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Characterizations of Hayashi-Samuel Spaces via Boundary Points

Sk Selim[1], Shyamapada Modak[2]*, Md. Monirul Islam[3]

Abstract
Some new closure operators in topological spaces with ideals are a part of this paper. A comparative study of a new type of boundary point, which is defined with the help of the local function and the boundary points will be discussed through this paper. Characterizations of Hayashi-Samuel spaces are also an object of this paper.

Keywords: Ideal topological space, Hayashi-Samuel space, Local function, ψ-operator.
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[1] *Department of Mathematics, University of Gour Banga, P.O. Mokdumpur, Malda, ORCID: 0000-0002-4226-2004*
[2] *Department of Mathematics, University of Gour Banga, P.O. Mokdumpur, Malda, ORCID:0000-0002-0226-2392*
[3] *Department of Mathematics, University of Gour Banga, P.O. Mokdumpur, Malda, ORCID: 0000-0003-4748-4690*
***Corresponding author**: spmodak2000@yahoo.co.in
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1. Introduction and preliminaries

A modification of closure operator in topological space is the local function in ideal topological space. This study was introduced by Kuratowski [1] and Vaidyanathswamy [2]. An ideal [1] $\mathscr{I}$ on a topological space (X,τ) is a nonempty collection of subsets of X which satisfies the following conditions:

(1) $A\in\mathscr{I}$ and $B\subseteq A$ implies $B\in\mathscr{I}$,

(2) $A\in\mathscr{I}$ and $B\in\mathscr{I}$ implies $A\cup B\in\mathscr{I}$.

A topological space (X,τ) with an ideal $\mathscr{I}$ on X is called an ideal topological space and is denoted by $(X,\tau,\mathscr{I})$. For a subset A of an ideal topological space $(X,\tau,\mathscr{I})$, the local function A^* is defined as: $A^*=\{x\in X : U_x\cap A\notin\mathscr{I},\, U_x\in\tau(x)\}$ (where $\tau(x)$ is the collection of all open sets which contains x) and it was defined by imposing extra condition on the closure operator. As a result, the mathematicians like Samuel [3], Pavlović [4], Hayashi [5], Hashimoto [6], Janković and Hamlett [7, 8], Ekici [9, 10, 11], Hatir [12], Noiri [11, 12, 13] have reached to obtain a new topology known as $*$-topology and it is finer topology than the original topology. In an ideal topological space $(X,\tau,\mathscr{I})$, the structures-"topology" and"ideal" played important roles simultaneously. The condition $\tau\cap\mathscr{I}=\{\emptyset\}$ is a remarkable part in ideal topological space and such ideal topological space is called Hayashi-Samuel space [14]. Modak and his associates studied this ideal topological space and introduced different types of generalized open sets and operators with the help of ideals (see [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26]). The complement operator of the local function is known as ψ-operator [8, 27] and it is defined by: $\psi(A)=X\setminus(X\setminus A)^*$, for a subset A of an ideal topological space $(X,\tau,\mathscr{I})$. ψ-operator is an important part for the study of ideal topological space.

In this paper, we introduce a new type of boundary points in ideal topological spaces by using $*$ - operator. We consider a comparative study of these boundary points with the boundary points in topological spaces. We also explore the characterizations of Hayashi-Samuel space which was established in [18, 19, 24]. We also obtain more closure operators in ideal topological spaces through this paper.

2. $*$ boundary points

Boundary operator [28] is a set valued set-function and we may consider it by the following way:

Let (X,τ) be a topological space and $A \subseteq X$. The boundary operator $Bd : \wp(X) \to C(\tau)$ is defined as $Bd(A) = Cl(A) \cap Cl(X \setminus A)$, where $C(\tau)$ denotes the collection of all closed sets and $Cl(A)$ denotes the closure of A in (X,τ).

Thus boundary point of a set $A \subseteq X$ is a common point between closure of A and closure of $(X \setminus A)$.

We modify the boundary operator with the help of the local function and call it $*$-boundary operator.

Definition 2.1. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space. The operator $Bd^* : \wp(X) \to C(\tau)$, defined by: $Bd^*(A) = A^* \cap (X \setminus A)^*$, for $A \in \wp(X)$, is called $*$-boundary operator on $(X,\tau,\mathscr{I})$.*

The point $x \in Bd^*(A)$ is called $*$-boundary point of A and it is the common point of A^* and $(X \setminus A)^*$.

We start with the following example which shows that there is some common points in A^* and $(X \setminus A)^*$.

Example 2.2. *Let $X = \mathbb{R}$, $\mathbb{R}_u$ be the usual topology on $\mathbb{R}$ and $\mathscr{I} = \{\emptyset\}$. Then $\mathbb{Q}^* = Cl(\mathbb{Q}) = \mathbb{R}$ and $(\mathbb{R} \setminus \mathbb{Q})^* = Cl(\mathbb{R} \setminus \mathbb{Q}) = \mathbb{R}$. This shows that there are common points between $\mathbb{Q}^*$ and $(\mathbb{R} \setminus \mathbb{Q})^*$.*

We know that boundary points of a set depends on the topology. For this, if we consider the indiscrete topology on $\mathbb{R}$, then $Bd(\mathbb{Q}) = \mathbb{R}$, where $\mathbb{Q}$ denotes the set of all rational numbers. But if we consider the discrete topology on $\mathbb{R}$, then $Bd(\mathbb{Q}) = \emptyset$.

$*$-boundary point of a set depends on not only the topology but the ideal also.

Followings examples show the role of ideal in $*$-boundary points:

Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A \subseteq X$.

(i) If we take $\mathscr{I} = \{\emptyset\}$, then $Bd^*(A) = Bd(A)$.

(ii) If the ideal $\mathscr{I} = \wp(X)$, $Bd^*(A) = \emptyset$.

Note that in discrete topological space, boundary points of any set is always empty. But in any ideal topological space, if the ideal is the collection of all subsets of the set then $*$-boundary points of any set is always empty.

(iii) When the ideal $\mathscr{I} = \mathscr{I}_f$, the ideal of finite subsets of X, then $Bd^*(A)$ is the ω-accumulation points of A and $X \setminus A$.

(iv) If one choose the ideal $\mathscr{I} = \mathscr{I}_c$, the ideal of countable subsets of X, then A^* is precisely the set of condensation points of A and boundary points accordingly.

(v) Let $\mathscr{I}_n$ be the collection of all nowhere dense subsets of (X,τ), then $\mathscr{I}_n$ is an ideal on X. If we take $\mathscr{I} = \mathscr{I}_n$, then $A^* = Cl(Int(Cl(A)))$ and $Bd^*(A) = Cl(Int(Cl(A))) \setminus Int(Cl(Int(A)))$.

(vi) Let (X,τ) be a topological space and $\mathscr{I}_m$ be the collection of all meager sets (or sets of first category). Then it forms an ideal on X and A^* is set the points of second category of A.

Note that for a subset $A \subseteq X$ in a topological space (X,τ) with an ideal $\mathscr{I}$, $x \in Bd^*(A)$ implies $U_x \notin \mathscr{I}$ for all $U_x \in \tau(x)$ but converse statement is not true in general.

Example 2.3. *Let $X = \{a,b,c\}$, $\tau = \{\emptyset, X, \{a\}, \{a,b\}\}$ and $\mathscr{I} = \{\emptyset, \{a\}\}$. Then $(\{b\})^* = \{b,c\}$ and all open sets containing a do not belongs to $\mathscr{I}$ but $a \notin Bd^*(\{b\})$.*

One of the characterizations of $*$-boundary point is:

Theorem 2.4. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A \subseteq X$. Then $x \in Bd^*(A)$ if and only if $x \in A^* \setminus \psi(A)$.*

Similar characterization of boundary point is:

Theorem 2.5. *[28] Let (X,τ) be a topological space and $A \subseteq X$. Then $x \in Bd(A)$ if and only if $x \in Cl(A) \setminus Int(A)$, where $Int(A)$ denotes the interior of A.*

Theorem 2.6. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A \subseteq X$. Then $Bd^*(A) = \emptyset$ if and only if $A^* \subseteq \psi(A)$.*

Similar characterization of boundary point is:

Theorem 2.7. *[28] Let (X,τ) be a topological space and $A \subseteq X$. Then $Bd(A) = \emptyset$ if and only if A is both open and closed.*

Note that $()^*$ is not a closure operator and ψ is not an interior operator, but both $A^* \setminus \psi(A)$ and $Cl(A) \setminus \psi(A)$ are closed set. In this regards, $A \cap \psi(A)$ is an interior operator [8] and $A \cup A^*$ is a closure operator [7] and both the operators induce the same topology which is above $*$-topology [7].

Corollary 2.8. *Let $(X,\tau,\mathscr{I})$ be a Hayashi-Samuel space and $A \subseteq X$. Then $Bd^*(A) = \emptyset$ if and only if $A^* = \psi(A)$.*

Proof. Proof is obvious from Theorem 2.6 and the following lemma. □

Lemma 2.9. *[16] Let $(X,\tau,\mathscr{I})$ be A be a Hayashi-Samuel space and $A \subseteq X$. Then $\psi(A) \subseteq A^*$.*

Theorem 2.10. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A \subseteq X$. Then $Bd^*(A) = (X\setminus A)^*$ if and only if $X\setminus A^* \subseteq \psi(A)$.*

Proof. Suppose $Bd^*(A) = (X\setminus A)^*$. Then $A^*\cap(X\setminus A)^* = (X\setminus A)^*$ implies $(X\setminus A)^* \subseteq A^*$. Therefore $X\setminus A^* \subseteq \psi(A)$.
Proof of the converse part is obvious. □

Theorem 2.11. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space and A be a $\mathscr{I}$-dense subset of X. Then $Bd^*(A) = (X\setminus A)^*$.*

Proof. Obvious from definition of $\mathscr{I}$-dense set (A subset A of X is said to be $\mathscr{I}$-dense [14] if $A^* = X$). □

Now we look how the $*$-boundary operator gives new closure operator:

Theorem 2.12. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A,B \subseteq X$. Then following statements hold:*

1. $Bd^*(\emptyset) = \emptyset$.
2. $Bd^*(X) = \emptyset$.
3. $Bd^*(I) = \emptyset$, *if* $I \in \mathscr{I}$.
4. $Bd^*(A)$ *is a closed set in* (X,τ).
5. $Bd^*(A\cup B) \subseteq Bd^*(A)\cup Bd^*(B)$.
6. $Bd^*(A)\cup Bd^*(B) = [A\cap Bd^*(B)]\cup[Bd^*(A\cup B)]\cup[Bd^*(A)\cap B]$.
7. $Bd^*(A) = A^*\setminus\psi(A)$.
8. $Cl^*(A) = Bd^*(A)\cup\psi(A)\cup A$ *(Cl^* denotes the closure operator of $*$-topology).*
9. $Bd^*(A) = \emptyset$ *implies* $Int^*(A) \supseteq A\cap A^*$ *(Int^* denotes the interior operator of $*$-topology).*
10. $Bd^*(Bd^*(A)) \subseteq Bd^*(A)$.
11. $Bd^*(A) = (X\setminus A)^*\setminus\psi(X\setminus A)$.
12. $Bd^*(X\setminus A) = Bd^*(A)$.
13. $Bd^*(A) \subseteq Bd_{\tau^*(\mathscr{I})}(A) \subseteq Bd(A)$ *($Bd_{\tau^*(\mathscr{I})}(A)$ denotes the set of all boundary points of A with respect to $*$-topology).*
14. $X\setminus Bd^*(A) = \psi(X\setminus A)\cup\psi(A)$.
15. $X = \psi(X\setminus A)\cup\psi(A)\cup Bd^*(A) = \psi(X\setminus A)\cup\psi(A)\cup Bd^*(X\setminus A)$.

Proof. The proofs of 1., 2., 3. and 4. are obvious.
5. $Bd^*(A\cup B) = (A\cup B)^*\cap(X\setminus A\cup B)^* = (A\cup B)^*\cap[(X\setminus A)\cap(X\setminus B)]^* \subseteq (A^*\cup B^*)\cap[(X\setminus A)^*\cap(X\setminus B)^*] = [[(X\setminus A)^*\cap(X\setminus B)^*]\cap A^*]\cup[[(X\setminus A)^*\cap(X\setminus B)^*]\cap B^*] \subseteq [A^*\cap(X\setminus A)^*]\cup[B^*\cap(X\setminus B)^*] = Bd^*(A)\cup Bd^*(B)$.
6. Note that $[A\cap Bd^*(B)]\cup[Bd^*(A)\cap B]\cup[Bd^*(A\cup B)] \subseteq Bd^*(B)\cup Bd^*(B)\cup Bd^*(A\cup B) = Bd^*(A)\cup Bd^*(B)$ (from 4.).
Again $Bd^*(A)\cup Bd^*(B) \subseteq Bd^*(A)\cup Bd^*(B)\cup Bd^*(A\cup B)\cup[A\cap Bd^*(B)]\cup[B\cap Bd^*(A)] = [(A^*\cap(X\setminus A)^*\cup(B^*\cap(X\setminus B)^*]\cup Bd^*(A\cup B)\cup[A\cap Bd^*(B)]\cup[B\cap Bd^*(A)] \subseteq [(A^*\cup B^*)\cup Bd^*(A\cup B)]\cup[A\cap Bd^*(B)]\cup[B\cap Bd^*(A)] = [(A\cup B)^*\cap(X\setminus(A\cup B)^*]\cup[A\cap Bd^*(B)]\cup[B\cap Bd^*(A)] = Bd^*(A\cup B)\cup[A\cap Bd^*(B)]\cup[B\cap Bd^*(A)]$.
7. $Bd^*(A) = A^*\cap(X\setminus A)^* = A^*\cap(X\setminus\psi(A)) = A^*\setminus\psi(A)$.
8. $Bd^*(A)\cup\psi(A)\cup A = (A^*\setminus\psi(A))\cup\psi(A)\cup A = A^*\cup A = Cl^*(A)$.
9. Given that $Bd^*(A) = \emptyset$. Then $A^* \subseteq \psi(A)$ and hence $A\cap A^* \subseteq Int^*(A)$.
10. $Bd^*(Bd^*(A)) = Bd^*[A^*\cap(X\setminus A)^*] = [A^*\cap(X\setminus A)^*]^*\cap(X\setminus[A^*\cap(X\setminus A)^*])^* \subseteq A^{**}\cap(X\setminus A)^{**} \subseteq A^*\cap(X\setminus A)^* = Bd^*(A)$.
11. $Bd^*(A) = (X\setminus A)^*\cap A^* = (X\setminus A)^*\cap[X\setminus\psi(X\setminus A)] = (X\setminus A)^*\setminus\psi(X\setminus A)$.
12. The proof of 12. is obvious from definition.
13. The proof is obvious from the fact $A^* \subseteq Cl^*(A) \subseteq Cl(A)$, for any subset A of X.
14. $X\setminus Bd^*(A) = (X\setminus A^*)\cup[X\setminus(X\setminus A)^*] = \psi(X\setminus A)\cup\psi(A)$.
15. The proof of 15. is obvious from definition. □

Definition 2.13. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space and $A,B \subseteq X$. The operator $k_1 : \wp(X) \to \wp(X)$ on X is defined by*

$$k_1(A) = A \cup T_1(A),$$

where $T_1 : \wp(X) \to \wp(X)$ is an operator which satisfies the following conditions:
(i) $T_1(\emptyset) = \emptyset$,
(ii) $T_1(A \cup B) \subseteq T_1(A) \cup T_1(B)$,
(iii) $Cl^*(A) = T_1(A) \cup \psi(A) \cup A$,
(iv) $T_1(T_1(A)) \subseteq T_1(A)$.

Then, k_1 is a closure operator on X,and $T_1(A) = Bd^*(A)$ for every subset A of X, in which the topology is induced by k.
The operator $k_1(A) = A \cup T_1(A)$, satisfies the following conditions:
(i) $k_1(\emptyset) = \emptyset \cup T_1(\emptyset) = \emptyset$;
(ii) $A \subseteq A \cup T_1(A) = k_1(A)$;
(iii) $k_1(k_1(A)) = k_1(A \cup T_1(A)) = A \cup T_1(A) \cup T_1(A \cup T_1(A)) \subseteq A \cup T_1(A) \cup T_1(A) \cup T_1(T_1(A)) \subseteq A \cup T_1(A) \cup T_1(A) = k_1(A)$;
(iv) $k_1(A \cup B) = A \cup B \cup T_1(A \cup B) \subseteq A \cup B \cup T_1(A) \cup T_1(B) = k_1(A) \cup k_1(B)$ and $k_1(A) \cup k_1(B) = A \cup T_1(A) \cup B \cup T_1(B) = A \cup B \cup (A \cap T_1(B)) \cup T_1(A \cup B) \cup (B \cap T_1(A)) \subseteq A \cup B \cup T_1(A \cup B) = k_1(A \cup B)$.
Recall the following lemma:

Lemma 2.14. *[20] An ideal topological space $(X,\tau,\mathscr{I})$ is Hayashi-Samuel if and only if, for each $O \in \tau$, $O^* = Cl(O)$.*

Theorem 2.15. *Let $(X,\tau,\mathscr{I})$ be a Hayashi-Samuel space. Then for each open set U, $Bd^*(U) \subseteq U^* \setminus U$.*

Proof. $Bd^*(U) = U^* \cap (X \setminus U)^* \subseteq Cl(U) \cap Cl(X \setminus U) = U^* \cap (X \setminus Int(U)) = U^* \setminus U$, since the space is Hayashi-Samuel. □

We recall the following theorem:

Theorem 2.16. *[19] Let $(X,\tau,\mathscr{I})$ be an ideal topological space. Then, the following properties are equivalent:*

1. *$\tau \cap \mathscr{I} = \{\emptyset\}$;*
2. *$I \in \mathscr{I}$, then $Int(I) = \emptyset$;*
3. *for every $G \in \tau$, $G \subseteq G^*$;*
4. *$X = X^*$;*
5. *if $O \in \tau$, then $O^* = Cl(O)$.*

Theorem 2.17. *An ideal topological space $(X,\tau,\mathscr{I})$ is Hayashi-Samuel if and only if, for each closed set $A \subseteq X$, $Bd^*(A) = A^* \setminus Int(A)$.*

Proof. $Bd^*(A) = A^* \cap (X \setminus A)^* = A^* \cap Cl(X \setminus A) = A^* \setminus Int(A)$, since the space is Hayashi-Samuel.
From the given condition, we have $Bd^*(X) = X^* \setminus Int(X)$. Then $\emptyset = X^* \setminus Int(X)$ (from Theorem 2.12) implies $X^* = X$. Thus, $\tau \cap \mathscr{I} = \{\emptyset\}$. □

Theorem 2.18. *An ideal topological space $(X,\tau,\mathscr{I})$ is Hayashi-Samuel if and only if, for each open set $U \subseteq X$, $Bd^*(U) = Bd(U)$.*

Proof. Suppose $(X,\tau,\mathscr{I})$ is Hayashi-Samuel. Then for $U \in \tau$, $Bd^*(U) = U^* \cap (X \setminus U)^* = Cl(U) \cap Cl(X \setminus U) = Bd(U)$.
Conversely suppose that $Bd^*(U) = Bd(U)$. Then $U^* \cap (X \setminus U)^* = Cl(U) \cap Cl(X \setminus U)$ implies $U^* \cap (X \setminus \psi(U)) = Cl(U) \setminus U$. Thus $U^* \setminus \psi(U) = Cl(U) \setminus U$ implies $Cl(U) \setminus U$. Thus $U^* \setminus \psi(U) = Cl(U) \setminus U$ implies $Cl(U) \setminus U \subseteq U^* \setminus U$ (since for open set U, $U \subseteq \psi(U)$ [8]). This implies that $Cl(U) \subseteq U^*$ and hence $U \subseteq Cl(U) \subseteq U^*$. Thus $U \subseteq U^*$. Therefore, $(X,\tau,\mathscr{I})$ is Hayashi-Samuel. □

Corollary 2.19. *Let $(X,\tau,\mathscr{I})$ be an ideal topological space. Then, the following properties are equivalent:*

1. *$\tau \cap \mathscr{I} = \{\emptyset\}$;*
2. *$I \in \mathscr{I}$, then $Int(I) = \emptyset$;*
3. *for every $G \in \tau$, $G \subseteq G^*$;*

4. $X = X^*$;

5. if $O \in \tau$, then $O^* = Cl(O)$;

6. $Bd^*(A) = A^* \setminus Int(A)$;

7. for each $U \in \tau$, $Bd^*(U) = Bd(U)$.

Theorem 2.20. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then for A, $B \subseteq X$, $Bd^*(A) \cup Bd^*(B) = Bd^*(A \setminus B) \cup Bd^*(A \cap B) \cup Bd^*(B \setminus A)$.*

Proof. We have:
(a) $Bd^*(A \cap B) = Bd^*(X \setminus (A \cap B)) = Bd^*[(X \setminus A) \cup (X \setminus B)] \subseteq Bd^*(X \setminus A) \cup Bd^*(X \setminus B) = Bd^*(A) \cup Bd^*(B)$.
(b) $Bd^*(A \setminus B) = Bd^*[A \cap (X \setminus B)] \subseteq Bd^*(A) \cup Bd^*(X \setminus B) = Bd^*(A) \cup Bd^*(B)$.
(c) $Bd^*(B \setminus A) \subseteq Bd^*(A) \cup Bd^*(B)$.
Thus from (a), (b) and (c) $Bd^*(A \setminus B) \cup Bd^*(A \cap B) \cup Bd^*(B \setminus A) \subseteq Bd^*(A) \cup Bd^*(B)$.
Further, we have $Bd^*(A) \cup Bd^*(B) = Bd^*[(A \setminus B) \cup (A \cap B)] \cup Bd^*[(B \setminus A) \cup (A \cap B)] \subseteq Bd^*(A \setminus B) \cup Bd^*(A \cap B) \cup Bd^*(B \setminus A)$.
Therefore, $Bd^*(A) \cup Bd^*(B) = Bd^*(A \setminus B) \cup Bd^*(A \cap B) \cup Bd^*(B \setminus A)$. □

Theorem 2.21. *Let A and B be subsets of a topological space (X, τ) with an ideal $\mathscr{I}$. Then the following properties hold:*
(1) $Bd^*(A) \cup Bd^*(B) = Bd^*(A \cap B) \cup Bd^*(A \setminus B) \cup Bd^*(A \cup B)$.
(2) $Bd^*(A) \cup Bd^*(B) = Bd^*(A \cup B) \cup Bd^*(B \setminus A) \cup Bd^*(A \cap B)$.
(3) $Bd^*(A) \cup Bd^*(B) = Bd^*(A \setminus B) \cup Bd^*(B \setminus A) \cup Bd^*(A \cap B)$.
(4) $Bd^*(A) \cup Bd^*(A \Delta B) = Bd^*(A \setminus B) \cup Bd^*(A \cap B) \cup Bd^*(B \setminus A)$ (Δ denotes the symmetric difference).
(5) $Bd^*(B) \cup Bd^*(A \Delta B) = Bd^*(A \setminus B) \cup Bd^*(A \cap B) \cup Bd^*(B \setminus A)$.

Proof. (1) If we put $X \setminus B$ in the relation of the Theorem 2.20 instead of B, then we get,

$$Bd^*(A) \cup Bd^*(X \setminus B) = Bd^*(A \setminus (X \setminus B)) \cup Bd^*(A \cap (X \setminus B)) \cup Bd^*((X \setminus B) \setminus A).$$

This implies that

$$Bd^*(A) \cup Bd^*(B) = Bd^*(A \cap B) \cup Bd^*(A \setminus B) \cup Bd^*(A \cup B).$$

(2) If we put $X \setminus A$ in the relation of the Theorem 2.20 instead of A, then we get,

$$Bd^*(X \setminus A) \cup Bd^*(B) = Bd^*((X \setminus A) \setminus B) \cup Bd^*((X \setminus A) \cap B) \cup Bd^*(B \setminus (X \setminus A)).$$

This implies that

$$Bd^*(A) \cup Bd^*(B) = Bd^*(A \cup B) \cup Bd^*(B \setminus A) \cup Bd^*(A \cap B).$$

(3) If we put $X \setminus A$ instead of A and $X \setminus B$ instead of B in the relation of the Theorem 2.20 we get,
$Bd^*(X \setminus A) \cup Bd^*(X \setminus B) = Bd^*[(X \setminus A) \setminus (X \setminus B)] \cup Bd^*[(X \setminus A) \cap (X \setminus B)] \cup Bd^*[(X \setminus B) \setminus (X \setminus A)]$.
This implies that

$$Bd^*(A) \cup Bd^*(B) = Bd^*(B \setminus A) \cup Bd^*(A \cup B) \cup Bd^*(A \setminus B).$$

(4) From Theorem 2.20,
$Bd^*(A) \cup Bd^*(A \Delta B) = Bd^*[A \setminus (A \Delta B)] \cup Bd^*[A \cap (A \Delta B)] \cup Bd^*[(A \Delta B) \setminus A] = Bd^*(A \cap B) \cup Bd^*(A \setminus B) \cup Bd^*(B \setminus A) = Bd^*(B) \cup Bd^*(A \Delta B)$.
(5) The proof of (5) is obvious from (4). □

We have from Theorem 2.21, the union of any two distinct elements of $\{Bd^*(A),\ Bd^*(B),\ Bd^*(A \Delta B)\}$ is equal to the union of any three distinct elements of $\{Bd^*(A \cup B),\ Bd^*(A \cap B),\ Bd^*(A \setminus B),\ Bd^*(B \setminus A)\}$

Definition 2.22. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. The operator $()^{*-} : \wp(X) \to \wp(X)$ is defined as:*

$$A^{*-} = A^* \setminus A, \text{ for } A \subseteq X.$$

Theorem 2.23. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and A, $B \subseteq X$, then following conditions hold:*

1. $\emptyset^{*-} = \emptyset$;
2. $A \cap A^{*-} = \emptyset$;
3. $(A \cup B)^{*-} = (A^{*-} \setminus B) \cup (B^{*-} \setminus A)$;
4. $(A^{*-})^{*-} \subseteq A$.

Proof. The proof of 1. and 2. are obvious from definition.
3. $(A \cup B)^{*-} = (A \cup B)^* \setminus (A \cup B) = (A^* \cup B^*) \setminus (A \cup B) = [(A^* \setminus A) \setminus B] \cup [(B^* \setminus B) \setminus A] = (A^{*-} \setminus B) \cup (B^{*-} \setminus A)$.
4. $(A^{*-})^{*-} = (A^{*-})^* \setminus A^{*-} = (A^* \setminus A)^* \setminus (A^* \setminus A) \subseteq (A^*)^* \setminus (A^* \setminus A) \subseteq A^* \setminus (A^* \setminus A) \subseteq A$. □

Definition 2.24. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A, B \subseteq X$. The operator $k_2 : \wp(X) \to \wp(X)$ on X is defined by*

$$k_2(A) = A \cup T_2(A),$$

where $T_2 : \wp(X) \to \wp(X)$ is an operator which satisfies the following conditions:
(i) $T_2(\emptyset) = \emptyset$,
(ii) $A \cap T_2(A) = \emptyset$,
(iii) $T_2(A \cup B) = (T_2(A) \setminus B) \cup (T_2(B) \setminus A)$,
(iv) $T_2(T_2(A)) \subseteq A$.

The operator k_2 satisfies the following conditions:
(i) $k_2(\emptyset) = \emptyset \cup T_2(\emptyset) = \emptyset$;
(ii) $A \subseteq A \cup T_2(A) = k_2(A)$;
(iii) $k_2(A \cup B) = (A \cup B) \cup T_2(A \cup B) = (A \cup B) \cup (T_2(A) \setminus B) \cup (T_2(B) \setminus A) = A \cup T_2(A) \cup B \cup T_2(B) = k_2(A) \cup k_2(B)$;
(iv) $k_2(k_2(A)) = k_2(A) \cup T_2(k_2(A)) = k_2(A) \cup T_2(A \cup T_2(A)) = k_2(A) \cup (T_2(A) \setminus T_2(A)) \cup (T_2(T_2(A)) \setminus A) = k_2(A)$.
Thus, the operator k_2 is a closure operator on $(X, \tau, \mathscr{I})$.

Theorem 2.25. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and A, $B \subseteq X$. Then, the following conditions hold:*

1. $A^{*-} \cup B^{*-} = (A \cap B^{*-}) \cup (A \cup B)^{*-} \cup (A^{*-} \cap B)$;
2. $(A^*)^{*-} = \emptyset$;
3. *A is $*$-open [5] if and only if $A^{*-} = Bd^*(A)$.*

Proof. 1. Note that $A^* \subseteq (A \cup B)^*$ if and only if $(A^* \setminus A) \setminus B \subseteq (A \cup B)^* \setminus (A \cup B)$ if and only if $A^{*-} \setminus B \subseteq (A \cup B)^{*-}$. Therefore, $(A^{*-} \setminus B) \cup (A^{*-} \cap B) \subseteq (A \cup B)^{*-} \cup (A^{*-} \cap B)$ and $A^{*-} \subseteq (A \cup B)^{*-} \cup (A^{*-} \cap B)$. Analogously, $B^{*-} \subseteq (A \cup B)^{*-} \cup (B^{*-} \cap A)$. So $A^{*-} \cup B^{*-} \subseteq (A \cup B)^{*-} \cup (B^{*-} \cap A) \cup (A^{*-} \cap B)$.

For the reverse inclusion we will only show that $(A \cup B)^{*-} \subseteq A^{*-} \cup B^{*-}$. Note that $(A \cup B)^* \setminus (A \cup B) \subseteq (A^* \setminus A) \cup (B^* \setminus B)$. Thus $(A \cup B)^{*-} \subseteq A^{*-} \cup B^{*-}$. This implies that $(A \cup B)^{*-} \cup (A \cap B^{*-}) \cup (B \cap A^{*-}) \subseteq A^{*-} \cup B^{*-}$.

2. Note that $(A^*)^{*-} = (A^*)^* \setminus A^* \subseteq A^* \setminus A^* = \emptyset$. □

Theorem 2.26. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space. Then a subset A of X is $*$-closed [5] if and only if $A^{*-} = \emptyset$.*

Proof. Suppose A is $*$-closed. Then $A \cup A^* \subseteq A$ and hence $A^* \subseteq A$. Now $A^{*-} = A^* \setminus A = \emptyset$.

Conversely suppose that $A^{*-} = \emptyset$. Then $A^* \setminus A = \emptyset$ implies $A^* \subseteq A$. Thus $A \cup A^* = A$. So A is $*$-closed. □

Theorem 2.27. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and $A \subseteq X$. If $A^{*-} = X$, then A is $\mathscr{I}$-dense.*

Proof. Given that $A^{*-} = X$, then $A^* \setminus A = X$. Thus $X \subseteq A^*$. □

Converse of the above theorem need not hold in general:

Example 2.28. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$ and $\mathscr{I} = \{\emptyset, \{c\}\}$. Then $(\{a, c\})^* = X$, but $(\{a, c\})^* \setminus \{a, c\} \neq X$.*

Definition 2.29. *We define the operator $()^{*\psi}$ on an ideal topological space $(X, \tau, \mathscr{I})$ in the following way: for a subset A of X, $A^{*\psi} = A \setminus \psi(A)$.*

Theorem 2.30. *Let $(X, \tau, \mathscr{I})$ be an ideal topological space and A, $B \subseteq X$. Then following conditions hold:*

1. $X^{*\psi} = \emptyset$;

2. $A^{*\Psi} \subseteq A$*;*

3. $(A \cap B)^{*\Psi} = (A^{*\Psi} \cap B) \cup (A \cap B^{*\Psi})$*;*

4. $(A^{*\Psi})^{*\Psi} = A^{*\Psi}$*, if the space is Hayashi-Samuel.*

Proof. The proofs of 1. and 2. hold trivially.

3. $(A \cap B)^{*\Psi} = (A \cap B) \setminus \psi(A \cap B) = (A \cap B) \cap [X \setminus \psi(A) \cap \psi(B)] = [A \cap (X \setminus \psi(A)) \cap B] \cup [A \cap B \cap (X \setminus \psi(B))] = (A^{*\Psi} \cap B) \cup (A \cap B^{*\Psi})$.

4. $(A^{*\Psi})^{*\Psi} = A^{*\Psi} \setminus \psi[A^{*\Psi}] = (A \setminus \psi(A)) \setminus \psi[A \setminus \psi(A)] = (A \setminus \psi(A)) \setminus \psi(A \setminus [X \setminus (X \setminus A)^*]) = (A \setminus \psi(A)) \setminus \psi[A \setminus X \cup (X \setminus A)^*] = (A \setminus \psi(A)) \setminus \emptyset = A^{*\Psi}$. □

Definition 2.31. *Let* $(X, \tau, \mathscr{I})$ *be an ideal topological space and* $A, B \subseteq X$*. The operator* $Int_\psi : \wp(X) \to \wp(X)$ *on* X *is defined by*

$$Int_\psi(A) = A \setminus T_3(A),$$

where $T_3 : \wp(X) \to \wp(X)$ *is an operator which satisfies the following conditions:*
(i) $T_3(X) = \emptyset$*,*
(ii) $T_3(A) \subseteq A$*,*
(iii) $T_3(A \cap B) = (T_3(A) \cap B) \cup (A \cap T_3(B))$*,*
(iv) $T_3(T_3(A)) = T_3(A)$*, if the space is Hayashi-Samuel.*

The operator Int_ψ satisfies the following conditions:
(i) $Int_\psi(X) = X \setminus T_3(X) = X$;
(ii) $Int_\psi(A) = A \setminus T_3(A) \subseteq A$;
(iii) $Int_\psi(A \cap B) = (A \cap B) \setminus T_3(A \cap B) = (A \cap B) \setminus [(T_3(A) \cap B) \cup (T_3(B) \cap A)] = [A \setminus T_3(A)] \cap (B \setminus T_3(B)) = Int_\psi(A) \cap Int_\psi(B)$ (from (ii));
(iv) $Int_\psi(Int_\psi(A)) = Int_\psi[A \setminus T_3(A)] = [A \setminus T_3(A)] \setminus T_3[A \setminus T_3(A)] = [A \setminus T_3(A)] \setminus T_3(A \cap T_3(A)^c) \supseteq (A \setminus T_3(A)) \setminus T_3(A) \cap T_3(X \setminus T_3(A))$ (from (iii)) $\supseteq (A \setminus T_3(A)) \setminus (T_3(A) \cap (X \setminus T_3(A))$ (from 3. of Theorem 2.30) $= Int_\psi(A)$.
This shows that Int_ψ is an interior operator on X.

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