Universal Journal of Mathematics and Applications, 2 (3) (2019) 148-151
Research paper

UJMA

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: http://dx.doi.org/10.32323/ujma.522420

On Some Properties of Integral-Type Operator in Weighted Herz Spaces with Variable Exponent Lebesgue Spaces

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Article Info

Keywords: *Variable exponent, Herz space, Operator theory*
2010 AMS: *42B25, 42B35, 47B38*
Received: *5 February 2019*
Accepted: *3 July 2019*
Available online: *30 September 2019*

Abstract

For the last quarter century a considerable number of research has been carried out on the study of Herz spaces, variable exponent Lebesgue spaces and Sobolev spaces. This studies also have played an important role in problems of elasticity, fluid dynamics, calculus of variations. Our aim in this work is to prove some properties of the integral-type operator on weighted Herz space with variable exponent Lebesgue space (VELS).

1. Introduction

Herz spaces and variable exponent spaces have played an important role in recent harmonic analysis because they have an interesting norm including both local and global properties.We refer to the book [1] for the history of Herz spaces. Based on the classical Muckenhoupt theory [2]-[4] some classes of weighted Herz spaces have been defined and the boundedness of many operators on those spaces have been proved [4]-[8] . Herz spaces can be generalized using variable exponents and many properties of them have been studied [9]-[11] . The boundedness or compactness of integral-type operators on weighted Lebesgue spaces has been obtained [12]-[19] . We prove the boundedness of the fractional maximal operator on the spaces with power weight.

2. Preliminaries, Definitions and Assertions

We use the following definitions and notations:

a) For a point $x \in R^n$ and a constant $r < 0$, we have $B(x,r) = \{y \in R^n : |x-y| < r\}$
b) The set N_0 consist of all non-negative integers. For every $k \in N_0$, we write

$$B_k = \overline{B(0,2^k)} = \{x \in R^n : |x| \le 2^k\}$$

c) We define a family $\{C_k\}_{k=0}^{\infty}$ by

$$C_k = \begin{cases} B_0 = \{x \in R^n : |x| \le 1\}, k = 0 \\ B_k \backslash B_{k-1} = \{x \in R^n : 2^{k-1} < |x| \le 2^k\}, k \ge 1 \end{cases}$$

Moreover χ_k denote the charecteristic function of C_k, namely $\chi_k = \chi_{C_k}$.
Definition 2.1 (See [16]) Let ω be a weight function on R^n and let $p(.) = R^n \longrightarrow [1,\infty)$ a bounded measurable function. For all f measurable function, weighted norm Lebesgue space $L_{\omega}^{p(.)}(R^n)$ with

$$\|f\|_{L_{\omega}^{p(.)}(R^n)} = inf\{\lambda > 0 : \int_{R^n} (\frac{|f(x)|}{\lambda})\omega(x)dx \le 1\}$$

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In the case that $\omega \equiv 1$, we simply write $L^{p(.)} = L^{p(.)}_{\omega}$ and $\|f\|_{L^{p(.)}} = \|f\|_{L^{p(.)}_{\omega}}$.

Definition 2.2(See [9], [11]) Given a, $0 < a < n$ and given an open set $\Omega \subset R^n$ the fractional maximal operator M_a is defined by

$$M_a f(x) = \sup_{B \ni x} \frac{1}{|B|^{1-\frac{a}{n}}} \int_{B\cap\Omega} f(y)dy$$

where the supremum is taken over all balls B which contain x. When $a = 0, M_0 = M$ is the Hardy-Littlewood maximal operator.

Definition 2.3 ([9], [16])

1. for measurable function $p(.) = R^n \longrightarrow (0,\infty)$, we write

$$p^+ = ess \sup_{x\in R^n} p(x)$$

$$p^- = \{(\frac{1}{p(.)})_+\}^{-1}$$

2. The set $P(R^n)$ consist of all measurable functions $p(.)$ defined on R^n satisfying $1 < p^- \leq p^+ < \infty$.
3. The set $LH(R^n)$ consist of $p(.)$ defined on R^n satisfying the following two inequalities

$$|p(x) - p(y)| \lesssim -\frac{1}{log(|x-y|)}, |x-y| \leq \frac{1}{2}$$

$$|p(x) - p(y)| \lesssim -\frac{1}{log(e+|x|)}, x \in R^n \tag{1}$$

for some real constant p_∞. In particular a measurable function $p(.)$ is said to be log-Hölder continuous at infinity when $p(.)$ satisfies (1).

3. Main Results

Let's start with the following definitions for proof of our main theorem.

Definition 3.1. *(See [11]) Let $0 < q < \infty$, $\beta(.) \in L^\infty$, ω be a weight function on R^n and $p(.) : R^n \longrightarrow [1,\infty)$ a bounded measurable function.*
1. The set $L^{p(.)}_{loc}(\omega^{\frac{1}{p(.)}})$ consist of all measurable function f such that $f\chi K \in L^{p(.)}(\omega^{\frac{1}{p(.)}})$ for any compact set $K \in R^n$.
2.The non-homogeneous Herz space $K^{\beta(.),q}_{p(.)}$ consist of all measurable functions $f \in L^{p(.)}_{loc}(1)$ such that

$$\|f\|_{K^{\beta(.),q}_{p(.)}} = (\sum_{k=0}^{\infty} \|2^{\beta(.)k} f\chi K\|^q_{L^{p(.)}}) \frac{1}{q} < \infty.$$

3. The critical weighted Herz space $B^{p(.)}(\omega)$ consist of all measurable functions $f \in L^{p(.)}_{loc}(\omega^{\frac{1}{p(.)}})$ such that

$$\|f\|_{B^{p(.)}(\omega)} = \sup_{k\geq 0} \|\omega(B_k)^{-\frac{1}{p(.)}} f\chi K\|_{L^{p(.)}(\omega)} < \infty. \tag{2}$$

Proposition 3.2. *(see [9, Proposition 3.8]). Let $p(.) \in P(R^n)$, $q \in (0,\infty)$ and $\beta(.) \in L^\infty$. If $\beta(.)$ is log-Hölder continuous at infinity, then we have*

$$K^{\beta(.),q}_{p(.)} = K^{\beta_\infty,q}_{p(.)}$$

with norm equivalence.

From now on, we consider a power weight $\omega(x) = |x|^m$ with a real constant m. It is easy to see that for all $k \in N_0$ and $R > 0$,

$$\omega(B_k) \sim 2^{(mn+n-1)k}, \omega(B(0,r)) \sim r^{mn+n-1}$$

where implicit constants are independant of k and r. Proposition 3.2 can be extended to the case $B^{p(.)}(\omega)$ by the same as the proof of [9, Proposition 3.8]. Herewith, we have following a corollary.

Corollary 3.3. *(See [11]) Let $\omega(x) = |x|^m$ with a real constant m, $p(.) \in P(R^n) \cap LH(R^n)$ and $\beta(.) \in L^\infty$. If $\omega \in \aleph_{p(.)}$, we have that for all measurable function $f \in B^{p(.)}(\omega)$,*

$$\|f\|_{B^{p(.)}(\omega)} = \sup_{k\geq 0} \|2^{-\frac{k(mn+n-1)}{p(.)}} f\chi_{C_k}\|_{L^{p(.)}(\omega)} \sim \sup_{k\geq 0} \|2^{-\frac{k(mn+n-1)}{p_\infty}} f\chi_{B_k}\|_{L^{p(.)}(\omega)}$$

Proposition 3.4. *(See [17, Proposition 1.2]). Let $\omega(x)=|x|^m$ with a real constant m, $p(.)\in P(R^n)$ and $f\in L^{p(.)}_{loc}(\omega^{\frac{1}{p(.)}})$. Then $f\in B^{p(.)}(\omega)$ holds if and only if*

$$\sup_{r\geq 1}\|\frac{|f|\chi_{B(0,r)}}{\omega(B(0,r))^{\frac{1}{p(.)}}}\|_{L^{p(.)}(\omega)} \tag{3}$$

is finite. If this is the case, then the quantity (3) is equivalent to the $B^{p(.)}(\omega)$ norm (2).

Our result is the boundedness of the weighted fractional maximal operator $M_{a,\omega}$ on the space $B^{p(.)}(\omega)$.

Theorem 3.5. *Let $m\in R$ and $\omega(x)=|x|^m$. Suppose that $p(.)\in P(R^n)\cap LH(R^n)$ and $\omega\in\aleph_{p(.)}$. Then $M_{a,\omega}$ is bounded on $B^{p(.)}(\omega)$.*

Proof. Let $f\in B^{p(.)}(\omega)$. Then Corollary 3.3 we have

$$\|M_{a,\omega}f\|_{B^{p(.)}(\omega)}\sim\sup_{k\geq 0}\|2^{-\frac{k(mn+n-1)}{p_\infty}}(M_{a,\omega}f)\chi_{B_k}\|_{L^{p(.)}(\omega)}$$

Thus we have only to estimate $\|2^{-\frac{k(mn+n-1)}{p_\infty}}(M_{a,\omega}f)\chi_{B_k}\|_{L^{p(.)}(\omega)}$ for each $k\in N_0$. Let constant $k\in N_0$ then we have

$$M_{a,\omega}f(x)\leq\sup_{x\in B\subset B_{k+1}}\frac{1}{|B|^{1-\frac{a}{n}}\omega(B)}\int_B|f(y)|\omega(y)dy+\sup_{x\in B\backslash B_{k+1}\neq 0}\frac{1}{|B|^{1-\frac{a}{n}}\omega(B)}\int_B|f(y)|\omega(y)dy$$

$$=M_{a,\omega,1,k}f(x)+M_{a,\omega,2,k}f(x)$$

(a) Note that $M_{a,\omega,1,k}f(x)=M_{a,\omega,1,k}f(x)(f\chi_{B_{k+1}})\chi_{B_{k+1}}(x)$.
By [10, Theorem 2.10] we have

$$\|2^{-\frac{k(mn+n-1)}{p_\infty}}(M_{a,\omega,1,k}f)\chi_{B_k}\|_{L^{p(.)}(\omega)}=\|2^{-\frac{k(mn+n-1)}{p_\infty}}M_{a,\omega,1,k}(f\chi_{B_{k+1}})\chi_{B(0,2^{k+1})}\|_{L^{p(.)}(\omega)}$$

$$\lesssim\|2^{-\frac{k(mn+n-1)}{p_\infty}}f\chi_{B_{k+1}}\|_{L^{p(.)}(\omega)}$$

(b) Next we estimate the function $M_{a,\omega,2,k}f(x)$. Let $|x|<2^k$. The smallest ball centered at 0 and containing B is called $B^{'}$. Then, there exist a constant C which is depended only on the dimension n such that $|B|\geq\frac{|B^{'}|}{C}$ by a geometric consideration.

$$M_{a,\omega,2,k}f(x)\leq CM_{a,\omega}f(0).$$

We note that $\|\omega(B^{'})^{-\frac{1}{p^{'}(.)}}\chi_{B^{'}}\|_{L^{p^{'}(.)}(\omega)}=1$ because

$$\int_{R^n}(\omega(B^{'})^{-\frac{1}{p^{'}(.)}}\chi_{B^{'}})^{p^{'}(x)}\omega(x)dx=\frac{1}{\omega(B^{'})}\int_{B^{'}}\chi_{B^{'}}(x)\omega(x)dx=1$$

If we use Hölder inequality and Proposition 3.4 we have

$$\frac{1}{|B|^{1-\frac{a}{n}}\omega(B^{'})}\int_{B^{'}}|f(y)|\omega(y)dy$$

$$=\frac{1}{|B|^{1-\frac{a}{n}}}\int_{B^{'}}\frac{|f(y)\chi_{B^{'}}(y)|\omega(y)^{\frac{1}{p(y)}}}{\omega(B^{'})^{\frac{1}{p(y)}}}.\frac{\chi_{B^{'}}(y)\omega(y)^{\frac{1}{p^{'}(y)}}}{\omega(B^{'})^{\frac{1}{p^{'}(y)}}}dy\leq C\|f\|_{B^{p(.)}(\omega)}\|\omega(B^{'})^{-\frac{1}{p^{'}(.)}}\chi_{B^{'}}\|_{L^{p^{'}(.)}(\omega)}\leq C\|f\|_{B^{p(.)}(\omega)}$$

Hence, we can see that $M_{a,\omega,2,k}f(x)\lesssim\|f\|_{B^{p(.)}(\omega)}$ holds.

Combining (a) and (b) we see that

$$\|M_{a,\omega}f\|_{B^{p(.)}(\omega)}\sim\sup_{k\geq 0}\|2^{-\frac{k(mn+n-1)}{p_\infty}}(M_{a,\omega}f)\chi_k\|_{L^{p(.)}(\omega)}$$

$$\leq\sup_{k\geq 0}(\|2^{-\frac{k(mn+n-1)}{p_\infty}}(M_{a,\omega,1,k}f)\chi_k\|_{L^{p(.)}(\omega)}+\|2^{-\frac{k(mn+n-1)}{p_\infty}}(M_{a,\omega,2,k}f)\chi_k\|_{L^{p(.)}(\omega)})$$

$$\leq\sup_{k\geq 0}(\|2^{-\frac{k(mn+n-1)}{p_\infty}}f\chi_{B(0,2^{k+1})}\|_{L^{p(.)}(\omega)}+\|f\|_{B^{p(.)}(\omega)}\|2^{-\frac{k(mn+n-1)}{p_\infty}}\chi_k\|_{L^{p(.)}(\omega)})$$

$$\lesssim \sup_{k\geq 0}\|2^{-\frac{k(mn+n-1)}{p_\infty}} f\chi_{B(0,2^{k+1})}\|_{L^{p(.)}(\omega)} + \|f\|_{B^{p(.)}(\omega)} \sup_{k\geq 0}\|2^{-\frac{k(mn+n-1)}{p_\infty}}\chi_k\|_{L^{p(.)}(\omega)}$$

$$\lesssim 2^{\frac{(mn+n-1)}{p_\infty}}\|f\|_{B^{p(.)}(\omega)} + \|f\|_{B^{p(.)}(\omega)} \lesssim \|f\|_{B^{p(.)}(\omega)}.$$

Theorem 3.5 is proved. □

Conclusion. We investigated and saw that is bounded the fractional maximal operator $M_{a,\omega}$ on the spaces with power weight in view of the given conditions. This method can be applied to different operators.

Acknowledgements. The author deeply respects both the journal and the referees valuable opinions.

References

[1] S. Lu, D. Yang and G. Hu, *Herz Type Spaces and Their Applications*, Science Press, Beijing, 2008.
[2] J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North–Holland Mathematics Studies 116, North–Holland Publishing Co., Amsterdam, 1985.
[3] B. Muckenhoupt, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165** (1972), 207–226.
[4] A. Torchinsky, *Real-Variable Methods in Harmonic Analysis*, Pure and Applied Mathematics 123, Academic Press, Inc., Orlando, FL, 1986.
[5] M. Izuki and Y. Sawano, *The Haar wavelet characterization of weighted Herz spaces and greediness of the Haar wavelet basis*, J. Math. Anal. Appl. **362**(1) (2010), 140–155.
[6] M. Izuki and K. Tachizawa, *Wavelet characterizations of weighted Herz spaces*, Sci. Math. Jpn. **67**(3)(2008), 353–363.
[7] S. Lu, K. Yabuta and D. Yang, *Boundedness of some sublinear operators in weighted Herz-type spaces*, Kodai Math. J. **23**(3)(2000), 391–410.
[8] K. Matsuoka, *On some weighted Herz spaces and the Hardy–Littlewood maximal operato*r, in: Proceedings of the International Symposium on Banach and Function Spaces II (Kitakyusyu, Japan, 2006), M. Kato et al. (eds.), pp. 375–384, Yokohama Publ., Yokohama, 2008.
[9] A. Almeida and D. Drihem, *Maximal, potential and singular type operators on Herz spaces with variable exponents*, J. Math.Anal. Appl. **394**(2)(2012), 781–795.
[10] M. Izuki and T. Noi, Duality of Besov, *Triebel-Lizorkin and Herz spaces with variable exponents*, Rend. Circ. Mat. Palermo (2) 63(2)(2014),221-245.
[11] M. Izuki and T. Noi, *Hardy spaces associated to critical Herz spaces with variable exponent*, Mediterranean J. Math.**13**(5)(2016), 29813013.
[12] D. Cruz-Uribe, L. Diening and P. Hasto, *The maximal operator on weighted variable Lebesgue spaces*, Fract. Calc. Appl. Anal. **14**(3)(2011), 361-374.
[13] D. Cruz-Uribe, A. Fiorenza and C. J. Neugebauer, *Weighted norm inequalities for the maximal operator on variable Lebesgue spaces*, J. Math. Anal. Appl. **394**(2)(2012), 744-760.
[14] D. Cruz-Uribe, SFO and D. Wang, *Extrapolation and weighted norm inequalities in the variable Lebesgue spaces*, Trans. Amer. Math Soc. **369**(2)(2017), 1205-1235.
[15] L. Diening and P. Hasto, *Muckenhoupt weights in variable exponent spaces*, preprint, available at $http : www.helsinki._/hasto/pp/p75submit.pdf$
[16] F. I. Mamedov, Y. Zeren and L. Akin, *Compactification of weighred Hardy operator in variable exponent Lebesgue spaces*, Asian J. Math. Comp. Res., **17**(1)(2017), 38-47.
[17] J. Garca-Cuerva, *Hardy spaces and Beurling algebras*, J. London Math. Soc. (2) **39**(3)(1989), 499-513.
[18] L. Akin, *A Characterization of Approximation of Hardy Operators in VLS*, Celal Bayar University Journal of Science, **14**(3)(2018), 333-336.
[19] L. Akin, *On two weight criterions for the Hardy-Littlewood maximal operator in BFS*, Asian J. Sci. Tech., **09** (5)(2018), 8085-8089.