Communications in Advanced Mathematical Sciences
Vol. II, No. 3, 213-218, 2019
Research Article
e-ISSN:2651-4001
DOI: 10.33434/cams.541413

Computable Proximity of ℓ_1-Discs on the Digital Plane

James F. Peters[1]*, K. Kordzaya[2] and I. Dochviri[3]

Abstract
This paper investigates problems in the characterization of the proximity of digital discs. Based on the l_1-metric structure for the 2D digital plane and using a Jaccard-like metric, we determine numerical characters for intersecting digital discs.

Keywords: Digital discs, Jaccard like metric, l_1-metric, Proximity
2010 AMS: Primary 65D18, Secondary, 68U05, 54E05

[1] *Computational Intelligence Laboratory, University of Manitoba, WPG, MB, R3T 5V6, Canada,*
Department of Mathematics, Faculty of Arts and Sciences, Adiyaman University, 02040 Adiyaman, Turkey,
The research has been supported by the Natural Sciences & Engineering Research Council of Canada (NSERC) discovery grant 185986, Instituto Nazionale di Alta Matematica (INdAM) Francesco Severi, Gruppo Nazionale per le Strutture Algebriche, Geometriche e Loro Applicazioni grant 9 920160 000362, n.prot U 2016/000036 and Scientific and Technological Research Council of Turkey (TÜBİTAK) Scientific Human Resources Development (BIDEB) under grant no: 2221-1059B211301223.
[2] *Department of Mathematics, Caucasus International University, 73, Chargali str., 0192 Tbilisi, Georgia, korka@ciu.edu.ge*
[3] *Department of Mathematics, Caucasus International University, 73, Chargali str., 0192 Tbilisi, Georgia, iraklidoch@yahoo.com*
***Corresponding author**: James.Peters3@umanitoba.ca
Received: 18 March 2019, **Accepted:** 1 August 2019, **Available online:** 30 September 2019

1. Introduction

In pure and applied mathematics, one of the important questions is connected to the discovery of proximal objects [1]. The objects often can be represented as sets of points and this stipulates that set-theoretic and topological methods are very useful tools in the study of proximity relations. Digital geometry is the study of geometric properties of shapes in digital pictures.

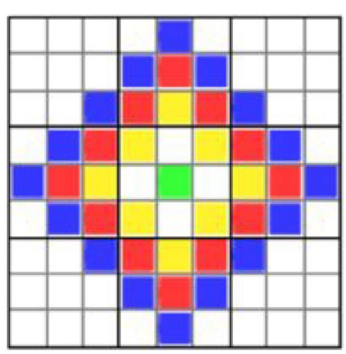


Figure 1.1. Structure of the Digital Discs

Many different computer screen images can be obtained via pixel lighting. A *pixel* is the smallest element on the digital plane and they are usually identified as points. In other words, we can describe images on the computer screen by their pixels that have digital valued coordinates, *i.e.*, a mathematical model of the computer screen is the digital plane $\mathbb{Z}^2$.

The importance of the notions of the circle and disc in Euclidean geometry is well known. In digital geometry, digital circles and digital discs have various important properties that are different from the Euclidean ones (see, *e.g.*, [2–4]). One of the reasonable realizations of metric structure on the digital plane $\mathbb{Z}^2$ can be determined via the so-called l_1 metric. This metric has the following view:

$$\begin{aligned}d(p_1,p_2) = |a_1-a_2|+|b_1-b_2|, &\text{where } p_1 \text{ and } p_2 \text{ are some matched points,}\\ &\text{with coordinates } (a_1,b_1) \text{ and } (a_2,b_2)\text{. respectively,}\end{aligned}$$

i.e., p_1 and p_2 are pixels for our future considerations. Since we can represent pixel coordinates as digital pairs, then it is obvious that $d(p_1,p_2)\in\mathbb{Z}$ (the integers).

Based on the l_1 metric, we define a digital circle with radius r and center x (denoted by $C_d(x,r)$) as follows:

$$C_d(x,r) = \left\{z\in\mathbb{Z}^2 : d(x,z) = r\right\}.$$

Moreover, we denote by $c(C_d(x,r))$ the circumference of the circle $C_d(x,r)$ where $r\in\mathbb{N}\cup\{0\}$.

It is well-known that if r is a natural number, we have $\pi_{l_1} = \frac{c(C_d(x,r))}{\mathrm{diam}(C_d(x,r))} = \frac{8r}{2r} = 4$, where $\mathrm{diam}(C_d(x,r))$ is the diameter of the circle $C_d(x,r)$. Using this fact, we easily obtain the following result.

Lemma 1.1. *Let $C_d(x,r)$ be a digital circle with center at point x and radius r relative to the l_1 metric. Then, for the number of pixels of $C_d(x,r)$, we have the formula*

$$card(C_d(x,r)) = \frac{2c(C_d(x,r))}{\pi_{l_1}} = 4r.$$

Fig. 1.1 demonstrates the structural property of the digital disc, namely,

$$D_d(x,R) = \left\{z\in\mathbb{Z}^2 \,|\, d(x,z)\le R\right\}, \text{ particularly:}$$

$$D_d(x,R) = \{x\}\cup\left(\bigcup_{r=1}^{R} C_d(x,r)\right), \text{ where } R\in\mathbb{N}.$$

Lemma 1.2. *If $D_d(x,R)$ is a digital disc relative to the l_1 metric d, then the number of pixels forming the disc $D_d(x,R)$ can be computed by the formula $card(D_d(x,R)) = 2R^2+2R+1$.*

Proof. Since $D_d(x,R) = \{x\}\cup\left(\bigcup_{r=1}^{R} C_d(x,r)\right)$, we can write

$$\mathrm{card}(D_d(x,R)) = 1+\mathrm{card}(C_d(x,1))+\mathrm{card}(C_d(x,2))+\cdots+\mathrm{card}(C_d(x,R)).$$

Now, applying Lemma 1.1, we get

$$\begin{aligned}\mathrm{card}(D_d(x,R)) &= 1+4+8+\cdots+4R =\\ &= 1+4\left(\frac{1+R}{2}R\right) =\\ &= 2R^2+2R+1.\end{aligned}$$

□

2. How near are digital discs?

To solve a wide class of the problems of computational proximity, the Hausdorff metric is appropriate. The Hausdroff metric (denoted by $d_H(A,B)$) measures the distance between the sets A,B in the given metric space (X,d) and is defined by

$$d_H(A,B) = \max\left\{\sup_{x\in A}\inf_{y\in B} d(x,y), \sup_{y\in B}\inf_{x\in A} d(x,y)\right\}.$$

If the sets A,B are finite, we obtain the simplication of the Hausdorff metric by maxima and minima, *i.e.*,

$$d_H(A,B) = \max\left\{\max_{x\in A}\min_{y\in B} d(x,y), \max_{y\in B}\min_{x\in A} d(x,y)\right\}.$$

It is clear that even in the case of finite sets, computation of the Hausdorff distances are quite capacitive. These difficulties can be bypassed in some special cases of analytical sets. Below, we are interested in characterization of intersecting digital discs.

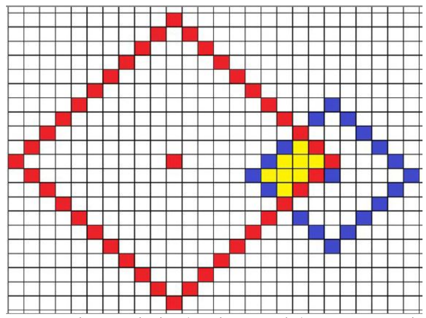


Figure 2.1. Intersecting Digital Discs with Intersecting Boundaries

Classification of images in computer science frequently need the application of Jaccard-like metrics [5], [6], [7] . We will use a simplified version to analyze proximity of intersecting digital discs. It must be especially noticed that the problem connected with the intersection of plane discs was considered from a computer science perspective in [8].

For the Jaccard-like metric m, we understand the distance function defined via the cardinality of the symmetric difference of two arbitrary nonempty finite sets A and B, *i.e.*,

$$\begin{aligned} m(A,B) &= \mathrm{card}(A \triangle B) \\ &= \mathrm{card}(A \setminus B) + \mathrm{card}(B \setminus A) \\ &= \mathrm{card}(A) + \mathrm{card}(B) - 2\mathrm{card}(A \cap B). \end{aligned}$$

It is obvious that if $\mathrm{card}(A) \neq \mathrm{card}(B)$ and both sets are finite while $A \cap B \neq \varnothing$, we get $m(A,B) \neq 0$. This raises the question of the computation of the proximity of intersecting digital discs such as the ones in Fig. 2.1.

Theorem 2.1. *Let $D_d(x,R_1)$ and $D_d(y,R_2)$ be digital discs such that $C_d(x,R_1) \cap C_d(y,R_2) \neq \varnothing$. Then*

$$m(D_d(x,R_1), D_d(y,R_2)) = 2\left(R_1^2 + R_2^2 + R_1 + R_2 - 2kn + k + n\right),$$

where k and n denote the number of pixels forming the width and height of the greatest rectangle subset of an intersection set.

Proof. Applying Lemma 1.2, we obtain the following cardinal equalities:

$$\begin{aligned} m(D_d(x,R_1), D_d(y,R_2)) &= \mathrm{card}(D_d(x,R_1)) + \mathrm{card}(D_d(y,R_2)) - 2\mathrm{card}(D_d(x,R_1) \cap D_d(y,R_2)) \\ &= 2\left(R_1^2 + R_2^2 + R_1 + R_2 + 1)\right) - 2\mathrm{card}(D_d(x,R_1) \cap D_d(y,R_2)) \\ &= 2\left(R_1^2 + R_2^2 + R_1 + R_2 + 1\right) - 2[kn + (k-1)(n-1)] \\ &= 2\left(R_1^2 + R_2^2 + R_1 + R_2 - 2kn + k + n\right) \end{aligned}$$

□

Notice that there is a situation in which two digital discs are intersecting but their boundaries are not intersecting (see, *e.g.*, Fig.2.2). Observe that in that case, we have $C_d(x,R_1-1) \cap C_d(y,R_2) \neq \varnothing$, or, equivalently, $C_d(x,R_1) \cap C_d(y,R_2-1) \neq \varnothing$.

Theorem 2.2. *Let $D_d(x,R_1)$ and $D_d(y,R_2)$ be digital discs such that $C_d(x,R_1) \cap C_d(y,R_2) = \varnothing$, but $C_d(x,R_1-1) \cap C_d(y,R_2) \neq \varnothing$. Then we have $m(D_d(x,R_1), D_d(y,R_2)) = 2\left(R_1^2 + R_2^2 + R_1 + R_2 + 1 - 2kn\right)$, where k and n denote the number of pixels forming the width and height of the greatest rectangle subset of an intersection set.*

Proof. In this case, we can easily note that $\mathrm{card}(D_d(x,R_1) \cap D_d(y,R_2)) = 2kn$. Hence, we have $\mathrm{m}(D_d(x,R_1), D_d(y,R_2)) = 2\left(R_1^2 + R_2^2 + R_1 + R_2 + 1 - 2kn\right)$. □

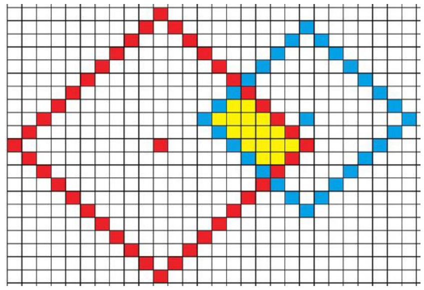
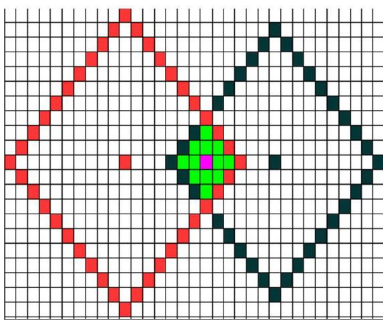
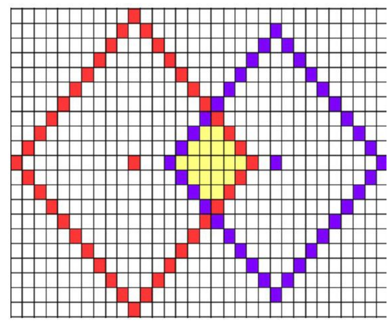


Figure 2.2. Intersecting Discs with Non-Intersecting Boundaries

Next, we need to represent the centers x and y of discs $D_d(x,R_1)$ and $D_d(y,R_2)$ by a couple of digital coordinates as follows: $x=(\alpha,\beta)$ and $y=(\gamma,\delta)$. If one of the following equalities hold $d(x,y)=|\alpha-\gamma|$ or $d(x,y)=|\beta-\delta|$, *i.e.*, the centers of the discs lie on horizontal or vertical axes (similar to the situations shown in Fig. 2.3.1 and Fig. 2.3.2), then we can measure the proximity of the discs via computation of the pixel cardinality of the intersections sets.



2.3.1: Intersecting Discs with Intersecting Boundaries



2.3.2: Intersecting Discs with Non-Intersecting Boundaries

Figure 2.3. Intersecting Discs on the Digital Plane

Theorem 2.3. *Let $D_d(x,R_1)$ and $D_d(y,R_2)$ be digital discs such that $x=(\alpha,0)$ and $y=(\gamma,0)$ with $\alpha<\gamma$ and $\gamma-\alpha\leq R_1+R_2$. If $C_d(x,R_1)\cap C_d(y,R_2)\neq\varnothing$, then*

$$m(D_d(x,R_1),D_d(y,R_2))=(R_1-R_2)^2+2(R_1+R_2+1)(\gamma-\alpha)-(\gamma-\alpha)^2.$$

Proof. Since $x=(\alpha,0)$, $y=(\gamma,0)$ and $C_d(x,R_1)\cap C_d(y,R_2)\neq\varnothing$, we claim that

$$C_d(x,R_1)\cap C_d(y,R_2)=C_d(k,r), \text{ where,}$$

$k=\left(\frac{\alpha+R_1+\gamma-R_2}{2},0\right)$ and
$r=R_1-(k-\alpha)=\frac{R_1+R_2+(\gamma-\alpha)}{2}\in\mathbb{N}\cup\{0\}$. Consequently, simplification of

$$\mathrm{m}(D_d(x,R_1),D_d(y,R_2))=2\left(R_1^2+R_2^2+R_1+R_2+1-2r^2-2r-1\right)$$

gives the needed expression

$$\mathrm{m}(D_d(x,R_1),D_d(y,R_2))=(R_1-R_2)^2+2(R_1+R_2+1)(\gamma-\alpha)-(\gamma-\alpha)^2.$$

$\square$

Observe that Theorem 2.3 can be applied in similar cases when the intersection set of the digital discs itself is a disc.

This leads us to consider two intersecting digital discs with non-intersecting boundaries (see, *e.g.*, Fig. 2.3.2) so that both centers lie on the horizontal or vertical axes. In such cases, we obtain the following result.

Corollary 2.4. *Let $D_d(x,R_1)$ and $D_d(y,R_2)$ be intersecting digital discs that satisfy the conditions of Theorem 2.3, but $C_d(x,R_1)\cap C_d(y,R_2)=\varnothing$. Then we have*

$$m(D_d(x,R_1),D_d(y,R_2)) = 2\left(R_1^2+R_2^2+R_1+R_2-2r_0^2-4r_0+1\right), where,$$
$$r_0=\frac{R_1-1+R_2+(\gamma-\alpha)}{2}.$$

Recall that a boundary point x_0 of a convex set C is called a *support point* [9, p. 27].

Lemma 2.5. [Bishop-Phelps [9]] *Suppose M is a closed subspace of finite co-dimension in a topological vector space X, and that C is a convex subset of X. Suppose x_0 is a support point of $C\cup M$ in the subspace M. Then x_0 is a support point of C.*

Let A,B be nonempty sets and let bdy$A$ denote the set of boundary points of a nonempty set A. Also, let $A\overset{\wedge\!\!\wedge}{\delta}B$ denote that A and B are overlapping sets. From Lemma 2.5, we obtain

Theorem 2.6. *Let A,B be nonempty sets of digital discs. If A,B are convex sets in a subspace M of the Euclidean plane intersect, then*
1^o $A\cap B$ is a convex set.
2^o $A\overset{\wedge\!\!\wedge}{\delta}B$ (A and B are strongly near).
3^o $x_0\in bdy(A\cap B)$ is a support point of $A\cap B$.

Proof.

1^o Immediate, since the intersection of any two convex sets is a convex set.
2^o Let int$A$,int$B$ be the interior of A,B, respectively. From Theorem 2.6.1, $A\cap B\neq\varnothing$ implies int$A\cap$int$B\neq\varnothing$. Hence, from [10, §2.3], $A\overset{\wedge\!\!\wedge}{\delta}B$.
3^o Immediate from Lemma 2.5.

□

Example 2.7. *The blue and red pixels on the boundary of the intersecting discs in Fig. 2.3.1 are support points.* ■

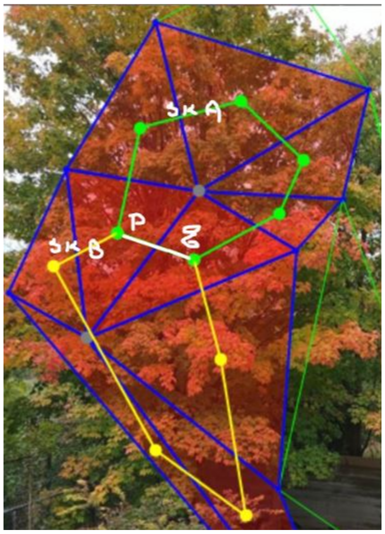


Figure 2.4. Intersecting sets of boundary points sk$A$,sk$B$

In what follows, we give an application of Theorem 2.6.2, namely, digital discs A,B are convex sets in a subspace M of the Euclidean plane intersect, provided

$A\overset{\wedge\!\!\wedge}{\delta}B$ (A and B are strongly near).

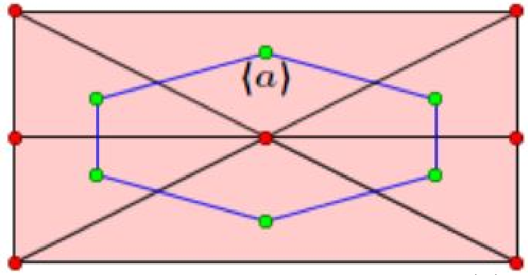


Figure 2.5. Boundary with support point $\langle a \rangle$

3. Application: Classifying triangulated digital images

Recall that an Alexandroff nerve on a triangulated 2D surface is a set of triangles with a common vertex (called the nucleus of the nerve), introduced by P. Alexandroff [Aleksandrov] [11, §31, p. 39], [12] and elaborated in [13, Vol. 3, p. 67], [14, §2.11, pp. 160-161]. Such a nerve with nucleus p is maximal, provided the number of triangles attached to p is highest [15]. It is possible for more than one Alexandroff nerve to be maximal on the same triangulated image (see, for example, Fig. 2.4). This observation leads to an application of Theorem 2.6.2 in classifying triangulated digital images.

Let $\mathrm{sk}A, \mathrm{sk}B$ be sets of boundary points on polygons whose vertexes are barycenters on an Alexandroff nerve in a triangulaged digital image *img* (see, for example, the set of boundary points that includes a support point $\langle a \rangle$ in Fig. 2.5). Also let I be a collection of triangulated digital images.

We can then derive a collection $\mathfrak{C}(I)$ of classified triangulated digital images containing intersecting support points on boundary sets on barycentric polygons on maximal Alexandroff nerves defined by

$$\mathfrak{C}(I) = \overbrace{\left\{ img \in I : \mathrm{sk}A, \mathrm{sk}B \in img \ \& \ A \overset{\Cap}{\delta} B \right\}}^{\text{Images containing overlapping boundary sets}}. \qquad \blacksquare$$

References

[1] I. Dochviri, J.F. Peters, *Topological sorting of finitely near sets*, Math. Comput. Sci., **10**(2) (2016), 273–277. 1

[2] E. Andres, T. Roussillon, *Analytical description of digital circles*, Lecture Notes Comput. Sci., **6607** (2011), 901–917. 1

[3] M.D. McIlroy, *Best approximate circles on integer grids*, ACM Transactions Graph., **2**(4) (1983), 237–263. 1

[4] J.-L. Toutant, E. Andres, T. Roussillon, *Digital circles, spheres and hyperspheres: from morphological models to analytical characterizations and topological properties*, Discrete Appl. Math., **161**(16-17) (2011), 2662–2677. 1

[5] O. Fujita, *Metrics based on average distance between sets*, Jpn. J. Ind. Appl. Math., **30**(1) (2013), 1–19. 2

[6] A. Gardner, J. Kanno, C.A. Duncan, R. Selmic, *Measuring distance between unordered sets of different sizes*, Proceeding 2014 IEEE Conference on Computer Vision and Pattern Recognition, (2014), 137–143. 2

[7] S. Kosub, *A note on the triangle inequality for the Jaccard distance*, Pattern Recognition Letters, **120** (2019), 36–38. 2

[8] M. Sharir, *Intersection and closest-pair problems for a set of planar discs*, SIAM J. Comput., **14**(2) (1985), 448–468. 2

[9] E. Bishop, R.R. Phelps, *The support functionals of a convex set*, 1963 Proc. Sympos. Pure Math., Amer. Math. Soc., **VII** (1985), 27–35. 2, 2.5

[10] J. F. Peters, *Proximal planar shape signatures. Homology nerves and descriptive proximity*, Adv. Math.: Sci. J., **6**(2) (2017), 71–85. 2

[11] P. Alexandroff, *Elementary Concepts of Topology*, Dover Publications, New York, 1965. 3

[12] P. Alexandroff, *Simpliziale approximationen in der allgemeinen topologie*, Math. Ann., **101**(1) (1926), 452–456. 3

[13] P.S. Alexandrov, *Combinatorial Topology*, Graylock Press, Baltimore, Md, USA, 1956. 3

[14] P. Alexandroff, H. Hopf, *Topologie. Band I*, Springer, Berlin, 1935. 3

[15] J.F. Peters, *Proximal planar shapes. Correspondence between triangulated shapes and nerve complexes*, Bull. Allahabad Math. Soc., **33**(1) (2018), 113–137. 3