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$\mathscr{I}_2$-Convergence of Double Sequences of Functions in 2-Normed Spaces

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Abstract

In this study, we introduced the concepts of $\mathscr{I}_2$-convergence and $\mathscr{I}_2^*$-convergence of double sequences of functions in 2-normed space. Also, were studied some properties about these concepts and investigated relationships between them for double sequences of functions in 2-normed spaces.

1. Introduction and Background

Throughout the paper, $\mathbb{N}$ denotes the set of all positive integers and $\mathbb{R}$ the set of all real numbers. The concept of convergence of a sequence of real numbers has been extended to statistical convergence independently by Fast [14] and Schoenberg [32]. Gökhan et al. [19] introduced the notion of pointwise and uniform statistical convergence of double sequences of real-valued functions.
The idea of $\mathscr{I}$-convergence was introduced by Kostyrko et al. [25] as a generalization of statistical convergence which is based on the structure of the ideal $\mathscr{I}$ of subset of $\mathbb{N}$ [14, 15]. Gezer and Karakuş [18] investigated $\mathscr{I}$-pointwise and uniform convergence and $\mathscr{I}^*$-pointwise and uniform convergence of function sequences and they examined the relation between them. Baláz et al. [4] investigated $\mathscr{I}$-convergence and $\mathscr{I}$-continuity of real functions. Das et al. [6] introduced the concept of $\mathscr{I}$-convergence of double sequences in a metric space and studied some properties of this convergence. Dündar and Altay [7, 9] studied the concepts of pointwise and uniformly $\mathscr{I}_2$-convergence and $\mathscr{I}_2^*$-convergence of double sequences of functions and investigated some properties about them. Furthermore, Dündar [11] investigated some results of $\mathscr{I}_2$-convergence of double sequences of functions. Also, a lot of development have been made about double sequences of functions (see [8], [10]-[12], [18], [27], [28], [34]-[36]).
The concept of 2-normed spaces was initially introduced by Gähler [16, 17] in the 1960's. Statistical convergence and statistical Cauchy sequence of functions in 2-normed space were studied by Yegül and Dündar [39]. Also, Yegül and Dündar [40] introduced concepts of pointwise and uniform convergence, statistical convergence and statistical Cauchy double sequences of functions in 2-normed space. Sarabadan and Talebi [29] presented various kinds of statistical convergence and $\mathscr{I}$-convergence for sequences of functions with values in 2-normed spaces and also defined the notion of $\mathscr{I}$-equistatistically convergence and study $\mathscr{I}$-equistatistically convergence of sequences of functions. Recently, Arslan and Dündar [1, 2] inroduced $\mathscr{I}$-convergence and $\mathscr{I}$-Cauchy sequences of functions in 2-normed spaces. Futhermore, a lot of development have been made in this area (see [3, 5, 13, 26, 30, 33]).

2. Definitions and Notations

Now, we recall the concept of density, statistical convergence, 2-normed space and some fundamental definitions and notations (See [1, 2, 4, 6, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26, 29, 30, 31, 37, 38, 40]).
Let X be a real vector space of dimension d, where $2 \leq d < \infty$. A 2-norm on X is a function $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ which satisfies the following statements:

(i) $\|x,y\| = 0$ if and only if x and y are linearly dependent.

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(ii) $\|x,y\| = \|y,x\|$.
(iii) $\|\alpha x,y\| = |\alpha|\|x,y\|$, $\alpha \in \mathbb{R}$.
(iv) $\|x,y+z\| \leq \|x,y\| + \|x,z\|$.

The pair $(X, \|\cdot,\cdot\|)$ is then called a 2-normed space. As an example of a 2-normed space we may take $X = \mathbb{R}^2$ being equipped with the 2-norm $\|x,y\| :=$ the area of the parallelogram based on the vectors x and y which may be given explicitly by the formula

$$\|x,y\| = |x_1y_2 - x_2y_1|; \;\; x = (x_1,x_2), y = (y_1,y_2) \in \mathbb{R}^2.$$

In this study, we suppose X to be a 2-normed space having dimension d; where $2 \leq d < \infty$.
Throughout the paper, we X and Y be two 2-normed spaces, $\{f_n\}_{n\in\mathbb{N}}$ and $\{g_n\}_{n\in\mathbb{N}}$ be two sequences of functions and f,g be two functions from X to Y.
The sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ is said to be convergent to f if $f_n(x) \overset{\|.,.\|_Y}{\longrightarrow} f(x)$ for each $x \in X$. We write $f_n \overset{\|.,.\|_Y}{\longrightarrow} f$. This can be expressed by the formula

$$(\forall y \in Y)(\forall x \in X)(\forall \varepsilon > 0)(\exists n_0 \in \mathbb{N})(\forall n \geq n_0)\|f_n(x) - f(x), y\| < \varepsilon.$$

A family of sets $\mathscr{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if and only if
(i) $\emptyset \in \mathscr{I}$, (ii) For each $A,B \in \mathscr{I}$ we have $A \cup B \in \mathscr{I}$, (iii) For each $A \in \mathscr{I}$ and each $B \subseteq A$ we have $B \in \mathscr{I}$.
An ideal is called nontrivial if $\mathbb{N} \notin \mathscr{I}$ and nontrivial ideal is called admissible if $\{n\} \in \mathscr{I}$ for each $n \in \mathbb{N}$.
A family of sets $\mathscr{F} \subseteq 2^{\mathbb{N}}$ is called a filter if and only if
(i) $\emptyset \notin \mathscr{F}$, (ii) For each $A,B \in \mathscr{F}$ we have $A \cap B \in \mathscr{F}$, (iii) For each $A \in \mathscr{F}$ and each $B \supseteq A$ we have $B \in \mathscr{F}$.
$\mathscr{I}$ is nontrivial ideal in $\mathbb{N}$ if and only if $\mathscr{F}(\mathscr{I}) = \{M \subset \mathbb{N} : (\exists A \in \mathscr{I})(M = \mathbb{N}\backslash A)\}$ is a filter in $\mathbb{N}$.
A nontrivial ideal $\mathscr{I}_2$ of $\mathbb{N}\times\mathbb{N}$ is called strongly admissible ideal if $\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\{i\}$ belong to $\mathscr{I}_2$ for each $i \in N$.
Throughout the paper we take $\mathscr{I}_2$ as a strongly admissible ideal in $\mathbb{N}\times\mathbb{N}$.
It is evident that a strongly admissible ideal is admissible also.
$\mathscr{I}_2^0 = \{A \subset \mathbb{N}\times\mathbb{N} : (\exists m(A) \in \mathbb{N})(i,j \geq m(A) \Rightarrow (i,j) \notin A)\}$. Then $\mathscr{I}_2^0$ is a strongly admissible ideal and clearly an ideal $\mathscr{I}_2$ is strongly admissible if and only if $\mathscr{I}_2^0 \subset \mathscr{I}_2$.
A sequence $\{f_n\}$ of functions is said to be $\mathscr{I}$-convergent (pointwise) to f on $D \subseteq \mathbb{R}$ if and only if for every $\varepsilon > 0$ and each $x \in D$,

$$\{n : |f_n(x) - f(x)| \geq \varepsilon\} \in \mathscr{I}.$$

In this case, we will write $f_n \overset{\mathscr{I}}{\to} f$ on D.
The sequence of functions $\{f_n\}$ is said to be $\mathscr{I}$-pointwise convergent to f, if for every $\varepsilon > 0$ and each nonzero $z \in Y$

$$A(\varepsilon,z) = \{n \in \mathbb{N} : \|f_n(x) - f(x), z\| \geq \varepsilon\} \in \mathscr{I},$$

or $\mathscr{I} - \lim\limits_{n\to\infty} \|f_n(x) - f(x), z\|_Y = 0$, for each $x \in X$. In this case, we write $f_n \overset{\|.,.\|_Y}{\longrightarrow}_{\mathscr{I}} f$. This can be expressed by the formula

$$(\forall z \in Y)(\forall \varepsilon > 0)(\exists M \in \mathscr{I})(\forall n_0 \in \mathbb{N}\backslash M)(\forall x \in X)(\forall n \geq n_0)\|f_n(x) - f(x), z\| \leq \varepsilon.$$

The sequence of functions $\{f_n\}$ is said to be (pointwise) $\mathscr{I}^*$-convergent to f, if there exists a set $M \in \mathscr{F}(\mathscr{I})$, (i.e., $\mathbb{N}\backslash M \in \mathscr{I}$), $M = \{m_1 < m_2 < \cdots < m_k < \cdots\}$, such that for each $x \in X$ and each nonzero $z \in Y$

$$\lim_{k\to\infty} \|f_{n_k}(x), z\| = \|f(x), z\|$$

and we write

$$\mathscr{I}^* - \lim_{n\to\infty} \|f_n(x), z\| = \|f(x), z\| \text{ or } f_n \overset{\mathscr{I}^*}{\to} f.$$

An admissible ideal $\mathscr{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ satisfies the property (AP2) if for every countable family of mutually disjoint sets $\{E_1, E_2, ...\}$ belonging to $\mathscr{I}_2$, there exists a countable family of sets $\{F_1, F_2, ...\}$ such that $E_j \Delta F_j \in \mathscr{I}_2^0$, i.e., $E_j \Delta F_j$ is included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each $j \in \mathbb{N}$ and $F = \bigcup_{j=1}^{\infty} F_j \in \mathscr{I}_2$ (hence $F_j \in \mathscr{I}_2$ for each $j \in \mathbb{N}$).
Throughout the paper, we let $\mathscr{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal, X and Y be two 2-normed spaces, $\{f_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$, $\{g_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ and $\{h_{mn}\}_{(m,n)\in\mathbb{N}\times\mathbb{N}}$ be three double sequences of functions, f, g and k be three functions from X to Y.
A double sequence $\{f_{mn}\}$ is said to be pointwise convergent to f if, for each point $x \in X$ and for each $\varepsilon > 0$, there exists a positive integer $k_0 = k_0(x,\varepsilon)$ such that for all $m,n \geq k_0$ implies $\|f_{mn}(x) - f(x), z\| < \varepsilon$, for every $z \in Y$. In this case, we write $f_{mn} \overset{\|.,.\|_Y}{\longrightarrow} f$.
A double sequence $\{f_{mn}\}$ is said to be (pointwise) statistical convergent to f, if for every $\varepsilon > 0$, $\lim\limits_{i,j\to\infty} \frac{1}{ij}\big|\{(m,n), m \leq i, n \leq j : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\}\big| = 0$, for each (fixed) $x \in X$ and each nonzero $z \in Y$. It means that for each (fixed) $x \in X$ and each nonzero $z \in Y$, $\|f_{mn}(x) - f(x), z\| < \varepsilon$, $a.a.$ (m,n). In this case, we write

$$st - \lim_{m,n\to\infty} \|f_{mn}(x) - z\| = \|f(x), z\| \;\; or \;\; f_{mn} \overset{\|.,.\|_Y}{\longrightarrow}_{st} f.$$

The double sequences of functions $\{f_{mn}\}$ is said to be statistically Cauchy sequence, if for every $\varepsilon > 0$ and each nonzero $z \in Y$, there exist a number $k = k(\varepsilon, z)$, $t = t(\varepsilon, z)$ such that $d_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f_{kt}(x), z\| \geq \varepsilon\}) = 0$, for each (fixed) $x \in X$, i.e., for each nonzero $z \in Y$, $\|f_{nm}(x) - f_{kt}(x), z\| < \varepsilon$, $a.a.$ (m,n).

3. Main Results

We introduced the concepts of $\mathscr{I}_2$-convergence and $\mathscr{I}_2^*$-convergence of double sequences of functions in 2-normed space. Also, were studied some properties about these concepts and investigated relationships between them for double sequences of functions in 2-normed spaces.

Definition 3.1. *$\{f_{mn}\}$ is said to be $\mathscr{I}_2$-convergent (pointwise sense) to f, if for every $\varepsilon > 0$ and each nonzero $z \in Y$*

$$A(\varepsilon, z) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\} \in \mathscr{I}_2,$$

for each $x \in X$. This can be expressed by the formula

$$(\forall z \in Y)\ (\forall x \in X)\ (\forall \varepsilon > 0)\ (\exists H \in \mathscr{I}_2)\ (\forall (m,n) \notin H)\ \|f_{mn}(x) - f(x), z\| < \varepsilon.$$

In this case, we write

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|, \text{ or } f_{mn} \xrightarrow{\|.,.\|_Y}_{\mathscr{I}_2} f.$$

Theorem 3.2. *For each $x \in X$ and each nonzero $z \in Y$,*

$$\lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \text{ implies } \mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$$

Proof. Let $\varepsilon > 0$ be given. Since

$$\lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$$

for each $x \in X$ and each nonzero $z \in Y$, therefore, there exists a positive integer $k_0 = k_0(\varepsilon, x)$ such that $\|f_{mn}(x) - f(x), z\| < \varepsilon$, whenever $m, n \geq k_0$. This implies that for each nonzero $z \in Y$,

$$\begin{aligned} A(\varepsilon, z) &= \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| < \varepsilon\} \\ &\subset ((\mathbb{N}\times\{1,2,..,k_0-1\}) \cup (\{1,2,..,k_0-1\}\times\mathbb{N})). \end{aligned}$$

Since $\mathscr{I}_2$ be an admissible ideal, therefore

$$((\mathbb{N}\times\{1,2,..,k_0-1\}) \cup (\{1,2,..,k_0-1\}\times\mathbb{N})) \in \mathscr{I}_2.$$

Hence, it is clear that $A(\varepsilon, z) \in \mathscr{I}_2$ and consequently, for each nonzero $z \in Y$ we have

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|.$$

□

Theorem 3.3. *If $\mathscr{I}_2$-limit of any double sequence of functions $\{f_{mn}\}$ exists, then it is unique.*

Proof. Assume that

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x_0), z\| = \|f(x_0), z\| \text{ and } \mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x_0), z\| = \|g(x_0), z\|,$$

where $f(x_0) \neq g(x_0)$ for a $x_0 \in X$ each nonzero $z \in Y$. Since $f(x_0) \neq g(x_0)$. So we may suppose that $f(x_0) \geq g(x_0)$. Now, select $\varepsilon = \frac{f(x_0)-g(x_0)}{3}$, so that neighborhoods $(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ and $(g(x_0) - \varepsilon, g(x_0) + \varepsilon)$ of points $f(x_0)$ and $g(x_0)$, respectively, are disjoints. Since for $x_0 \in X$ and each nonzero $z \in Y$

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x_0), z\| = \|f(x_0), z\| \text{ and } \mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x_0), z\| = \|g(x_0), z\|,$$

then for each nonzero $z \in Y$, we have

$$A(\varepsilon, z) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x_0) - f(x_0), z\| \geq \varepsilon\} \in \mathscr{I}_2$$

and

$$B(\varepsilon, z) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x_0) - g(x_0), z\| \geq \varepsilon\} \in \mathscr{I}_2.$$

This implies that the sets

$$A^c(\varepsilon, z) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x_0) - f(x_0), z\| < \varepsilon\}$$

and

$$B^c(\varepsilon, z) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x_0) - g(x_0), z\| < \varepsilon\}$$

belongs to $\mathscr{F}(\mathscr{I}_2)$ and $A^c(\varepsilon, z) \cap B^c(\varepsilon, z)$ is nonempty set in $\mathscr{F}(\mathscr{I}_2)$ for $x_0 \in X$ and each nonzero $z \in Y$. Since $A^c(\varepsilon, z) \cap B^c(\varepsilon, z) \neq \emptyset$, we obtain a contradiction to the fact that the neighborhoods $(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ and $(g(x_0) - \varepsilon, g(x_0) + \varepsilon)$ of points $f(x_0)$ and $g(x_0)$ respectively are disjoints. Hence, it is clear that for $x_0 \in X$ and each nonzero $z \in Y$,

$$\|f(x_0), z\| = \|g(x_0), z\|$$

and consequently, we have $\|f(x), z\| = \|g(x), z\|$, (i.e., $f = g$) for each $x \in X$ and each nonzero $z \in Y$. □

Theorem 3.4. *For each $x \in X$ and each nonzero $z \in Y$, If*

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \text{ and } \mathscr{I}_2 - \lim_{m,n\to\infty} \|g_{mn}(x), z\| = \|g(x), z\|,$$

then
(i) $\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x) + g_{mn}(x), z\| = \|f(x) + g(x), z\|$,
(ii) $\mathscr{I}_2 - \lim_{m,n\to\infty} \|cf_{mn}(x), z\| = \|cf(x), z\|$, $c \in \mathbb{R}$,
(iii) $\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x)g_{mn}(x), z\| = \|f(x)g(x), z\|$.

Proof. (i) Let $\varepsilon > 0$ be given. Since

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \text{ and } \mathscr{I}_2 - \lim_{m,n\to\infty} \|g_{mn}(x), z\| = \|g(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$, then

$$A\left(\frac{\varepsilon}{2}, z\right) = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{2}\right\} \in \mathscr{I}_2$$

and

$$B\left(\frac{\varepsilon}{2}, z\right) = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|g_{mn}(x) - g(x), z\| \geq \frac{\varepsilon}{2}\right\} \in \mathscr{I}_2$$

and by the definition of ideal we have

$$A\left(\frac{\varepsilon}{2}, z\right) \cup B\left(\frac{\varepsilon}{2}, z\right) \in \mathscr{I}_2.$$

Now, for each $x \in X$ and each nonzero $z \in Y$ we define the set

$$C(\varepsilon, z) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|(f_{mn}(x) + g_{mn}(x)) - (f(x) + g(x)), z\| \geq \varepsilon\}$$

and it is sufficient to prove that $C(\varepsilon, z) \subset A\left(\frac{\varepsilon}{2}, z\right) \cup B\left(\frac{\varepsilon}{2}, z\right)$. Let $(m,n) \in C(\varepsilon, z)$, then for each $x \in X$ and each nonzero $z \in Y$, we have

$$\varepsilon \leq \|(f_{mn}(x) + g_{mn}(x)) - (f(x) + g(x)), z\| \leq \|f_{mn}(x) - f(x), z\| + \|g_{mn}(x) - g(x), z\|.$$

As both of $\{\|f_{mn}(x) - f(x), z\|, \|g_{mn}(x) - g(x), z\|\}$ can not be (together) strictly less than $\frac{\varepsilon}{2}$ and therefore either

$$\|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{2} \text{ or } \|g_{mn}(x) - g(x), z\| \geq \frac{\varepsilon}{2},$$

for each $x \in X$ and each nonzero $z \in Y$. This shows that $(m,n) \in A\left(\frac{\varepsilon}{2}, z\right)$ or $(m,n) \in B\left(\frac{\varepsilon}{2}, z\right)$ and so we have

$$(m,n) \in A\left(\frac{\varepsilon}{2}, z\right) \cup B\left(\frac{\varepsilon}{2}, z\right).$$

Hence, we have

$$C(\varepsilon, z) \subset A\left(\frac{\varepsilon}{2}, z\right) \cup B\left(\frac{\varepsilon}{2}, z\right)$$

and so

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x) + g_{mn}(x), z\| = \|f(x) + g(x), z\|.$$

(ii) Let $c \in \mathbb{R}$ and $\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$, for each $x \in X$ and each nonzero $z \in Y$. If $c = 0$, there is nothing to prove. We assume that $c \neq 0$. Then,

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{|c|}\right\} \in \mathscr{I}_2$$

for each $x \in X$ and each nonzero $z \in Y$ and by the definition we have

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|cf_{mn}(x) - cf(x), z\| \geq \varepsilon\} = \left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \frac{\varepsilon}{|c|}\right\}.$$

Hence, the right side of above equality belongs to $\mathscr{I}_2$ and so

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|cf_{mn}(x), z\| = \|cf(x), z\|,$$

for each $x \in X$ and each nonzero $z \in Y$.
(iii) Since $\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$ for each $x \in X$ and each nonzero $z \in Y$, then for $\varepsilon = 1 > 0$,

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq 1\} \in \mathscr{I}_2,$$

and so

$$A=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-f(x),z\|<1\}\in\mathscr{F}(\mathscr{I}_2).$$

Also, for any $(m,n)\in A$, $\|f_{mn}(x),z\|<1+\|f(x),z\|$ for each $x\in X$ and each nonzero $z\in Y$. Let $\varepsilon>0$ be given. Choose $\delta>0$ such that

$$0<2\delta<\frac{\varepsilon}{\|f(x),z\|+\|g(x),z\|+1}$$

for each $x\in X$ and each nonzero $z\in Y$. It follows from the assumption that

$$B=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-f(x),z\|<\delta\}\in\mathscr{F}(\mathscr{I}_2)$$

and

$$C=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|g_{mn}(x)-g(x),z\|<\delta\}\in\mathscr{F}(\mathscr{I}_2)$$

for each $x\in X$ and each nonzero $z\in Y$. Since $\mathscr{F}(\mathscr{I}_2)$ is a filter, therefore $A\cap B\cap C\in\mathscr{F}(\mathscr{I}_2)$. Then, for each $(m,n)\in A\cap B\cap C$ we have

$$\begin{aligned}\|f_{mn}(x)g_{mn}(x)-f(x).g(x),z\| &= \|f_{mn}(x)g_{mn}(x)-f_{mn}(x)g(x)+f_{mn}(x)g(x)-f(x)g(x),z\|\\ &\leq \|f_{mn}(x),z\|\|g_{mn}(x)-g(x),z\|+\|g(x),z\|\|f_{mn}(x)-f(x),z\|\\ &< (\|f(x),z\|+1)\delta+(\|g(x),z\|)\delta\\ &= (\|f(x),z\|+\|g(x),z\|+1)\delta\\ &< \varepsilon\end{aligned}$$

and so, we have

$$\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x).g_{mn}(x)-f(x).g(x),z\|\geq\varepsilon\}\in\mathscr{I}_2,$$

for each $x\in X$ and each nonzero $z\in Y$. This completes the proof of theorem. □

Theorem 3.5. *For each $x\in X$ and each nonzero $z\in Y$, if*
(i) $\{f_{mn}\}\leq\{g_{mn}\}\leq\{h_{mn}\}$, for every $(m,n)\in K$, where $\mathbb{N}\times\mathbb{N}\supseteq K\in\mathscr{F}(\mathscr{I}_2)$
and
(ii) $\mathscr{I}_2-\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|k(x),z\|$ and $\mathscr{I}_2-\lim_{m,n\to\infty}\|h_{mn}(x),z\|=\|k(x),z\|$,
then we have

$$\mathscr{I}_2-\lim_{m,n\to\infty}\|g_{mn}(x),z\|=\|k(x),z\|.$$

Proof. Let $\varepsilon>0$ be given. By condition (ii) we have

$$\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-k(x),z\|\geq\varepsilon\}\in\mathscr{I}_2$$

and

$$\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|h_{mn}(x)-k(x),z\|\geq\varepsilon\}\in\mathscr{I}_2,$$

for each $x\in X$ and each nonzero $z\in Y$. This implies that the sets

$$P=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|f_{mn}(x)-k(x),z\|<\varepsilon\}$$

and

$$R=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|h_{mn}(x)-k(x),z\|<\varepsilon\}$$

belong to $\mathscr{F}(\mathscr{I}_2)$, for each $x\in X$ and each nonzero $z\in Y$. Let

$$Q=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|g_{mn}(x)-k(x),z\|<\varepsilon\},$$

for each $x\in X$ and each nonzero $z\in Y$. It is clear that the set $P\cap R\cap K\subset Q$. Since $P\cap R\cap K\in\mathscr{F}(\mathscr{I}_2)$ and $P\cap R\cap K\subset Q$, then from the definition of filter, we have $Q\in\mathscr{F}(\mathscr{I}_2)$ and so

$$\{(m,n)\in\mathbb{N}\times\mathbb{N}:\|g_{mn}(x)-k(x),z\|\geq\varepsilon\}\in\mathscr{I}_2,$$

for each $x\in X$ and each nonzero $z\in Y$. Hence,

$$\mathscr{I}_2-\lim_{m,n\to\infty}\|g_{mn}(x),z\|=\|k(x),z\|.$$

□

Theorem 3.6. *For each $x\in X$ and each nonzero $z\in Y$, we let*

$$\mathscr{I}_2-\lim_{m,n\to\infty}\|f_{mn}(x),z\|=\|f(x),z\|\ \text{and}\ \mathscr{I}_2\lim_{m,n\to\infty}\|g_{mn}(x),z\|=\|g(x),z\|.$$

Then, for every $(m,n)\in K$ we have
(i) If $f_{mn}(x)\geq 0$ then, $f(x)\geq 0$ and
(ii) If $f_{mn}(x)\leq g_{mn}(x)$ then $f(x)\leq g(x)$, where $K\subseteq\mathbb{N}\times\mathbb{N}$ and $K\in\mathscr{F}(\mathscr{I}_2)$.

Proof. (i) Suppose that $f(x) < 0$. Select $\varepsilon = -\frac{f(x)}{2}$, for each $x \in X$. Since

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|,$$

so there exists the set M such that

$$M = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| < \varepsilon\} \in \mathscr{F}(\mathscr{I}_2),$$

for each $x \in X$ and each nonzero $z \in Y$. Since $M, K \in \mathscr{F}(\mathscr{I}_2)$, then $M \cap K$ is a nonempty set in $\mathscr{F}(\mathscr{I}_2)$. So we can find out point $(m_0, n_0) \in K$ such that

$$\|f_{m_0 n_0}(x) - f(x), z\| < \varepsilon.$$

Since $f(x) < 0$ and $\varepsilon = -\frac{f(x)}{2}$ for each $x \in X$, then we have $f_{m_0 n_0}(x) \leq 0$. This is a contradiction to the fact that $f_{mn}(x) > 0$ for every $(m,n) \in K$. Hence, we have $f(x) > 0$, for each $x \in X$.
(ii) Suppose that $f(x) > g(x)$. Select $\varepsilon = \frac{f(x)-g(x)}{3}$, for each $x \in X$. So that the neighborhoods $(f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ and $(g(x_0) - \varepsilon, g(x_0) + \varepsilon)$ of $f(x)$ and $g(x)$, respectively, are disjoints. Since for each $x \in X$ and each nonzero $z \in Y$,

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \text{ and } \mathscr{I}_2 - \lim_{m,n\to\infty} \|g_{mn}(x), z\| = \|g(x), z\|$$

and $\mathscr{F}(\mathscr{I}_2)$ is a filter on $\mathbb{N}\times\mathbb{N}$, therefore we have

$$A = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| < \varepsilon\} \in \mathscr{F}(\mathscr{I}_2)$$

and

$$B = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|g_{mn}(x) - g(x), z\| < \varepsilon\} \in \mathscr{F}(\mathscr{I}_2).$$

This implies that $\emptyset \neq A \cap B \cap K \in \mathscr{F}(\mathscr{I}_2)$. There exists a point $(m_0, n_0) \in K$ such that

$$\|f_{mn}(x) - f(x), z\| < \varepsilon \text{ and } \|g_{mn}(x) - g(x), z\| < \varepsilon.$$

Since $f(x) > g(x)$ and $\varepsilon = \frac{f(x)-g(x)}{3}$ for each $x \in X$, then we have

$$f_{m_0 n_0}(x) > g_{m_0 n_0}(x).$$

This is a contradiction to the fact $f_{mn}(x) \leq g_{mn}(x)$ for every $(m,n) \in K$. Thus, we have $f(x) \leq g(x)$, for each $x \in X$. □

Definition 3.7. *The double sequence of functions $\{f_{mn}\}$ in 2-normed space $(X, \|.,.\|)$ is said to be $\mathscr{I}_2^*$-convergent (pointwise sense) to f, if there exists a set $M \in \mathscr{F}(\mathscr{I}_2)$ (i.e., $H = \mathbb{N}\times\mathbb{N} \setminus M \in \mathscr{I}_2$) such that for each $x \in X$, each nonzero $z \in Y$ and all $(m,n) \in M$*

$$\lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$$

and we write

$$\mathscr{I}_2^* - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \quad or \quad f_{mn} \xrightarrow{\|.,.\|_Y}_{\mathscr{I}_2^*} f.$$

Theorem 3.8. *For each $x \in X$ and nonzero $z \in Y$,*

$$\mathscr{I}_2^* - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \text{ implies } \mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|.$$

Proof. Since for each $x \in X$ and each nonzero $z \in Y$,

$$\mathscr{I}_2^* - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|,$$

so there exists a set $H \in \mathscr{I}_2$ such that for $M \in \mathscr{F}(\mathscr{I}_2)$ $(i.e., H = \mathbb{N}\times\mathbb{N} \setminus M \in \mathscr{I}_2)$ we have

$$\lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|, \ (m,n) \in M.$$

Let $\varepsilon > 0$. Then, for each $x \in X$ there exists a $k_0 = k_0(\varepsilon, x) \in \mathbb{N}$ such that for each nonzero $z \in Y$, $\|f_{mn}(x) - f(x), z\| < \varepsilon$, for all $(m,n) \in M$ such that $m, n \geq k_0$. Then, clearly we have

$$\begin{aligned} A(\varepsilon, z) &= \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\} \\ &\subset H \cup [M \cap ((\{1,2,3,...,(k_0-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,3,...,(k_0-1)\}))], \end{aligned}$$

for each $x \in X$, for each nonzero $z \in Y$. Since $\mathscr{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$be a strongly admissible ideal then

$$H \cup [M \cap ((\{1,2,3,...,(k_0-1)\} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,3,...,(k_0-1)\}))] \in \mathscr{I}_2$$

and so, $A(\varepsilon, z) \in \mathscr{I}_2$. This implies that $\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$. □

Theorem 3.9. *Let $\mathscr{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be an admissible ideal having the property $(AP2)$. For each $x \in X$ and nonzero $z \in Y$,*

$$\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\| \text{ implies } \mathscr{I}_2^* - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|.$$

Proof. Let $\mathscr{I}_2 \subset 2^{\mathbb{N}\times\mathbb{N}}$ be an admissible ideal having the property $(AP2)$ and $\mathscr{I}_2 - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$, for each $x \in X$ and each nonzero $z \in Y$. Then, for any $\varepsilon > 0$

$$A(\varepsilon, z) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \varepsilon\} \in \mathscr{I}_2$$

for each $x \in X$ and each nonzero $z \in Y$. Now, put

$$A_1(\varepsilon, z) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq 1\}$$

and

$$A_k(\varepsilon, z) = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{k} \leq \|f_{mn}(x) - f(x), z\| < \frac{1}{k-1}\}$$

for $k \geq 2$. $A_i \cap A_j = \emptyset$ for $i \neq j$ and $A_i \in \mathscr{I}_2$ for each $i \in \mathbb{N}$. By property $(AP2)$ there exists a sequence $\{B_k\}_k \in \mathbb{N}$ of sets such that $A_j \triangle B_j$ is finite union of rows and columns in $\mathbb{N}\times\mathbb{N}$ for each $j \in \mathbb{N}$ and $B = \cup_{j=1}^{\infty} B_j \in \mathscr{I}_2$.
We shall prove that, for each $x \in X$ and each nonzero $z \in Y$

$$\lim_{m,n\to\infty} \|f_{mn}(x) - f(x), z\| = \|f(x), z\|, \ (m,n) \in M,$$

for $M = \mathbb{N}\times\mathbb{N} \setminus B \in \mathscr{F}(\mathscr{I}_2)$. Let $\delta > 0$ be given. Choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \delta$. Then, we have

$$\{(m,n) \in \mathbb{N}\times\mathbb{N} : \|f_{mn}(x) - f(x), z\| \geq \delta\} \subset \bigcup_{j=1}^{k} A_j.$$

Since $A_j \triangle B_j$, $j = 1, 2, ..., k$ are included in finite union of rows and columns, there exis

$$\left(\bigcup_{j=1}^{k} B_j\right) \cap \{(m,n) \in \mathbb{N}\times\mathbb{N} : m \geq n_0 \wedge n \geq n_0\} = \left(\bigcup_{j=1}^{k} A_j\right) \cap \{(m,n) \in \mathbb{N}\times\mathbb{N} : m \geq m_0 \wedge n \geq n_0\}.$$

If $m, n \geq n_0$ and $(m,n) \notin B$ then

$$(m,n) \notin \bigcup_{j=1}^{k} B_j \text{ and so } (m,n) \notin \bigcup_{j=1}^{k} A_j.$$

Thus, we have $\|f_{mn}(x) - f(x), z\| < \frac{1}{k} < \delta$ for each $x \in X$ and each nonzero $z \in Y$. This implies that

$$\lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|, \ (m,n) \in M$$

and so we have

$$\mathscr{I}_2^* - \lim_{m,n\to\infty} \|f_{mn}(x), z\| = \|f(x), z\|$$

for each $x \in X$ and each nonzero $z \in Y$. □

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