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Inequalities of Hermite-Hadamard and Bullen Type for AH-Convex Functions

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Abstract

In this paper, by using an integral identity some new general inequalities of the Hermite-Hadamard and Bullen type for functions whose second derivatives in absolute value at certain power are arithmetically-harmonically convex are obtained. Some applications to special means of real numbers are also given.

1. Introduction

Definition 1.1. *A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

valids for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on interval $I \neq \varnothing$. This definition is well known in the literature.

Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences.

Theorem 1.2. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The inequality*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

holds.

The inequality (1.1) is known in the literature as Hermite-Hadamard integral inequality for convex functions. Moreover, it is known that some of the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the function f. See [3, 5, 8, 9], for the generalizations, improvements and extensions of the Hermite-Hadamard integral inequality.

Theorem 1.3. *Suppose that $f : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$. Then, the inequalities are obtained:*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] \\ &\leq \frac{1}{b-a}\int_a^b f(x)dx \\ &\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}. \end{aligned} \tag{1.2}$$

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The third inequality in (1.2) is known in the literature as Bullen's inequality.

Definition 1.4 ([2, 10]). *A function $f : I \subset \mathbb{R} \to (0,\infty)$ is said to be arithmetic-harmonically (AH) convex function if for all $x,y \in I$ and $t \in [0,1]$ the inequality*

$$f(tx+(1-t)y) \le \frac{f(x)f(y)}{tf(y)+(1-t)f(x)} \tag{1.3}$$

holds. If the inequality (1.2) is reversed then the function $f(x)$ is said to be arithmetic-harmonically (AH) concave function.

Readers can find more informations on arithmetic-harmonically convex functions in [1, 2, 4, 6, 7, 10] and references therein.
In order to establish some integral inequalities of Hermite-Hadamard type for arithmetic-harmonically convex functions, the following lemma [4] will be used.

Lemma 1.5 ([4]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable mapping on I° such that $f'' \in L[a,b]$, where $a,b \in I^\circ$ with $a < b$, then the following identity holds:*

$$J_n(f,a,b) = \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3}\left[\int_0^1 t(1-t) f''\left(\begin{array}{c} t\left(\frac{1+n-k}{n}a+\frac{k-1}{n}b\right) \\ +(1-t)\left(\frac{n-k}{n}a+\frac{k}{n}b\right)\end{array}\right)dt\right] \tag{1.4}$$

for all $n \in \mathbb{N}$, where

$$J_n(f,a,b) = \sum_{k=1}^{n} \frac{1}{2n}\left[f\left(a+\frac{(k-1)(b-a)}{n}\right)+f\left(a+\frac{k(b-a)}{n}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx.$$

In this study, using Hölder integral inequality and the identity (1.4) in order to provide inequality for functions whose first derivatives in absolute value at certain power are arithmetic-harmonically-convex functions.
Throughout this paper, for shortness, the following notations will be used for special means of two nonnegative numbers a,b with $b > a$:
1. The arithmetic mean

$$A := A(a,b) = \frac{a+b}{2}, \quad a,b > 0,$$

2. The geometric mean

$$G := G(a,b) = \sqrt{ab}, \quad a,b \ge 0$$

3. The harmonic mean

$$H := H(a,b) = \frac{2ab}{a+b}, \quad a,b > 0,$$

4. The logarithmic mean

$$L := L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a}, & a \ne b \\ a, & a = b \end{cases} ; \; a,b > 0$$

5. The p-logarithmic mean

$$L_p := L_p(a,b) = \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & a \ne b, p \in \mathbb{R}\backslash\{-1,0\} \\ a, & a = b \end{cases} ; \; a,b > 0.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \le G \le L \le I \le A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$. In addition,

$$A_{n,k} = A_{n,k}(a,b) = \frac{1+n-k}{n}a + \frac{k-1}{n}b, n \in \mathbb{N}, k = 1,2,...,n,$$

and $B(\alpha,\beta)$ is the classical Beta function which may be defined by

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt, \;\; \alpha,\beta > 0.$$

2. Main results

Theorem 2.1. *Let $f: I \subset (0,\infty) \to (0,\infty)$ be a twice differentiable mapping on I°, $n \in \mathbb{N}$ and $a,b \in I^\circ$ with $a<b$ such that $f'' \in L_1[a,b]$ and $|f''|$ are an arithmetic-harmonically convex function on the interval $[a,b]$, then the following inequalities hold:*
i) If $|f''(A_{n,k+1})| - |f''(A_{n,k})| \neq 0$, then

$$|J_n(f,a,b)| \leq \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3} \frac{|f''(A_{n,k})||f''(A_{n,k+1})|}{\left(|f''(A_{n,k+1})| - |f''(A_{n,k})|\right)^2} \tag{2.1}$$

$$\times \left[A\left(|f''(A_{n,k})|, |f''(A_{n,k+1})|\right) - |f''(A_{n,k})||f''(A_{n,k+1})| L^{-1}\left(|f''(A_{n,k})|, |f''(A_{n,k+1})|\right)\right]$$

ii) If $|f''(A_{n,k+1})| - |f''(A_{n,k})| = 0$, then

$$|J_n(f,a,b)| \leq \sum_{k=1}^{n} \frac{(b-a)^2}{12n^3} |f''(A_{n,k+1})|. \tag{2.2}$$

Proof. i) Let $|f''(A_{n,k+1})| - |f''(A_{n,k})| \neq 0$. From the Lemma 1.5 and the properties of modulus, the inequality can be written:

$$\begin{aligned}
|J_n(f,a,b)| &= \left| \sum_{k=1}^{n} \frac{1}{2n} \left[f\left(a + \frac{(k-1)(b-a)}{n}\right) + f\left(a + \frac{k(b-a)}{n}\right) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \\
&= \left| \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3} \left[\int_0^1 t(1-t) f''(tA_{n,k} + (1-t)A_{n,k+1})\, dt \right] \right| \\
&\leq \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3} \left[\int_0^1 |t(1-t)| \left| f''(tA_{n,i} + (1-t)A_{n,i+1}) \right| dt \right].
\end{aligned} \tag{2.3}$$

Since $|f''|$ is an arithmetic-harmonically convex function on the interval $[a,b]$, the inequality

$$|f''(tA_{n,k} + (1-t)A_{n,k+1})| \leq \frac{|f''(A_{n,k})||f''(A_{n,k+1})|}{t|f''(A_{n,k+1})| + (1-t)|f''(A_{n,k})|}$$

holds. By using the above inequality in (2.3), the inequality

$$|J_n(f,a,b)| = \leq \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3} \int_0^1 \frac{t(1-t)|f''(A_{n,k})||f''(A_{n,k+1})|}{t|f''(A_{n,k+1})| + (1-t)|f''(A_{n,k})|} dt \tag{2.4}$$

is obtained. By changing variable as $u = t|f''(A_{n,k+1})| + (1-t)|f''(A_{n,k})|$ in the last integral, it is easily seen that

$$\begin{aligned}
\int_0^1 \frac{t(1-t)}{t|f''(A_{n,k+1})| + (1-t)|f''(A_{n,k})|} dt &= \frac{1}{\left(|f''(A_{n,k+1})| - |f''(A_{n,k})|\right)^3} \int_{|f''(A_{n,k})|}^{|f''(A_{n,k+1})|} \frac{\left(u - |f''(A_{n,k})|\right)\left(|f''(A_{n,k+1})| - u\right)}{u} du \\
&= \frac{\left[-\frac{u^2}{2} + \left(|f''(A_{n,k+1})| + |f''(A_{n,k})|\right) u - |f''(A_{n,k})||f''(A_{n,k+1})| \ln u \right]}{\left(|f''(A_{n,k+1})| - |f''(A_{n,k})|\right)^3} \Bigg|_{|f''(A_{n,k})|}^{|f''(A_{n,k+1})|} \\
&= \frac{1}{\left(|f''(A_{n,k+1})| - |f''(A_{n,k})|\right)^3} \left[- \frac{|f''(A_{n,k+1})|^2 - |f''(A_{n,k})|^2}{2} \right. \\
&\quad + \left(|f''(A_{n,k+1})| + |f''(A_{n,k})|\right)\left(|f''(A_{n,k+1})| - |f''(A_{n,k})|\right) \\
&\quad \left. - |f''(A_{n,k})||f''(A_{n,k+1})| \left(\ln|f''(A_{n,k+1})| - \ln|f''(A_{n,k})|\right) \right] \\
&= \frac{A\left(|f''(A_{n,k})|, |f''(A_{n,k+1})|\right) - |f''(A_{n,k})||f''(A_{n,k+1})| L^{-1}\left(|f''(A_{n,k})|, |f''(A_{n,k+1})|\right)}{\left(|f''(A_{n,k+1})| - |f''(A_{n,k})|\right)^2}
\end{aligned} \tag{2.5}$$

Substituting (2.5) in (2.4), the inequality

$$|J_n(f,a,b)| \leq \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3} \frac{|f''(A_{n,k})||f''(A_{n,k+1})|}{\left(|f''(A_{n,k+1})| - |f''(A_{n,k})|\right)^2}$$
$$\times \left[A\left(|f''(A_{n,k})|, |f''(A_{n,k+1})|\right) - |f''(A_{n,k})||f''(A_{n,k+1})| L^{-1}\left(|f''(A_{n,k})|, |f''(A_{n,k+1})|\right)\right],$$

is obtained which is the desired result.
ii) Let $|f''(A_{n,k+1})| - |f''(A_{n,k})| = 0$. Then, substituting $|f''(A_{n,k+1})| = |f''(A_{n,k})|$ in the inequality (2.4), the following holds:

$$|J_n(f,a,b)| \leq \sum_{k=1}^{n} \frac{(b-a)^2}{12n^3} |f''(A_{n,k+1})|.$$

This completes the proof of theorem. □

Corollary 2.2. *By choosing $n=1$ in Theorem 2.1, the following inequalities are obtained:*
i) If $\left|f''\left(A_{1,k+1}\right)\right|-\left|f''\left(A_{1,k}\right)\right|\neq 0$ for $k=1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2}{2}\frac{|f''(a)||f''(b)|\left[A\left(|f''(a)|,|f''(b)|\right)-|f''(a)||f''(b)|L^{-1}\left(|f''(a)|,|f''(b)|\right)\right]}{\left(|f''(b)|-|f''(a)|\right)^2},$$

ii) If $\left|f''\left(A_{1,k+1}\right)\right|-\left|f''\left(A_{1,k}\right)\right|=0$ for $k=1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2}{12}\left|f''(b)\right|.$$

Corollary 2.3. *By choosing $n=2$ in Theorem 2.1, the following Bullen type inequalities are obtained:*
i) If $\left|f''\left(A_{2,k+1}\right)\right|-\left|f''\left(A_{2,k}\right)\right|\neq 0$ for $k=1,2$, then

$$\begin{aligned}\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|&\leq\frac{(b-a)^2}{16}\frac{|f''(a)|\left|f''\left(\frac{a+b}{2}\right)\right|}{\left(\left|f''\left(\frac{a+b}{2}\right)\right|-|f''(a)|\right)^2}\left[A\left(|f''(a)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right)\right.\\&\left.-|f''(a)|\left|f''\left(\frac{a+b}{2}\right)\right|-L^{-1}\left(|f''(a)|,\left|f''\left(\frac{a+b}{2}\right)\right|\right)\right]\\&+\frac{(b-a)^2}{16}\frac{\left|f''\left(\frac{a+b}{2}\right)\right||f''(b)|}{\left(|f''(b)|-\left|f''\left(\frac{a+b}{2}\right)\right|\right)^2}\left[A\left(\left|f''\left(\frac{a+b}{2}\right)\right|,|f''(b)|\right)\right.\\&\left.-\left|f''\left(\frac{a+b}{2}\right)\right||f''(b)|-L^{-1}\left(\left|f''\left(\frac{a+b}{2}\right)\right|,|f''(b)|\right)\right],\end{aligned}$$

ii) If $\left|f''\left(A_{2,k+1}\right)\right|-\left|f''\left(A_{2,k}\right)\right|=0$ for $k=1,2$, then

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)^2}{96}\left[\left|f''\left(\frac{a+b}{2}\right)\right|+|f''(b)|\right].$$

Theorem 2.4. *Let $f:I\subset(0,\infty)\to(0,\infty)$ be a twice differentiable mapping on I°, $n\in\mathbb{N}$ and $a,b\in I^\circ$ with $a<b$ such that $f''\in L_1[a,b]$ and $|f''|^q$ are an arithmetic-harmonically convex function on the interval $[a,b]$ for some fixed $q>1$, then the following inequalities hold:*
i) If $\left|f''\left(A_{n,k+1}\right)\right|^q-\left|f''\left(A_{n,k}\right)\right|^q\neq 0$, then

$$|J_n(f,a,b)|\leq\sum_{k=1}^{n}\frac{(b-a)^2}{2n^3}\frac{\left[B(p+1,p+1)\right]^{\frac{1}{p}}G^2\left(\left|f''\left(A_{n,k}\right)\right|,\left|f''\left(A_{n,k+1}\right)\right|\right)}{L^{\frac{1}{q}}\left(\left|f''\left(A_{n,k}\right)\right|^q,\left|f''\left(A_{n,k+1}\right)\right|^q\right)},\tag{2.6}$$

ii) If $\left|f''\left(A_{n,k+1}\right)\right|^q-\left|f''\left(A_{n,k}\right)\right|^q=0$, then

$$|J_n(f,a,b)|\leq\sum_{k=1}^{n}\frac{(b-a)^2}{2n^3}\left[B(p+1,p+1)\right]^{\frac{1}{p}}\left|f''\left(A_{n,k+1}\right)\right|.\tag{2.7}$$

where $B(\alpha,\beta)$ is the classical Beta function and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. i) Let $\left|f''\left(A_{n,k+1}\right)\right|^q-\left|f''\left(A_{n,k}\right)\right|^q\neq 0$. From the Lemma 1.5 and the properties of modulus, the following inequality can be writtten

$$|J_n(f,a,b)|\leq\sum_{k=1}^{n}\frac{(b-a)^2}{2n^3}\left[\int_0^1|t(1-t)|\left|f''\left(tA_{n,k}+(1-t)A_{n,k+1}\right)\right|dt\right].\tag{2.8}$$

Since $|f''|^q$ is an arithmetic-harmonically convex function on the interval $[a,b]$, the inequality

$$\left|f''\left(tA_{n,k}+(1-t)A_{n,k+1}\right)\right|^q\leq\frac{\left|f''\left(A_{n,k}\right)\right|^q\left|f''\left(A_{n,k+1}\right)\right|^q}{t\left|f''\left(A_{n,k+1}\right)\right|^q+(1-t)\left|f''\left(A_{n,k}\right)\right|^q}\tag{2.9}$$

holds. By applying the well known Hölder integral inequality and the inequality (2.9) on (2.8), the inequality

$$\begin{aligned}|J_n(f,a,b)|&\leq\sum_{k=1}^{n}\frac{(b-a)^2}{2n^3}\left(\int_0^1[t(1-t)]^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''\left(tA_{n,k}+(1-t)A_{n,k+1}\right)\right|^q dt\right)^{\frac{1}{q}}\\&\leq\sum_{k=1}^{n}\frac{(b-a)^2}{2n^3}\left(\int_0^1 t^p(1-t)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\frac{\left|f''\left(A_{n,k}\right)\right|^q\left|f''\left(A_{n,k+1}\right)\right|^q dt}{t\left|f''\left(A_{n,k+1}\right)\right|^q+(1-t)\left|f''\left(A_{n,k}\right)\right|^q}\right)^{\frac{1}{q}}\qquad(2.10)\\&=\sum_{k=1}^{n}\frac{(b-a)^2}{2n^3}\frac{\left[B(p+1,p+1)\right]^{\frac{1}{p}}G^2\left(\left|f''\left(A_{n,k}\right)\right|,\left|f''\left(A_{n,k+1}\right)\right|\right)}{L^{\frac{1}{q}}\left(\left|f''\left(A_{n,k}\right)\right|^q,\left|f''\left(A_{n,k+1}\right)\right|^q\right)},\qquad(2.11)\end{aligned}$$

is obtained, where

$$\int_0^1 t^p (1-t)^p dt = B(p+1,p+1)$$

$$\int_0^1 \frac{1}{t\left|f''\left(A_{n,k+1}\right)\right|^q + (1-t)\left|f''\left(A_{n,k}\right)\right|^q} dt = L^{-1}\left(\left|f''\left(A_{n,k}\right)\right|^q, \left|f''\left(A_{n,k+1}\right)\right|^q\right).$$

ii) Let $\left|f''\left(A_{n,k+1}\right)\right|^q - \left|f''\left(A_{n,k}\right)\right|^q = 0$. Then, substituting $\left|f''\left(A_{n,k+1}\right)\right|^q = \left|f''\left(A_{n,k}\right)\right|^q$ in the inequality (2.10), the following inequality is found:

$$|J_n(f,a,b)| \leq \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3} \left[B(p+1,p+1)\right]^{\frac{1}{p}} \left|f''\left(A_{n,k+1}\right)\right|.$$

This completes the proof of theorem. □

Corollary 2.5. *By choosing $n = 1$ in Theorem 2.4, the following inequalities are obtained:*
i) If $\left|f''\left(A_{1,k+1}\right)\right| - \left|f''\left(A_{1,k}\right)\right| \neq 0$ for $k = 1$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{2} \frac{\left[B(p+1,p+1)\right]^{\frac{1}{p}} G^2\left(\left|f''(a)\right|, \left|f''(b)\right|\right)}{L^{\frac{1}{q}}\left(\left|f''(a)\right|^q, \left|f''(b)\right|^q\right)},$$

ii) If $\left|f''\left(A_{1,k+1}\right)\right| - \left|f''\left(A_{1,k}\right)\right| = 0$ for $k = 1$, then

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{2} \left[B(p+1,p+1)\right]^{\frac{1}{p}} \left|f''(b)\right|.$$

Corollary 2.6. *By choosing $n = 2$ in Theorem 2.4, the following Bullen type inequalities are obtained:*
i) If $\left|f''\left(A_{2,k+1}\right)\right| - \left|f''\left(A_{2,k}\right)\right| \neq 0$ for $k = 1,2$, then

$$\begin{aligned}\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{(b-a)^2}{16}\left[B(p+1,p+1)\right]^{\frac{1}{p}} \\ &\times \left[\frac{G^2\left(\left|f''(a)\right|, \left|f''\left(\frac{a+b}{2}\right)\right|\right)}{L^{\frac{1}{q}}\left(\left|f''(a)\right|^q, \left|f''\left(\frac{a+b}{2}\right)\right|^q\right)} + \frac{G^2\left(\left|f''\left(\frac{a+b}{2}\right)\right|, \left|f''(b)\right|\right)}{L^{\frac{1}{q}}\left(\left|f''\left(\frac{a+b}{2}\right)\right|, \left|f''(b)\right|\right)}\right],\end{aligned}$$

ii) If $\left|f''\left(A_{2,k+1}\right)\right| - \left|f''\left(A_{2,k}\right)\right| = 0$ for $k = 1,2$, then

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{16}\left[B(p+1,p+1)\right]^{\frac{1}{p}}\left[\left|f''\left(\frac{a+b}{2}\right)\right| + \left|f''(b)\right|\right].$$

Theorem 2.7. *Let $f : I \subset (0,\infty) \to (0,\infty)$ be a twice differentiable mapping on I°, $n \in \mathbb{N}$ and $a,b \in I^\circ$ with $a < b$ such that $f'' \in L_1[a,b]$ and $\left|f''\right|^q$ are an arithmetic-harmonically convex function on the interval $[a,b]$ for some fixed $q \geq 1$, then the following inequalities hold:*
i) If $\left|f''\left(A_{n,k+1}\right)\right|^q - \left|f''\left(A_{n,k}\right)\right|^q \neq 0$, then

$$\begin{aligned}|J_n(f,a,b)| &\leq \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3}\left(\frac{1}{6}\right)^{1-\frac{1}{q}} \frac{G^2\left(\left|f''\left(A_{n,k}\right)\right|, \left|f''\left(A_{n,k+1}\right)\right|\right)}{\left(\left|f''\left(A_{n,k+1}\right)\right|^q - \left|f''\left(A_{n,k}\right)\right|^q\right)^{\frac{2}{q}}} \\ &\times \left[A\left(\left|f''\left(A_{n,k}\right)\right|^q, \left|f''\left(A_{n,k+1}\right)\right|^q\right) - \frac{G^2\left(\left|f''\left(A_{n,k}\right)\right|^q, \left|f''\left(A_{n,k+1}\right)\right|^q\right)}{L\left(\left|f''\left(A_{n,k}\right)\right|^q, \left|f''\left(A_{n,k+1}\right)\right|^q\right)}\right]^{\frac{1}{q}},\end{aligned} \tag{2.12}$$

ii) If $\left|f''\left(A_{n,k+1}\right)\right|^q - \left|f''\left(A_{n,k}\right)\right|^q = 0$, then

$$|J_n(f,a,b)| \leq \sum_{k=1}^{n} \frac{(b-a)^2}{12n^3}\left|f''\left(A_{n,k+1}\right)\right|, \tag{2.13}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. i) Let $\left|f''\left(A_{n,k+1}\right)\right|^q - \left|f''\left(A_{n,k}\right)\right|^q \neq 0$. From the Lemma 1.5 and the properties of modulus, the inequality can be written:

$$|J_n(f,a,b)| \leq \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3}\left[\int_0^1 |t(1-t)|\left|f''\left(tA_{n,k} + (1-t)A_{n,k+1}\right)\right| dt\right]. \tag{2.14}$$

Since $\left|f''\right|^q$ is an arithmetic-harmonically convex function on the interval $[a,b]$, the inequality

$$\left|f''\left(tA_{n,k} + (1-t)A_{n,k+1}\right)\right|^q \leq \frac{\left|f''\left(A_{n,k}\right)\right|^q \left|f''\left(A_{n,k+1}\right)\right|^q}{t\left|f''\left(A_{n,k+1}\right)\right|^q + (1-t)\left|f''\left(A_{n,k}\right)\right|^q}$$

holds. By applying the last inequality and the well known power-mean integral inequality on (2.14), the inequality

$$\begin{aligned}
|J_n(f,a,b)| &\le \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3}\left(\int_0^1 |t(1-t)|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |t(1-t)|\left|f''\left(tA_{n,k}+(1-t)A_{n,k+1}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\le \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3}\left(\int_0^1 t(1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{t(1-t)\left|f''\left(A_{n,k}\right)\right|^q\left|f''\left(A_{n,k+1}\right)\right|^q}{t\left|f''\left(A_{n,k+1}\right)\right|^q+(1-t)\left|f''\left(A_{n,k}\right)\right|^q}dt\right)^{\frac{1}{q}} \\
&= \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3}\left|f''\left(A_{n,k}\right)\right|\left|f''\left(A_{n,k+1}\right)\right|\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\left(\int_0^1 \frac{t(1-t)\,dt}{t\left|f''\left(A_{n,k+1}\right)\right|^q+(1-t)\left|f''\left(A_{n,k}\right)\right|^q}\right)^{\frac{1}{q}} \\
&= \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\frac{G^2\left(\left|f''\left(A_{n,k}\right)\right|,\left|f''\left(A_{n,k+1}\right)\right|\right)}{\left(\left|f''\left(A_{n,k+1}\right)\right|^q-\left|f''\left(A_{n,k}\right)\right|^q\right)^{\frac{2}{q}}} \\
&\quad\times\left[A\left(\left|f''\left(A_{n,k}\right)\right|^q,\left|f''\left(A_{n,k+1}\right)\right|^q\right)-\frac{G^2\left(\left|f''\left(A_{n,k}\right)\right|^q,\left|f''\left(A_{n,k+1}\right)\right|^q\right)}{L\left(\left|f''\left(A_{n,k}\right)\right|^q,\left|f''\left(A_{n,k+1}\right)\right|^q\right)}\right]^{\frac{1}{q}},
\end{aligned} \tag{2.15}$$

is obtained, where

$$\begin{aligned}
&\int_0^1 t(1-t)\,dt=\frac{1}{6}, \\
&\int_0^1 \frac{t(1-t)}{t\left|f''\left(A_{n,k+1}\right)\right|^q+(1-t)\left|f''\left(A_{n,k}\right)\right|^q}dt=\frac{1}{\left(\left|f''\left(A_{n,k+1}\right)\right|^q-\left|f''\left(A_{n,k}\right)\right|^q\right)^2} \\
&\times\left[A\left(\left|f''\left(A_{n,k}\right)\right|^q,\left|f''\left(A_{n,k+1}\right)\right|^q\right)-\frac{G^2\left(\left|f''\left(A_{n,k}\right)\right|^q,\left|f''\left(A_{n,k+1}\right)\right|^q\right)}{L\left(\left|f''\left(A_{n,k}\right)\right|^q,\left|f''\left(A_{n,k+1}\right)\right|^q\right)}\right].
\end{aligned}$$

ii) Let $\left|f''\left(A_{n,k+1}\right)\right|^q-\left|f''\left(A_{n,k}\right)\right|^q=0$. Then, substituting $\left|f''\left(A_{n,k+1}\right)\right|^q=\left|f''\left(A_{n,k}\right)\right|^q$ in the inequality (2.15), the following inequality is found:

$$\begin{aligned}
|J_n(f,a,b)| &\le \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3}\left(\int_0^1 t(1-t)\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 t(1-t)\left|f''\left(A_{n,k+1}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&= \sum_{k=1}^{n} \frac{(b-a)^2}{12n^3}\left|f''\left(A_{n,k+1}\right)\right|.
\end{aligned}$$

This completes the proof of theorem. □

Corollary 2.8. *By choosing $n=1$ in Theorem 2.7, the following inequalities are obtained:*
i) If $\left|f''\left(A_{1,k+1}\right)\right|^q-\left|f''\left(A_{1,k}\right)\right|^q\neq 0$ for $k=1$, then

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| &\le \frac{(b-a)^2}{2}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\frac{G^2\left(\left|f''(a)\right|,\left|f''(b)\right|\right)}{\left(\left|f''(b)\right|^q-\left|f''(a)\right|^q\right)^{\frac{2}{q}}} \\
&\quad\times\left[A\left(\left|f''(a)\right|^q,\left|f''(b)\right|^q\right)-\frac{G^2\left(\left|f''(a)\right|^q,\left|f''(b)\right|^q\right)}{L\left(\left|f''(a)\right|^q,\left|f''b\right|^q\right)}\right]^{\frac{1}{q}},
\end{aligned}$$

ii) If $\left|f''\left(A_{1,k+1}\right)\right|^q-\left|f''\left(A_{1,k}\right)\right|^q=0$ for $k=1$, then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{(b-a)^2}{12}\left|f''(b)\right|.$$

Corollary 2.9. *By choosing $n=2$ in Theorem 2.7, the following Bullen type inequalities are obtained:*
i) If $\left|f''\left(A_{2,k+1}\right)\right|^q-\left|f''\left(A_{2,k}\right)\right|^q\neq 0$ for $k=1,2$, then

$$\begin{aligned}
&\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^2}{16}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\frac{G^2\left(\left|f''(a)\right|,\left|f''\left(\frac{a+b}{2}\right)\right|\right)}{\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q-\left|f''(a)\right|^q\right)^{\frac{2}{q}}} \\
&\times\left[A\left(\left|f''(a)\right|^q,\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)-\frac{G^2\left(\left|f''(a)\right|^q,\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)}{L\left(\left|f''(a)\right|^q,\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)}\right]^{\frac{1}{q}}+\frac{(b-a)^2}{16}\left(\frac{1}{6}\right)^{1-\frac{1}{q}}\frac{G^2\left(\left|f''\left(\frac{a+b}{2}\right)\right|,\left|f''(b)\right|\right)}{\left(\left|f''(b)\right|^q-\left|f''\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{2}{q}}} \\
&\times\left[A\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(b)\right|^q\right)-\frac{G^2\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(b)\right|^q\right)}{L\left(\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(b)\right|^q\right)}\right]^{\frac{1}{q}},
\end{aligned}$$

ii) If $\left|f''\left(A_{2,k+1}\right)\right|^q-\left|f''\left(A_{2,k}\right)\right|^q=0$ for $k=1,2$, then

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{(b-a)^2}{16}\left[\left|f''\left(\frac{a+b}{2}\right)\right|+\left|f''(b)\right|\right].$$

Corollary 2.10. *Taking $q=1$ in the inequality (2.12), the following inequality is obtained:*

$$|J_n(f,a,b)| \le \sum_{k=1}^{n} \frac{(b-a)^2}{2n^3} \frac{G^2\left(\left|f''(A_{n,k})\right|,\left|f''(A_{n,k+1})\right|\right)}{\left(\left|f''(A_{n,k+1})\right|-\left|f''(A_{n,k})\right|\right)^2} \times \left[A\left(\left|f''(A_{n,k})\right|,\left|f''(A_{n,k+1})\right|\right) - \frac{G^2\left(\left|f''(A_{n,k})\right|,f''(A_{n,k+1})\right)}{L\left(\left|f''(A_{n,k})\right|,\left|f''(A_{n,k+1})\right|\right)}\right].$$

3. Applications for special means

If $p\in(-1,0)$ then the function $f(x)=x^p, x>0$ is an arithmetic harmonically-convex [2]. Using this function, the following propositions are obtained:

Proposition 3.1. *Let $0<a<b$ and $p\in(-1,0)$. Then, the following inequality holds:*

$$\frac{1}{(p+1)(p+2)}\left|\sum_{k=1}^{n}\frac{1}{n}\left[(A_{n,k})^{p+2}+(A_{n,k+1})^{p+2}\right]-L_{p+2}^{p+2}(a,b)\right|$$
$$\le \sum_{k=1}^{n}\frac{(b-a)^2}{2n^3}\frac{(A_{n,k})^p(A_{n,k+1})^p}{\left[(A_{n,k+1})^p-(A_{n,k})^p\right]^2}\left[A\left((A_{n,k})^p,(A_{n,k+1})^p\right)-\frac{(A_{n,k})^p(A_{n,k+1})^p}{L\left((A_{n,k})^p,(A_{n,k+1})^p\right)}\right].$$

Proof. It is known that if $p\in(-1,0)$ then the function $f(x)=\frac{x^{p+2}}{(p+1)(p+2)}, x>0$ is an arithmetic harmonically-convex function. Therefore, the assertion follows from the inequality (2.1) in the Theorem 2.1, for $f:(0,\infty)\to\mathbb{R}$, $f(x)=\frac{x^{p+2}}{(p+1)(p+2)}$. □

Corollary 3.2. *Taking $n=1$ in Proposition 3.1, the following inequality is obtained:*

$$\frac{1}{(p+1)(p+2)}\left|A\left((A_{1,1})^{p+2},(A_{1,2})^{p+2}\right)-L_{p+2}^{p+2}(a,b)\right|$$
$$\le \frac{(b-a)^2}{2}\frac{(A_{1,1})^p(A_{1,2})^p}{\left[(A_{1,2})^p-(A_{1,1})^p\right]^2}\left[A\left((A_{1,1})^p,(A_{1,2})^p\right)-\frac{a^pb^p}{L(a^p,b^p)}\right],$$

that is,

$$\frac{1}{(p+1)(p+2)}\left|A\left(a^{p+1},b^{p+1}\right)-L_{p+2}^{p+2}(a,b)\right| \le \frac{(b-a)^2}{2}\frac{a^pb^p}{[a^p-b^p]^2}\left[A(a^p,b^p)-\frac{G^{2p}(ab)}{L(a^p,b^p)}\right].$$

Proposition 3.3. *Let $a,b\in(0,\infty)$ with $a<b$, $q>1$ and $m\in(-1,0)$. Then, the following inequality is obtained:*

$$\frac{1}{\left(\frac{m}{q}+1\right)\left(\frac{m}{q}+2\right)}\left|\sum_{k=1}^{n}\frac{1}{n}A\left((A_{n,k})^{\frac{m}{q}+2},(A_{n,k+1})^{\frac{m}{q}+2}\right)-L_{\frac{m}{q}+2}^{\frac{m}{q}+2}(a,b)\right| \le \sum_{k=1}^{n}\frac{(b-a)^2}{2n^3}\frac{[B(p+1,p+1)]^{\frac{1}{p}}G^{\frac{2m}{q}}\left((A_{n,k}),(A_{n,k+1})\right)}{L^{\frac{1}{q}}\left((A_{n,k})^m,(A_{n,k+1})^m\right)}.$$

Proof. The assertion follows from the inequality (2.6) in the Theorem 2.4. Let

$$f(x)=\frac{1}{\left(\frac{m}{q}+1\right)\left(\frac{m}{q}+2\right)}x^{\frac{m}{q}+2},\quad x\in(0,\infty).$$

Then

$$\left|f''(x)\right|^q=x^m$$

is an arithmetic harmonically-convex on $(0,\infty)$ and the result follows directly from Theorem 2.4. □

Corollary 3.4. *Taking $n=1$ in Proposition 3.3, the following inequality is obtained:*

$$\frac{1}{\left(\frac{m}{q}+1\right)\left(\frac{m}{q}+2\right)}\left|A\left(a^{\frac{m}{q}+2},b^{\frac{m}{q}+2}\right)-L_{\frac{m}{q}+2}^{\frac{m}{q}+2}(a,b)\right| \le \frac{(b-a)^2}{2}\frac{[B(p+1,p+1)]^{\frac{1}{p}}G^{\frac{2m}{q}}(a,b)}{L^{\frac{1}{q}}(a^m,b^m)}.$$

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