathbb{N}$ and $P = \mathbb{N} \cup \{0\}$ be a partially ordered set with the relation $<$ where $\mathbb{N}$ denote the set of all natural numbers. Let $A = \{1,2,3\}$ be a subset of S. Define

$$G(p,q) = \begin{cases} 1 & if \quad p \notin A, q \in S, \\ 1 & if \quad p \in S, q \notin A, \\ \{0\} & if \quad p \notin A, q \notin A, \end{cases}.$$

and $p \in A, q \in A$ define $G(p,q)$ as follows: $G(1,1) = 0, G(1,2) = 2, G(1,3) = 3, G(2,1) = 4, G(2,2) = 0, G(2,3) = 6, G(3,1) = 1, G(3,2) = 2, G(3,3) = 0$. Case 1: $p \notin A, q \in S$. Then $G(p,q) = 1$. Let $f = 1$. Then $N_p(1) = \emptyset$. For $f \geq 2, N_p(f) = S$. Case 2: $p \in S, q \notin A$. Then

$G(p,q)=1$. Let $f=1$. Then $N_p(1)=\emptyset$ and $N_p(f)=S-A$ for $f\geq 2$. Case 3: $p\notin A, q\notin A$. Then $G(p,q)=0$ and so $N_p(f)=S-A$ for $f>0$. Case 4: $p\in A, q\in A$. Then $N_1(1)=N_1(2)=\{1\}; N_1(3)=\{1,2\}; N_1(f)=A$ for $f\geq 4$. Now $N_2(1)=N_2(2)=N_2(3)=N_2(4)=\{2\}; N_2(5)=N_2(6)=\{1,2\}; N_2(f)=A$ for $f\geq 7$ and $N_3(1)=\{3\}; N_3(2)=\{1,3\}; N_3(f)=A$ for $f\geq 3$.

Given a statistical metric space (S,F), for each pair of points p and r in S, the r-sphere with center p, $N_p(r)$ is defined to be the sphere $N_p(G_{pr})=\{q\mid G_{pq}<G_{pr}\}$. The *R-g-topology* [9] for (S,F) is the structure whose family of neighbourhoods at each point p is the collection $\mathscr{N}_p(r)=\{N_p(r)\mid r\in S\}$.

Example 2.4. Consider the SM space (S,F) where $S=\mathbb{N}$ and the distribution function

$$F_{pq}(x)=\begin{cases}\frac{x}{d(p,q)} & if \quad 0<x<d(p,q), d(p,q)\neq 0\\ 1 & if \quad x\geq d(p,q)\end{cases}.$$

Fix $p=1$ and $r=2$ are in S. Let $x=\frac{1}{4}$. Then $G_{pr}(x)=0.75$. Now $N_1(0.75)=\{1\}$.

Observation 2.5. *In a statistical metric space, $N_p(G_{pr})=\emptyset$ if $p=r$.*

Notations 2.6. *In a SM space (S,F), we use the following notations:*
(a) Let τ denote the g-topology of $typeV$.
(b) Let τ_D denote the g-topology of $typeV_D$.
(c) Let τ_α denote the g-topology of $typeV_\alpha$.
(d) Let τ_e denote the g-écart g-topology.
(e) Let τ_R denote the R-g-topology.
(f) Each element in $\mathscr{N}(X)$ is called a τ-neighborhood.
(f) Each element in $\mathscr{N}_p(P)$ is called a τ_e-neighborhood.
(f) Each element in $\mathscr{N}_p(r)$ is called a τ_R-neighborhood.

3. Behaviour of various g-topology

In this section, we give some properties and find the relations between four types of neighborhoods in a SM space. Also, we give the answer for some of the questions raised by Throp [9].

Theorem 3.1. *Let (S,F) be a statistical metric space. Then the following hold.*
(a) *If $u_1\leq u$ and $v_1\leq v$, then $N_p(u_1,v_1)\subset N_p(u,v)$ where $u,v,u_1,v_1>0$.*
(b) *If $\kappa=\{\mathscr{N}(Z),\mathscr{U}(Z),\mathscr{N}_p(P),\mathscr{N}_p(r)\}$ and $A\in\kappa$, then there exist $B,C\in\kappa$ such that $B\subset A\subset C$.*

Proof.

(a) Let $q\in N_p(u_1,v_1)$. Then $F_{pq}(u_1)>1-v_1$. Since $u_1\leq u$ and $v_1\leq v$, $F_{pq}(u)\geq F_{pq}(u_1)>1-v_1\geq 1-v$. Thus, $F_{pq}(u)>1-v$. Therefore, $q\in N_p(u,v)$. Hence $N_p(u_1,v_1)\subset N_p(u,v)$.

(b) We give the detailed proof only for $\kappa=\mathscr{N}(Z)$ and $\kappa=\mathscr{N}_p(r)$. Suppose that $\kappa=\mathscr{N}(Z)$ and $A\in\kappa$. Then $A=N_p(u,v)$ where $u,v>0$. Take $0<u_1\leq u, 0<v_1\leq v$ and $B=N_p(u_1,v_1)$. By (a), $B\subset A$. If $u_2\geq u, v_2\geq v$, then $u_2>0, v_2>0$. Define $C=N_p(u_2,v_2)$. By (a), $A\subset C$. Thus, there exist $B,C\in\mathscr{N}(Z)$ such that $B\subset A\subset C$.

Suppose that $\kappa=\mathscr{N}_p(r)$. Let $A\in\kappa$. Then $A=\{q\in S\mid G_{pq}<G_{pr}\}$ and so $A=\{q\in S\mid G_{pq}(u)<G_{pr}(u)\}$ where $u>0$. Take $u_1\geq u$. Define $B=\{q\in S\mid G_{pq}(u)<G_{pr}(u_1)\}$. Then $B\in\mathscr{N}_p(r)$. Let $s\in B$. Then $G_{ps}(u)<G_{pr}(u_1)$ and so $G_{ps}(u)<G_{pr}(u)$, since $G_{pr}(u_1)\leq G_{pr}(u)$. Therefore, $s\in A$. Hence $B\subset A$. Define $C=\{q\in S\mid G_{pq}(u_1)<G_{pr}(u)\}$. Then $C\in\mathscr{N}_p(r)$. Let $s\in A$. Then $G_{ps}(u)<G_{pr}(u)$ and so $G_{ps}(u_1)<G_{pr}(u)$, since $G_{ps}(u_1)\leq G_{ps}(u)$. Therefore, $s\in C$. Hence $A\subset C$.

□

From the definition of g-topology of $typeV_D$, it is observed that every g-topological space of $typeV_D$ is a g-topological space of $typeV$. The following Theorem 3.2 discusses the converse of the question that "What are the necessary and sufficient conditions that the g-topology of $typeV$ to be of $typeV_D$?" which is raised by Thorp [9].

Theorem 3.2. *Let (S,F) be a statistical metric space. Then the following hold.*
(a) *τ satisfies N_2.*
(b) *The g-topology of $typeV$ is a g-topology of $typeV_D$.*

Proof. (a) Let U_p and W_p be two neighbourhoods of p. Then $U_p=\{q\in S\mid G_{pq}(u)<v\}$ and $W_p=\{q\in S\mid G_{pq}(u_1)<v_1\}$. Define $V_p=\{q\in S\mid G_{pq}(min(u,u_1))<min(v,v_1)\}$. Then $p\in V_p$ and so V_p is a neighbourhood of p. Since $min(u,u_1)\leq u$, $min(v,v_1)\leq v$,

we have $V_p \subset U_p$, by Theorem 3.1(a). Also, $min(u,u_1) \leq u_1$ and $min(v,v_1) \leq v_1$. Therefore, $V_p \subset W_p$, by Theorem 3.1(a). Hence $V_p \subset U_p \cap W_p$. Therefore, τ satisfies N_2.

(b) By (a) and the definition of g-topology of $typeV_D$, it follows that every g-topology of $typeV$ is a g-topology of $typeV_D$.

□

The following two questions are raised by Thorp [9].
(I) "What are the necessary and sufficient conditions that the g-topology of $typeV_\alpha$ to be the g-topology of $typeV_D$?".
(II) "What conditions are both necessary and sufficient for the g-topology of $typeV_\alpha$ to be a topology?".
The following Corollary 3.3 (a) gives a necessary condition for the given space to be a g-topological space of $typeV_D$ which also gives a partial answer to the question (I) and Corollary 3.3 (b) gives the answer to the question (II).

Corollary 3.3. *Let (S,F) be a statistical metric space. Then the following hold.*
(a) *The g-topology of $typeV_\alpha$ is a g-topology of $typeV_D$.*
(b) *The g-topology of $typeV_\alpha$ is a topology and conversely.*

Proof. (a) By the definition of g-topology of $typeV_\alpha$, g-topology of $typeV_\alpha$ is a g-topology of $typeV$. Therefore, g-topology of $typeV_\alpha$ is a g-topology of $typeV_D$, by Theorem 3.2(b).

(b) By the definition of g-topology of $typeV_\alpha$, g-topology of $typeV_\alpha$ is a g-topology of $typeV$ and satisfies the condition N_1. By (a), g-topology of $typeV_\alpha$ satisfies the condition N_2, by the definition g-topology of $typeV_D$. Hence a g-topology of $typeV_\alpha$ is a topology. Converse follows from the definition of topology in a statistical metric space.

□

Theorem 3.4. *Let (S,F) be a statistical metric space. If $U \in \mathscr{U}(Z)$, then there exists $V \subset S$ such that $V \in \mathscr{N}(Z)$.*

Proof. Let $U \in \mathscr{U}(Z)$. Define $V = \{q \in S \mid (p,q) \in U\}$. Since $U \in \mathscr{U}(Z), V = \{q \in S \mid G_{pq}(u) < v\}$. Hence $V = N_p(u,v)$, by the definition of $N_p(u,v)$. Therefore, $V \in \mathscr{N}(Z)$. □

Theorem 3.5. *Let (S,F) be a statistical metric space. Then the following hold.*
(a) $\tau_e \subset \tau$.
(b) τ_e satisfies N_2.

Proof. (a) Let $p \in S$ and U be a τ_e-neighbourhood of a point p. Then $U = \{q \in S \mid G(p,q) < f\}$. Since in a statistical metric space $G(p,q) = G_{pq}, U = \{q \in S \mid G_{pq} < f\}$. Here $0 < f \in P$ where P is a partially ordered set. Then there is an element $g \in P$ such that $g < f$. Take $g(u) = v$ for all $u > 0$. Then $v > 0$. Define $V = \{q \in S \mid G_{pq}(u) < v\}$. Then V is a τ-neighbourhood such that $p \in V \subset U$. Hence $\tau_e \subset \tau$.

(b) Suppose that U_p and W_p are two neighbourhoods of p. Then $U_p = \{q \in S \mid G(p,q) < f_1\}$ and $W_p = \{q \in S \mid G(p,q) < f_2\}$. Consider $V_p = \{q \in S \mid G(p,q) < min(f_1,f_2)\}$. Then $p \in V_p$ and so V_p is a neighbourhood of p. Also, $V_p \subset U_p \cap W_p$. Therefore, τ_e satisfies N_2. □

The following Corollary 3.6 gives a necessary condition for the given space to be a g-topological space of $typeV_D$ which also gives a partial answer to the question that "What are the necessary and sufficient conditions for τ_e to be of $typeV_D$?" raised by Thorp [9].

Corollary 3.6. *Let (S,F) be a statistical metric space. Then $\tau_e \subset \tau_D$.*

Proof. Follows from Theorem 3.5 and the definition of $typeV_D$. □

Theorem 3.7. *Let (S,F) be a statistical metric space. Then the following hold.*
(a) τ_R satisfies N_2.
(b) $\tau_R \subset \tau$.

Proof. (a) Let U_p and W_p be τ_R-neighbourhoods of p. Then $U_p = \{q \in S \mid G_{pq} < G_{pr_1}\}$ and $W_p = \{q \in S \mid G_{pq} < G_{pr_2}\}$ where $r_1, r_2 \in S$. Define $V_p = \{q \in S \mid G_{pq} < inf(G_{pr_1}, G_{pr_2})\}$. It follows that V_p is a neighbourhood of p and $p \in V_p$. Also, $V_p \subset U_p \cap W_p$. Therefore, τ_R satisfies N_2.

(b) If $t \in S$ and B is a τ_R-neighbourhood of t, then $B = \{q \mid G_{pq} < G_{pr}\}$ and so $B = \{q \mid G_{pq}(u) < G_{pr}(u)\}$ where $u > 0$. Choose an element v such that $u < v$. Take $v_1 = G_{pr}(v)$. Since $B \neq \emptyset$ we have $p \neq r$ and so $G_{pr}(v) \neq 0$ so that $v_1 > 0$. Define $B_1 = \{q \mid G_{pq}(u) < v_1\}$. Then $t \in B_1$ and B_1 is a τ-neighbourhood contained in B. Hence $\tau_R \subset \tau$.

□

The following Corollary 3.8 gives a necessary condition for the given space to be a g-topological space of $typeV_D$ which also gives a partial answer to the question that "What are the necessary and sufficient conditions for the R-g-topology to be g-topology of $typeV_D$?" raised by Thorp in [9].

Corollary 3.8. *Let (S,F) be a statistical metric space. Then $\tau_R \subset \tau_D$.*

Proof. Follows from Theorem 3.7 and the definition of $typeV_D$. □

Lemma 3.9. *A function $T : I \times I \to I$ is defined by $T(x,y) = max(x,y)$ where $I = [0,1]$. Then T satisfies the conditions (T-II) and (T-IV).*

Proof. (a) Suppose that $c \geq a, d \geq b$ where $a,b,c,d \in [0,1]$. Now $T(c,d) = max(c,d)$ and $T(a,b) = max(a,b)$. Case-1: If $T(c,d) = c$ and $T(a,b) = a$, then $T(c,d) \geq T(a,b)$. Case-2: Suppose $T(c,d) = c$ and $T(a,b) = b$. Since $b \leq d \leq c$, $T(c,d) \geq T(a,b)$. Case-3: If $T(c,d) = d$ and $T(a,b) = b$, then $T(c,d) \geq T(a,b)$. Case-4: Suppose $T(c,d) = d$ and $T(a,b) = a$. Since $a \leq c \leq d$, $T(c,d) \geq T(a,b)$. Therefore, T satisfies the condition (T-II).

(b) Now $T(1,1) = max(1,1) = 1$. Hence T satisfies the condition (T-IV).

□

The following Theorem 3.10 gives the answer to the question that "What are the necessary and sufficient conditions for the g-topology of $typeV$ to be a topology?" raised by Thorp in [9].

Theorem 3.10. *Let (S,F) be a statistical metric space with the g-topology of typeV. If SM-IVm satisfies under $T : I \times I \to I$ defined by $T(x,y) = max(x,y)$, then the g-topology on S is a topology and conversely.*

Proof. Given that (S,F) is a statistical metric space with a g-topology of $typeV$. Then by Theorem 3.2, N_0 and N_2 are satisfied. Let $p \in S$ and U_p be a neighbourhood for p. Then $U_p = \{r \in S \mid F_{pr}(u) > 1 - v\}$. Choose $u_1 = \frac{u}{2}$ and $v_1 < v$ with $0 \leq v_1 \leq 1$. Taking $W_p = \{s \in S \mid F_{ps}(u_1) > 1 - v_1\}$, we get that W_p is a neighbourhood of p. For $q \in W_p$, define $V_q = \{t \in S \mid F_{qt}(u_1) > 1 - v_1\}$ so that V_q is a neighbourhood of q. Since $q \in W_p$, $F_{pq}(u_1) > 1 - v_1$ and so $F_{qp}(u_1) > 1 - v_1$, by the condition (SM-III). Hence $p \in V_q$. If $a \in V_q$, then $F_{qa}(u_1) > 1 - v_1$. Since $p \in V_q, F_{qp}(u_1) > 1 - v_1$. By Lemma 3.9, T satisfies the condition (T-II). Thus, $T(F_{pq}(u_1), F_{qa}(u_1)) \geq T(1 - v_1, 1 - v_1)$. By (SM-IVm), $F_{pa}(u) \geq T(F_{pq}(u_1), F_{qa}(u_1))$, since $u_1 = \frac{u}{2}$ which implies that $F_{pa}(u) \geq T(1 - v_1, 1 - v_1)$ which in turn implies that $F_{pa}(u) \geq 1 - v_1$, by hypothesis. Hence $F_{pa}(u) > 1 - v$ and so $a \in U_p$. Therefore, $V_q \subset U_p$ and consequently N_1 is satisfied. Thus, g-topology of $typeV$ is a topology. Converse part follows from the definition of topology in a statistical metric space. □

The following Corollary 3.11 gives the answer to the question "What are the necessary and sufficient conditions for the g-topology of $typeV_D$ to be a topology?" raised by Thorp [9].

Corollary 3.11. *Let (S,F) be a statistical metric space with the g-topology of $typeV_D$. If SM-IVm satisfies under a function $T : I \times I \to I$ defined by $T(x,y) = max(x,y)$, then the g-topology of $typeV_D$ is a topology and conversely.*

Proof. By the definition of $typeV_D$, it follows that it is of $typeV$. By hypothesis and Theorem 3.10, g-topology of $typeV_D$ is a topology. Converse follows from the definition of topology in a statistical metric space. □

The following Corollary 3.12 gives a sufficient condition for g-topology of $typeV_D$ to be a g-topology of $typeV_\alpha$ which also gives a partial answer to the question "What conditions are both necessary and sufficient for the g-topology of $typeV_\alpha$ to be of $typeV_D$?" raised by Thorp in [9].

Corollary 3.12. *Let (S,F) be a statistical metric space with the g-topology of $typeV_D$. If SM-IVm satisfies under a function $T : I \times I \to I$ defined by $T(x,y) = max(x,y)$, then g-topology of $typeV_D$ is a g-topology of $typeV_\alpha$.*

Proof. By the definition of $typeV_D$, g-topology of $typeV_D$ is of $typeV$. As in the proof of Theorem 3.10, $typeV$ satisfies the condition N_1. Therefore, g-topology of $typeV_D$ is a g-topology of $typeV_\alpha$. □

The following Theorem 3.13 gives a necessary condition for the g-écart-g-topology to be a topology which also gives a partial answer to the questions "What are the necessary and sufficient conditions for the g-écart-g-topology to be a topology?" raised by Thorp [9].

Theorem 3.13. *Let (S,F) be a statistical metric space with g-écart-g-topology. If SM-IVm holds under a function T satisfying T-IV, T-II and $sup_{x<1}T(x,x) = 1$, then the g-écart g-topology is a topology on S.*

Proof. By Corollary 3.6, g-écart g-topology is a g-topology of $typeV_D$ and hence the conditions N_0 and N_2 are satisfied. Let $p \in S$ and U_p be a neighbourhood of p. Then $U_p = \{r \in S \mid G_{pr} < f\}$. Let f_1 be a tail with $L < f_1 < f$. If $W_p = \{s \in S \mid G_{ps} < f_1\}$, then W_p is a neighbourhood of p. Choose $q \in W_p$ and take $V_q = \{t \in S \mid G_{qt} < f_1\}$. Then V_q is a neighbourhood of q. Since $q \in W_p$, $G_{pq} < f_1$ and so $G_{qp} < f_1$ which implies that $p \in V_q$ which in turn implies that $G_{qp} < f_1$ and hence $F_{qp}(x) > 1 - f_1(x)$. Let $m \in V_q$. Then $G_{qm} < f_1$ and so $F_{qm}(x) > 1 - f_1(x)$. By T-II, $T(F_{pq}(x), F_{qm}(x)) \geq T(1 - f_1(x), 1 - f_1(x))$. Also, $F_{pm}(2x) \geq T(F_{pq}(x), F_{qm}(x))$, by SM-IVm. Hence it suffices to find a f_1 such that $T(1-f_1(x),1-f_1(x)) \geq 1-f_1(2x)$ for some x. Since $f > L$, there exists $a > 0$ such that $1-f(2a) < 1$. By hypothesis, there is a number $b < 1$ such that $T(b,b) > 1-f(2a)$. Now we define $f_1(x)$ using a and b by

$$f_1(x) = \begin{cases} 0 & \text{if} \quad x > a, \\ 1-b & \text{if} \quad 0 < x \leq a. \end{cases}$$

If $x > a$, then $T(1-f_1(x),1-f_1(x)) = T(1,1)$. Again, using T-IV, $T(1-f_1(x),1-f_1(x)) = 1$. Therefore, $T(1-f_1(x),1-f_1(x)) \geq 1-f(2x)$. If $0 < x \leq a$, then $T(1-f_1(x),1-f_1(x)) = T(b,b) > 1-f(2a) \geq 1-f(2x)$, since f is a left continuous function. Thus, $T(1-f_1(x),1-f_1(x)) > 1-f(2x)$ for $0 < x \leq a$. Hence $F_{pm}(2x) > 1-f(2x)$ for $0 < x \leq a$. Thus, $m \in U_p$ so that $V_q \subset U_p$. Therefore, N_1 is satisfied and hence g-écart g-topology is a topology. □

Theorem 3.14 below gives a necessary condition for an R-g-topology to be a topology which also gives a partial answer to the question "What are the necessary and sufficient conditions for the R-g-topology to be a topology?" raised by Thorp in [9].

Theorem 3.14. *Let (S,F) be a statistical metric space with R-g-topology. If SM-IVm satisfies under a function $T: I \times I \to I$ defined by $T(x,y) = max(x,y)$, then the R-g-topology is a topology.*

Proof. By hypothesis and Corollary 3.8, R-g-topology is a g-topology of $typeV_D$ and hence the conditions N_0 and N_2 are satisfied. Let $p \in S$ and U_p be a neighbourhood for p. Then $U_p = \{s \in S \mid G_{ps} < G_{pr}\}$. Take $0 < c \leq 1$ and define $W_p = \{t \in S \mid G_{pt} < cG_{pr}\}$. Then W_p is a neighbourhood of p. If $q \in W_p$, then $G_{pq} < cG_{pr}$ and so $G_{qp} < cG_{pr}$. Hence $p \in \{u \in S \mid G_{qu} < cG_{pr}\}$. Take $V_q = \{u \in S \mid G_{qu} < cG_{pr}\}$. Then $p \in V_q$ and V_q is a neighbourhood of q. If $n \in V_q$, then $G_{qn} < cG_{pr}$ and so $G_{qn} < G_{pr}$ so That $F_{qn} > F_{pr}$. Since $p \in V_q$, $G_{qp} < cG_{pr} < G_{pr}$ and hence $F_{qp} > F_{pr}$. By SM-IVm, $F_{pn}(x) \geq T(F_{pq}(0),F_{qn}(x)) = Max(0,F_{qn}(x)) = F_{qn}(x)$, by hypothesis and SM-II. Thus, $F_{pn}(x) \geq F_{qn}(x)$ so that $F_{pn}(x) > F_{pr}(x)$ and hence $G_{pn} < G_{pr}$. Therefore, $n \in U_p$ and so $V_q \subset U_p$. Thus, N_1 is satisfied. Therefore, the R-g-topology is a topology. □

In [6], Min introduced stack as in the following way: A collection $\mathscr{C}$ of subsets of S is called a *stack* [6] if $A \in \mathscr{C}$ whenever $B \in \mathscr{C}$ and $B \subset A$. Also, he analyzes whether a neighbourhood collections are stack or not in generalized topological spaces. Here we prove that different types of the neighbourhood collections become stack in statistical metric spaces.

Theorem 3.15. *Let (S,F) be a statistical metric space. Then $\mathscr{N}(Z)$ is a stack.*

Proof. Let $A \in \mathscr{N}(Z)$ and $A \subseteq B$. Then $A = \{q \in S \mid G_{pq}(u) < v\}$. Take $u_1 > u$ and

$$v_1 = \begin{cases} v & \text{if} \quad s \in A, \\ G_{ps}(u) & \text{if} \quad s \in B-A, \\ G_{ps}(u_1) & \text{if} \quad s \in S-B. \end{cases}$$

Then $u_1 > 0$ and $v_1 > 0$. If $U = \{q \in S \mid G_{pq}(u_1) < v_1\}$, then $U \in \mathscr{N}(Z)$. Choose $t \in B$. Then $t \in A$ or $t \in B-A$. Suppose $t \in A$. Then $G_{pt}(u) < v$. Since $u_1 > u$, $G_{pt}(u_1) < G_{pt}(u)$ which implies that $G_{pt}(u_1) < v = v_1$ and hence $t \in U$. If $t \in B-A$, then $G_{pt}(u) > v$. Since $u_1 > u$, $G_{pt}(u_1) < G_{pt}(u) = v_1$ and so $t \in U$. Hence $B \subset U$. Let $s \in U$. Then $G_{ps}(u_1) < v_1$. By the definition of v_1, $s \in A$ or $s \in B-A$. This implies that $s \in B$ which implies that $U \subset B$. Therefore, $B = U$. Since $U \in \mathscr{N}(Z), B \in \mathscr{N}(Z)$. Hence $\mathscr{N}(Z)$ is a stack. □

Theorem 3.16. *Let (S,F) be a statistical metric space. Then $\mathscr{U}(Z)$ is a stack.*

Proof. Let $A \in \mathscr{U}(Z)$ and $A \subseteq B$. Then $A = \{(p,q) \in S \times S \mid G_{pq}(u) < v\}$. Take $u_1 > u$ and

$$v_1 = \begin{cases} v & \text{if} \quad (p,q) \in A, \\ G_{pq}(u) & \text{if} \quad (p,q) \in B-A, \\ G_{pq}(u_1) & \text{if} \quad (p,q) \in S-B. \end{cases}$$

Then u_1 and $v_1 > 0$. Define $U = \{(p,q) \in S \times S \mid G_{pq}(u_1) < v_1\}$ so that $U \in \mathscr{U}(Z)$. If $(s,t) \in B$, then $(s,t) \in A$ or $(s,t) \in B-A$. If $(s,t) \in A$, then $G_{st}(u) < v$. Since $u_1 > u, G_{st}(u_1) < G_{st}(u)$ which implies that $G_{st}(u_1) < v = v_1$ and hence $(s,t) \in U$. Suppose that $(s,t) \in B-A$. Then $G_{st}(u) > v$. Since $u_1 > u, G_{st}(u_1) < G_{st}(u) = v_1$ and so $(s,t) \in U$. Hence $B \subset U$. Let $(l,m) \in U$. Then $G_{lm}(u_1) < v_1$. By definition of $v_1, (l,m) \in A$ or $(l,m) \in B-A$. This implies that $(l,m) \in B$ which implies that $U \subset B$. Therefore, $B = U$. Since $U \in \mathscr{U}(Z), B \in \mathscr{U}(Z)$. Hence $\mathscr{U}(Z)$ is a stack. □

Theorem 3.17. *Let (S,F) be a statistical metric space. Then $\mathscr{N}_p(P)$ is a stack.*

Proof. Let $A \in \mathscr{N}_p(P)$ and $A \subseteq B$. Then $A = \{q \in S \mid G(p,q) < f\}$. In a statistical metric space, $G_{pq} = G(p,q)$ so that $A = \{q \in S \mid G_{pq}(u) < f(u)\}$ where $u > 0$. Take $u_1 > u$ and

$$f_1(u_1) = \begin{cases} f(u) & \text{if} \quad s \in A, \\ G_{ps}(u) & \text{if} \quad s \in B-A, \\ G_{ps}(u_1) & \text{if} \quad s \in S-B. \end{cases}$$

Define $U = \{q \in S \mid G(p,q) < f_1\}$. Then $U \in \mathscr{N}_p(P)$. Since (S,F) is a statistical metric space, $U = \{q \in S \mid G_{pq}(u_1) < f_1(u_1)\}$. Let $t \in B$. Then $t \in A$ or $t \in B-A$. If $t \in A$, then $G_{pt}(u) < f(u)$. Since $u_1 > u$, $G_{pt}(u_1) < G_{pt}(u)$ which implies that $G_{pt}(u_1) < f(u) = f_1(u_1)$ and hence $t \in U$. If $t \in B-A$, then $G_{pt}(u) > v$. Since $u_1 > u$, $G_{pt}(u_1) < G_{pt}(u) = f_1(u_1)$ and so $t \in U$. Hence $B \subset U$. Let $s \in U$. Then $G_{ps}(u_1) < f_1(u_1)$. By definition of $v_1, s \in A$ or $s \in B-A$. This implies that $s \in B$ which implies that $U \subset B$. Therefore, $B = U$ and so $B \in \mathscr{N}_p(P)$, since $U \in \mathscr{N}_p(P)$. Hence $\mathscr{N}_p(P)$ is a stack. □

The following Theorem 3.18 shows that a neighbourhood collection $\mathscr{N}(Z)$ is closed under finite intersection in a statistical metric space.

Theorem 3.18. *Let (S,F) be a statistical metric space and $\kappa = \{\mathscr{N}(Z), \mathscr{N}_p(P)\}$. If $W_1, W_2, \ldots, W_n \in Q$ with $W_1 \cap W_2 \cap \ldots \cap W_n \neq \emptyset$, then $W_1 \cap W_2 \cap \ldots \cap W_n \in Q$ where $Q \in \kappa$.*

Proof. We will give a detailed proof only for $Q = \mathscr{N}(Z)$ where $Q \in \kappa$. Suppose that $V_1, V_2, \ldots, V_n \in Q$ with $V_1 \cap V_2 \cap \ldots \cap V_n \neq \emptyset$. Let $x \in V_1 \cap V_2 \cap \ldots \cap V_n$. Then $x \in V_i$ for $i = 1$ to n. Since V_1 and V_2 are τ-neighbourhoods containing x, there exists τ-neighbourhood W_1 containing x such that $W_1 \subset V_1 \cap V_2$, by Theorem 3.2(a). Again, W_1 and V_3 are τ-neighbourhoods containing x implies that there exists a τ-neighbourhood W_2 containing x such that $W_2 \subset W_1 \cap V_3 \subset V_1 \cap V_2 \cap V_3$. Proceeding like this, we get a τ-neighbourhood W_{n-1} containing x such that $W_{n-1} \subset W_{n-2} \cap V_n \subset V_1 \cap V_2 \cap V_3 \cap \ldots \cap V_n$. Since $W_{n-1} \in Q$ and $W_{n-1} \subset V_1 \cap V_2 \cap V_3 \cap \ldots \cap V_n$ we have $V_1 \cap V_2 \cap V_3 \cap \ldots \cap V_n \in Q$, by Theorem 3.15. □

4. Relation between GMS and SM space

In this section, we find the relations between λ_Ω-open sets in generalized metric spaces and various g-topology neighbourhoods defined in statistical metric spaces. Also, we give some properties of λ_Ω-open sets, kernel and perfect kernel in generalized metric spaces.

The notion of a generalized metric space was introduced by Korczak-Kubiak et al. in [2]. Let $X \neq \emptyset$. The symbol Ω to denote the family consisting of metrics defined on subsets of X, that is, if $\rho \in \Omega$, then there exists a non-null set $A_\rho \subset X$ such that ρ is a metric on A_ρ where A_ρ is a domain of ρ and it will be denoted by $dom(\rho)$. The space (X,Ω) is called a *generalized metric space* (GMS) [2]. We will write Ω_X if we want to point out that all the metrics $\rho \in \Omega_X$ defined on X [2].

Denote λ_Ω is the family of Ω-open sets in (X,Ω), more precisely, $V \in \lambda_\Omega$ if and only if for each $x \in V$, there exist $\rho \in \Omega$ and $\varepsilon > 0$ such that $B_\rho(x,\varepsilon) \subset V$ where $B_\rho(x,\varepsilon) = \{y \in dom(\rho) : \rho(x,y) < \varepsilon\}$ [2].

Let (X,Ω) be a GMS. A *kernel* [2] of the space (X,Ω) is a finite family $\Omega_0 \subset \Omega$ with the following property: for any set $V \in \tilde{\lambda}_\Omega$, there exists $\rho \in \Omega_0$ such that $i_\rho V \neq \emptyset$. A finite family $\Omega_0 \subset \Omega$ is called a *perfect kernel* [2] of the space (X,Ω) if for any $V_1, V_2, \ldots, V_m \in \mu_\Omega$ such that $V_1 \cap V_2 \cap \ldots \cap V_m \neq \emptyset$, there exists $\rho \in \Omega_0$ such that $i_\rho(V_1 \cap V_2 \cap \ldots \cap V_m) \neq \emptyset$ [2]. Every perfect kernel is a kernel [2].

A GMS (X,Ω) is said to be a *weakly complete space* [2] if there exists a kernel $\Omega_0 \subset \Omega$ consisting of complete metrics. A GMS (X,Ω) is said to be a *complete space* [2] if there exists a perfect kernel $\Omega_0 \subset \Omega$ consisting of complete metrics. Every complete space is a weakly complete space [2].

Definition 4.1. Let (X,Ω) be a generalized metric space. Then Ω is said to satisfy $\mathscr{V}$*-property* if $\sigma_1, \sigma_2 \in \Omega$ and $x, y \in X$, then $\sigma(x,y) = max\{\sigma_1(x,y), \sigma_2(x,y)\}$ is a metric and hence $\sigma \in \Omega$.

Theorem 4.2. *Let (X,Ω) be a generalized metric space. Then λ_Ω satisfies the condition N_1.*

Proof. Let $p \in X$ and U_p be a neighbourhood of p. Then $U_p \in \tilde{\lambda}_\Omega$. Since $p \in U_p$, there is a metric $\sigma_1 \in \Omega$ and $\varepsilon_1 > 0$ such that $B_{\sigma_1}(p,\varepsilon_1) \subset U_p$. Since $B_{\sigma_1}(p,\varepsilon_1) \in \tilde{\lambda}_\Omega$, for every $q \in B_{\sigma_1}(p,\varepsilon_1)$, there exist $\sigma \in \Omega$ and $\varepsilon > 0$ such that $B_\sigma(q,\varepsilon) \subset B_{\sigma_1}(p,\varepsilon_1) \subset U_p$. Therefore, every λ_Ω satisfies the condition N_1. □

Theorem 4.3. *Let (X,Ω) be a generalized metric space and Ω satisfies the $\mathscr{V}$-property. Then the following hold.*
(a) *λ_Ω satisfies N_2.*
(b) *If $W_1, W_2, \ldots, W_n \in \lambda_\Omega$ with $W_1 \cap W_2 \cap \ldots \cap W_n \neq \emptyset$, then $W_1 \cap W_2 \cap \ldots \cap W_n \in \lambda_\Omega$.*

Proof.
(a) Let $p \in X$ and $U_p, W_p \in \lambda_\Omega$. Then there exist $\sigma_1, \sigma_2 \in \Omega$ and $\varepsilon_1, \varepsilon_2 > 0$ such that $B_{\sigma_1}(p,\varepsilon_1) \subset U_p$, $B_{\sigma_2}(p,\varepsilon_2) \subset W_p$. For $y \in X$, define $\sigma_3(x,y) = max\{\sigma_1(x,y), \sigma_2(x,y)\}$. Then $\sigma_3 \in \Omega$ and $\sigma_3(x,y) \geq \sigma_1(x,y), \sigma_3(x,y) \geq \sigma_2(x,y)$. This implies that $B_{\sigma_3}(p,\varepsilon_1) \subset B_{\sigma_1}(p,\varepsilon_1)$ and $B_{\sigma_3}(p,\varepsilon_2) \subset B_{\sigma_2}(p,\varepsilon_2)$ which implies that $B_{\sigma_3}(p,\varepsilon_1) \cap B_{\sigma_3}(p,\varepsilon_2) \subset B_{\sigma_1}(p,\varepsilon_1) \cap B_{\sigma_2}(p,\varepsilon_2)$. Choose $\varepsilon = min\{\varepsilon_1, \varepsilon_2\}$ so that $\varepsilon > 0$. Then $B_{\sigma_3}(p,\varepsilon) \subset B_{\sigma_3}(p,\varepsilon_1) \cap B_{\sigma_3}(p,\varepsilon_2)$ and so $B_{\sigma_3}(p,\varepsilon) \subset B_{\sigma_1}(p,\varepsilon_1) \cap B_{\sigma_2}(p,\varepsilon_2)$. Therefore, $B_{\sigma_3}(p,\varepsilon) \subset U_p \cap W_p$. Take $V_p = B_{\sigma_3}(p,\varepsilon)$. Then V_p is a λ_Ω-neighbourhood of p such that $V_p \subset U_p \cap W_p$. Hence λ_Ω satisfies N_2.
(b) The proof is similar to that of (a).

□

Theorem 4.4. *Let (X,Ω) be a generalized metric space and Ω satisfy the $\mathscr{V}$-property. Then every kernel in (X,Ω) is a perfect kernel.*

Proof. Suppose that $\Omega_0 \subset \Omega$ is a kernel in (X,Ω). Let $W_1, W_2, W_3, \ldots, W_n \in \lambda_\Omega$ with $W_1 \cap W_2 \cap \ldots \cap W_n \neq \emptyset$. By Theorem 4.3, $W_1 \cap W_2 \cap \ldots \cap W_n \in \tilde{\lambda}_\Omega$. Since Ω_0 is a kernel, there exists a metric $\sigma_1 \in \Omega_0$ such that $i_{\sigma_1}(\cap_{i=1}^n W_i) \neq \emptyset$. Therefore, Ω_0 is a perfect kernel in (X,Ω). □

Theorem 4.5. *Let (X,Ω) be a generalized metric space and Ω satisfy the $\mathscr{V}$-property. Then (X,Ω) is a weakly complete metric space if and only if (X,Ω) is a complete metric space.*

Proof. Suppose (X,Ω) is a weakly complete metric space. Then there exists a kernel $\Omega_0 \subset \Omega$ consisting of all complete metrics on X. By Theorem 4.4, Ω_0 is a perfect kernel on X. Thus, there exists a perfect kernel $\Omega_0 \subset \Omega$ consisting of all complete metrics on X. Therefore, (X,Ω) is a complete space. Since every complete metric space is a weakly complete metric space, the converse follows. □

The following Theorem 4.6 gives the relations between λ_Ω-open sets and neighbourhoods defined in a statistical metric space.

Theorem 4.6. *Let (S,F) be a statistical metric space. If the distribution function $F_{pq}(x_i) = 1 - \sigma_i(p,q)$ for $x_i > 0, \sigma_i \in \Omega_S$ and $i \in \mathbb{N}$ where Ω_S is the collection of all metrics defined on S, then the following hold.*
(a) *Every τ-neighbourhood on S is a λ_{Ω_S}-open set.*
(b) *Every τ_e-neighbourhood on S is a λ_{Ω_S}-open set.*
(c) *Every τ_R-neighbourhood on S is a λ_{Ω_S}-open set.*

Proof.
(a) Let U be an arbitrary τ-neighbourhood on S. Then $U = \{q \in S \mid F_{pq}(u_1) > 1 - v_1\}$ where $u_1, v_1 > 0$. By hypothesis, $U = \{q \in S \mid 1 - \sigma_1(p,q) > 1 - v_1\} = \{q \in S \mid \sigma_1(p,q) < v_1\} = \{q \in S \mid q \in B_{\sigma_1}(p,v_1)\}$. Hence $U = B_{\sigma_1}(p,v_1)$ and so for each $x \in U$, there is a metric $\sigma \in \Omega_S$ and $\varepsilon > 0$ such that $B_\sigma(x,\varepsilon) \subset U$. Therefore, $U \in \lambda_{\Omega_S}$. Hence every τ-neighbourhood is a λ_{Ω_S}-open set.

(b) By Theorem 3.5, every τ_e-neighbourhood on S is a τ-neighbourhood on S. Therefore, by (a), every τ_e-neighbourhood on S is a λ_{Ω_S}-open set on S.
(c) Every τ_R-neighbourhood on S is a τ-neighbourhood on S, by Theorem 3.7. By (a), every τ_R-neighbourhood on S is a λ_{Ω_S}-open set on S.

□

Theorem 4.7. *Let (S,F) be a statistical metric space. If the distribution function $F_{pq}(x_i) = 1-\sigma_i(p,q)$ for $x_i > 0, \sigma_i \in \Omega_S$ and $i \in \mathbb{N}$, then the following hold.*
(a) Every λ_{Ω_S}-open set contains a τ-neighbourhood on S.
(b) Every λ_{Ω_S}-open set contains a τ_e-neighbourhood on S.

Proof. We will present the detailed proof only for (b). Let $A \in \tilde{\lambda}_{\Omega_S}$ and $x \in A$. Then there is a metric $\sigma_1 \in \Omega_S$ and $\varepsilon > 0$ such that $B_{\sigma_1}(x,\varepsilon) \subset A$. Let $y \in B_{\sigma_1}(x,\varepsilon)$. Then $\sigma_1(x,y) < \varepsilon$ implies that $1 - F_{xy}(u_1) < \varepsilon$ where $u_1 > 0$, by hypothesis. Take $f(u_1) = \varepsilon$. Then $F_{xy}(u_1) > 1 - f(u_1)$ and so $y \in \{z \in S \mid F_{xz}(u_1) > 1 - f(u_1)\}$. Take $U = \{z \in S \mid F_{xz}(u_1) > 1 - f(u_1)\}$. Then $U = \{z \in S \mid G_{xz}(u_1) < f(u_1)\}$ and $B_{\sigma_1}(x,\varepsilon) \subseteq U$. Since in a statistical metric space $G(p,q) = G_{pq}$, $U = \{z \in S \mid G(x,z) < f\}$. Therefore, U is a τ_e-neighbourhood on S. Let $t \in U$. Then $F_{xt}(u_1) > 1 - f(u_1)$ and so $1 - \sigma_1(x,t) > 1 - f(u_1)$, by hypothesis. This implies that $\sigma_1(x,t) < f(u_1)$ which implies that $\sigma_1(x,t) < \varepsilon$, since $f(u_1) = \varepsilon$. Therefore, $t \in B_{\sigma_1}(x,\varepsilon)$. Hence $U = B_{\sigma_1}(x,\varepsilon)$. Thus, $U \subset A$. Hence A contains a τ_e-neighbourhood on S. □

The following Theorem 4.8 shows that a collection of all λ_Ω-open sets is a stack in statistical metric spaces.

Theorem 4.8. *Let (S,F) be a statistical metric space with a g-topology ν. If the distribution function $F_{pq}(x_i) = 1-\sigma_i(p,q)$ for $x_i > 0, \sigma_i \in \Omega_S, i \in \mathbb{N}$ where $\nu \in \{\tau, \tau_e\}$, then the following hold.*
(a) *The collection λ_{Ω_S} is a stack.*
(b) *If $W_1, W_2, \ldots, W_n \in \lambda_{\Omega_S}$ with $W_1 \cap W_2 \cap \ldots \cap W_n \neq \emptyset$, then $W_1 \cap W_2 \cap \ldots \cap W_n \in \lambda_{\Omega_S}$.*

Proof. We will give a detailed proof only for $\nu = \tau$.

(a) Let $A \in \lambda_{\Omega_S}$ and $A \subset B$. By hypothesis and Theorem 4.7, A contains a τ-neighbourhood W on S. This implies that $W \subset B$ which implies that $B \in \mathscr{N}(Z)$, since $\mathscr{N}(Z)$ is stack (Theorem 3.15). Therefore, $B \in \lambda_{\Omega_S}$, by hypothesis and Theorem 4.6. Hence λ_{Ω_S} is a stack.

(b) Let $V_1, V_2, \ldots, V_n \in \lambda_{\Omega_S}$ with $V_1 \cap V_2 \cap \ldots \cap V_n \neq \emptyset$. Choose $x \in V_1 \cap V_2 \cap \ldots \cap V_n$. Then there exist $\sigma_i \in \Omega_S, \varepsilon_i > 0$ such that $B_{\sigma_i}(x,\varepsilon_i) \subset V_i$ for $i = 1$ to n and so $\cap_{i=1}^n B_{\sigma_i}(x,\varepsilon_i) \subset \cap_{i=1}^n V_i$. As in the proof of Theorem 4.7, we get that $B_i = W_i$ where $B_i = B_{\sigma_i}(x,\varepsilon_i)$ and W_i is a τ-neighbourhood on S for $i = 1$ to n. Therefore, $\cap_{i=1}^n W_i \subset \cap_{i=1}^n V_i$. By Theorem 3.18, $\cap_{i=1}^n W_i$ is a τ-neighbourhood on S. Thus, $\cap_{i=1}^n W_i$ is a λ_{Ω_S}-open set, by Theorem 4.6 and hence $\cap_{i=1}^n V_i \in \lambda_{\Omega_S}$, by (a).

□

Theorem 4.9. *Let (S,F) be a statistical metric space with a g-topology τ or τ_e. If the distribution function $F_{pq}(x_i) = 1-\sigma_i(p,q)$ for $x_i > 0, \sigma_i \in \Omega_S, i \in \mathbb{N}$ and if $\Omega_0 \subset \Omega_S$ is a kernel in (S,Ω_S), then it is a perfect kernel in (S,Ω_S).*

Proof. Let (S,F) be a statistical metric space with τ. Suppose $\Omega_0 \subset \Omega_S$ is a kernel in (S,Ω_S). Let $V_1, V_2, V_3, \ldots, V_n \in \lambda_{\Omega_S}$ with $V_1 \cap V_2 \cap \ldots \cap V_n \neq \emptyset$. By hypothesis and Theorem 4.8, $V_1 \cap V_2 \cap \ldots \cap V_n \in \tilde{\lambda}_{\Omega_S}$. Since Ω_0 is a kernel, there exists a metric $\sigma_1 \in \Omega_0$ such that $i_{\sigma_1}(\cap_{i=1}^n V_i) \neq \emptyset$. Therefore, Ω_0 is a perfect kernel in (S,Ω_S).
Let (S,F) be a statistical metric space with τ_e. By the same argument as in above and Theorem 4.8, we can prove that Ω_0 is a perfect kernel in (S,Ω_S). □

Theorem 4.10. *Let (S,F) be a statistical metric space with a g-topology τ or τ_e. If the distribution function $F_{pq}(x_i) = 1-\sigma_i(p,q)$ for $x_i > 0, \sigma_i \in \Omega_S, i \in \mathbb{N}$ and if (S,Ω_S) is a weakly complete metric space, then (S,Ω_S) is a complete metric space.*

Proof. Let (S,F) be a statistical metric space with τ. Suppose (S,Ω_S) is a weakly complete space. Then there exists a kernel $\Omega_0 \subset \Omega_S$ consisting of all complete metrics on S. By hypothesis and Theorem 4.9, Ω_0 is a perfect kernel on S. Thus, there exists a perfect kernel $\Omega_0 \subset \Omega_S$ consisting of all complete metrics on S. Therefore, (S,Ω_S) is a complete metric space.
Suppose that (S,F) is a statistical metric space with τ_e. By the same argument as in above and Theorem 4.9, we can prove that (S,Ω_S) is a complete metric space. □

5. Conclusion

This article provide the basis for carrying out analysis in statistical metric spaces, in particular for the development of various g-topologies, neighbourhoods defined in a statistical metric space and also the improvement of λ_Ω-open sets in a generalized metric space. We have given more examples of the neighbourhoods defined in a statistical metric space and the special kind of relationship between various g-topologies defined by Thorp in a SM space. Also, new properties for λ_Ω-open sets in a generalized metric space have presented. We have given partial answer to the following questions raised by Thorp in statistical metric spaces:
What are the necessary and sufficient conditions that the g-topology of $typeV$ to be of $typeV_D$?
What are the necessary and sufficient conditions that the g-topology of $typeV_\alpha$ to be the g-topology of $typeV_D$?
What conditions are both necessary and sufficient for the g-topology of $typeV_\alpha$ to be a topology?

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