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Pseudo-Valuations on UP-Algebras

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Abstract

Looking at pseudo-valuations on some classes of abstract algebras, such as BCK, BCI, BCC and KU, in this article we introduce the concept of pseudo-valuations on UP-algebras and analyze the relationship of these mappings with UP-substructures.

1. Introduction

The idea that universal algebra should be analyzed by means of pseudo-valuation was first developed by D. Busneag in 1996 [1]. This author has expanded the perception of pseudo-valuation on Hilbert's algebras [2]. Logical algebras and pseudo-valuations on them have become an object of interest for researchers in recent years. For example, Doh and Kang [3, 4] introduced in the concept of pseudo-valuation on BCK/BCI - algebras. Ghorbani in 2010 [5] determined a congruence on BCI - algebras based on pseudo-valuation and describe the obtained factorial structure generated by this congruence. Song, Roh and Jun described pseudo-valuation on BCK/BCI - algebras [12] and Song, Bordbar and Jun have described the quotient structure on such algebras generated by pseudo-valuation [13]. Jun, Lee and Song analyzed in article [8] several types of quasi-valuation maps on BCK - algebra and their interactions. Also, Mehrshad and Kouhestani were interested in pseudo-valuations on BCK - algebra [10]. Jun, Ahn and Roh. in [7] described pseudo-valuation on the BCC - algebras. Koam, Haider and Ansari described in 2019 pseudo-valuations on KU algebras [9].
The concept of UP-algebras is introduced and analyzed by Iampan in 2017 [6] as a generalization of the concept of KU - algebras. In this note, we offer one way of determining of pseudo-evaluation on PU - algebras. Apart from showing the features of this pseudo-valuation on UP-algebras, we have demonstrated how to construct a pseudo-metric space by such mapping.

2. Preliminaries

Here we give the definition of UP-algebra and some of its substructures necessary for further work.

Definition 2.1 ([6]). *An algebra* $A = (A, \cdot, 0)$ *of type* $(2,0)$ *is called a UP- algebra if it satisfies the following axioms:*
(UP-1) $(\forall x,y,z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP-2) $(\forall x \in A)(0 \cdot x = x)$,
(UP-3) $(\forall x \in A)(x \cdot 0 = 0)$, *and*
(UP-4) $(\forall x,y \in A)((x \cdot y = 0 \wedge y \cdot x = 0) \implies x = y)$.

In A we can define a binary relation $' \leqslant '$ by

$$(\forall x,y \in A)(x \leqslant y \iff x \cdot y = 0).$$

Definition 2.2 ([6]). *A non-empty subset J of a UP-algebra A is called a UP-ideal of A if it satisfies the following conditions:*
(1) $0 \in J$, *and*
(2) $(\forall x,y,z \in A)((x \cdot (y \cdot z) \in J \wedge y \in J) \implies x \cdot z \in J)$.

Definition 2.3 ([11]). *Let A be a UP-algebra. A subset G of A is called a* proper UP-filter *of A if it satisfies the following properties:*
(3) $\neg(0 \in G)$, *and*
(4) $(\forall x,y,z \in A)((\neg(x \cdot (y \cdot z) \in G) \wedge x \cdot z \in G) \implies y \in G)$.

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3. The concept of pseudo-valuations on UP-algebras

In this section, we introduce the concept of pseudo-valuations on UP-algebras, describe the basics properties of such pseudo-valuation and construct a pseudo-metric space based on this mapping.

Definition 3.1. *A real-valued function v on a UP-algebra A is called a pseudo-valuation on A if it satisfies the following two conditions:*
(1) $v(0) = 0$*, and*
(2) $(\forall x,y,z \in A)(v(x \cdot z) \leqslant v(x \cdot (y \cdot z)) + v(y))$.
A pseudo-valuation v on a UP-algebra A satisfying the following condition:
(3) $(\forall x \in A)(v(x) = 0 \Longrightarrow x = 0)$
is called a valuation on X.

Theorem 3.2. *Let v be a pseudo-valuation on a UP-algebra A. Then the following are valid:*
(4) $(\forall x,y \in A)(v(y) \leqslant v(x \cdot y) + v(x))$,
(5) $(\forall x,y \in A)(v(x \cdot y) \leqslant v(y))$,

Proof. If we put $x = 0$, $y = x$ and $z = y$ in formula (2), we get

$$v(y) \leqslant v(x \cdot y) + v(x).$$

Thus, formula (4) is valid.
If we put $z = y$ in formula (2), we have $v(x \cdot y) \leqslant v(x \cdot (y \cdot y)) + v(y)$ from which follows $v(x \cdot y) \leqslant v(y)$ due to the assertion (1) of Proposition 1.7 in [6], (UP-3) and (1). So, (5) is proven. □

Corollary 3.3. *Let v be a pseudo-valuation on a UP-algebra A. Then*
(6) $(\forall x,y \in A)(x \leqslant y \Longrightarrow v(y) \leqslant v(x))$.

Proof. Let x and y be arbitrary elements of a UP-algebra A such that $x \leqslant y$. Then $x \cdot y = 0$ and $v(x \cdot y) = 0$ by (1). From here follows $v(y) \leqslant v(x \cdot y) + v(x)$ according to (4). Thus $v(y) \leqslant v(x)$. Thus, any pseudo-valuation on a UP-algebra is an inversely monotone mapping. □

Corollary 3.4. *Let v be a pseudo-valuation on a UP-algebra A. Then*
(7) $(\forall x \in A)(0 \leqslant v(x))$.

Proof. Since $x \cdot 0 = 0$ according to (UP-3), i.e. as always $x \leqslant 0$ in UP-algebra A, we have $0 = v(0) \leqslant v(x)$ according to Corollary 3.3. □

Corollary 3.5. *Let v be a pseudo-valuation on a UP-algebra A. Then*
(8) $(\forall x,y \in A)(v(x \cdot y) \leqslant v(x) + v(y))$.

Proof. Let x and y be arbitrary elements of A. Thus $v(x \cdot y) \leqslant y(y)$ by (5). Thus $v(x \cdot y) \leqslant v(x) + v(y)$ by Corollary 3.4. □

Theorem 3.6. *Let v be a pseudo-valuation on a UP-algebra A. Then the set $J_v = \{x \in A : v(x) = 0\}$ is an UP-ideal of A and the set $G = \{x \in A : 0 < v(x)\}$ is a proper UP-filter of A.*

Proof. Since $v(0) = 0$, follows $0 \in J_v$.
Let x, y and z be arbitrary elements of A such that $x \cdot (y \cdot z) \in J_v$ and $y \in J_v$. Then $v(x \cdot (y \cdot z)) = 0$ and $v(y) = 0$. By (2) we have

$$v(x \cdot z) \leqslant v(x \cdot (y \cdot z)) + v(y) = 0 + 0 = 0.$$

Thus $v(x \cdot z) = 0$ according to Corollary 3.4. Hence $x \cdot z \in J_v$. So, the set J_v is a UP-ideal of UP-algebra A.
The set G is a proper UP-filter of A by Theorem 3.7 in [11]. □

Corollary 3.7. *Let v be a pseudo-valuation on a UP-algebra A. Then v is a valuation on A if and only if $J_v = \{0\}$.*

Proof. The claim follows from the definition of the concept of valuations on a UP-algebra A. □

Remark 3.8. *The previous corollary suggested that a valuation on an UP-algebra A can be defined if $\{0\}$ is a UP-ideal at A.*

Example 3.9. *For any ideal J of a UP-algebra A, define a map $v_J : A \longrightarrow \mathbb{R}$ by $(\forall x \in J)(v_J(x) = 0)$ and $(\forall x \in A \backslash J)(v_J(x) \in \mathbb{R}^+)$. Then, v_J is a pseudo-valuation of A.*

Example 3.10. *Let $A = \{0,1,2,3,4\}$ be given and an operations on it as in Example 2.2 in [6]. Then $(A, \cdot, 0)$ is a UP-algebra. It is easy to directly verified that $v : A \longrightarrow \mathbb{R}$, given with $v(0) = v(1) = v(2) = 0$, $v(3) = v(4) = 3$, is a pseudo-valuation on A.*

Theorem 3.11. *Let $f : (A, \cdot, 0_A) \longrightarrow (B, *, 0_B)$ be a homomorphism of UP-algebras. If v is a pseudo-valuation on B, then the composition $v \circ f$ is a pseudo-valuation on A.*

Proof. First, we have $(v \circ f)(0_A) = v(f(0_A)) = v(0_B) = 0$.
For any $x,y,z \in A$, we get $(v \circ f)(x \cdot z) = v(f(x \cdot z)) = v(f(x) * f(z)) \leqslant v(f(x) * (f(y) * f(z))) + v(f(y)) = (v \circ f)(x \cdot (y \cdot z)) + (v \circ f)(y)$.
Hence, $v \circ f$ is a pseudo-valuation on A. □

Lemma 3.12. *Suppose that A is a UP-algebra. Then every pseudo- valuation v on A satisfies the following inequality:*
(9) $(\forall x,y,z \in A)(v(x \cdot z) \leqslant v(x \cdot y) + v(y \cdot z))$.

Proof. From (UP-1) follows $y\cdot z \leqslant (x\cdot y)\cdot(x\cdot z)$. Thus $v(y\cdot z) \geqslant v((x\cdot y)\cdot(x\cdot z))$ by (6) and $v(y\cdot z) \geqslant v(x\cdot z) - v(x\cdot y)$ by (4). Therefore, $v(x\cdot z) \leqslant v(x\cdot y) + v(y\cdot z)$. □

Now, we define pseudo-metric on UP-algebras and show related results.

Theorem 3.13. *Let A be a UP-algebra and v be a pseudo-valuation on A. Then the mapping* $d_v : A\times A \ni (x,y) \longmapsto v(x\cdot y) + v(y\cdot x) \in \mathbb{R}$ *is a pseudo-metric on A.*

Proof. Clearly, $0 \leqslant d_v(x,y)$; $d_v(x,x) = 0$ and $d_v(x,y) = d_v(y,x)$ for any $x,y \in A$. For any $x,y,z \in A$ from Lemma 3.12, we get that

$$\begin{aligned} d_v(x,y) + d_v(y,z) = \\ (v(x\cdot y) + v(y\cdot x)) + (v(y\cdot z) + v(z\cdot y)) = \\ (v(x\cdot y) + v(y\cdot z)) + (v(z\cdot y) + v(y\cdot x)) \geqslant \\ v(x\cdot z) + v(z\cdot x) = d_v(x\cdot z). \end{aligned}$$

Hence (A, d_v) is a pseudo-metric space. □

4. Conclusion

The aim of this paper was to study the concept of pseudo-valuation and their induced pseudo-metrics on UP - algebras. This work can be the basis for further and deeper research of the properties of UP - algebras.

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References

[1] D. Busneag. *Hilbert algebras with valuations*. Math. Japon., **44** (2)(1996), 285–289.
[2] D. Busneag. *On extensions of pseudo-valuations on Hilbert algebras*. Discrete Math., **263**(1-3)(2003), 11–24.
[3] M. I. Doh and M. S. Kang. *BCK/BCI-algebras with pseudo-valuation*. Honam Math. J., **32**(2)(2010), 217–226.
[4] M. I. Doh and M. S. Kang. *Commutative pseudo valuations on BCK-algebras*. Int. J. Math. Math. Sci., Vol. 2011, Article ID 754047, 6 pages
[5] S. Ghorbani. *Quotient BCI-algebras induced by pseudo-valuations*. Iranian J. Math. Sc. Inform., **5**(2010), 13–24.
[6] A. Iampan. *A new branch of the universal algebra: UP-algebras*. Journal of Algebra and Related Topics, **5**(1)(2017), 35–54.
[7] Y. B. Jun, S. S. Ahn amd E. H. Roh. *BCC-algebras with pseudo-valuations*. Filomat, **26**(2)(2012), 243-–252.
[8] Y. B. Jun, K. J. Lee and S. Z. Song. *Quasi-valuation maps based on positive implicative ideals in BCK-algebras*. Algebra and Discrete Mathematics, **26**(1)(2018), 65-–75.
[9] A. N.A. Koam, A. Haider and M. A. Ansari. *Pseudo-metric on KU-algebras*. Korean J. Math., **27**(1)(2019), 131–140.
[10] S. Mehrshad and N. Kouhestani. *On pseudo-valuations on BCK-algebras*. Filomat, **32**(12)(2018), 4319-–4332.
[11] D. A. Romano. *Proper UP-filters in UP-algebra*. Universal Journal of Mathematics and Applications, **1**(2)(2018), 98–100.
[12] S.-Z. Song, E. H. Roh and Y. B. Jun. *Quasi-valuation maps on BCK/BCI-algebras*. Kyungpook Math. J., **55**(4)(2015), 859–870.
[13] S.-Z. Song, H. Bordbar and Y. B. Jun. *Quotient structures of BCK/BCI-algebras induced by quasi-valuation maps*. Axioms 2018, **7**(2): 26. 11pages