



A NEW FAMILY OF LIFETIME DISTRIBUTIONS IN TERMS OF CUMULATIVE HAZARD RATE FUNCTION

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ABSTRACT. In the present paper, a new family of lifetime distributions is introduced according to cumulative hazard rate function, the well-known concept in survival analysis and reliability engineering. Some important properties of proposed model including survival function, quantile function, hazard function, order statistic and some results of stochastic ordering are obtained in general setting. An especial case of this new family is introduced by considering Weibull distribution as the parent distribution; in addition estimating unknown parameters of specialized model will be examined from the perspective of Bayesian and classic statistics. Moreover, three examples of real data sets: complete, right-censored and progressively type-I interval-censored data are studied; point and interval estimations of all parameters are obtained. Finally, the superiority of proposed model in terms of parent Weibull distribution over other fundamental statistical distributions is shown via complete real observations.

1. INTRODUCTION

The statistical distribution theory has been widely explored by researchers in recent years. Given the fact that the data from our surrounding environment follow various statistical models, is necessary to extract and develop appropriate high-quality models.

Recently, Nadarajah and Haghighi (2011) have introduced a new model of lifetime distributions, which the researchers refer it as NH distribution. It is an extended form of exponential distribution and attracted the attention of some researchers. We refer the reader to (Lemonte (2013), Dey et al. (2017) and Kumar et al. (2017)). This model has a number of desirable features and is comprehensively studied by the authors. For example, whenever the data contains zero values, NH model can be a strong competitor for other well-known lifetime distributions such

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as gamma, Weibull and generalized exponential distribution. The cumulative distribution function (*cdf*) and probability distribution function (*pdf*) related to *NH* model is given respectively as;

$$F(x) = 1 - e^{1-(1+\lambda x)^\alpha}, \quad x > 0$$

and

$$f(x) = \alpha\lambda(1 + \lambda x)^{\alpha-1}e^{1-(1+\lambda x)^\alpha}, \quad x > 0$$

where the parameters $\alpha > 0$ controls the shapes of the distribution and the parameter $\lambda > 0$ is the scale parameter. It is easy to see that the *NH* model has increasing, decreasing and constant hazard shapes.

In the present paper, we introduce a New family of Lifetime distributions based on the Cumulative Hazard rate quantity of a parent distribution G , so-called *NLCH* – G distribution. One of our main motivation to introduce this new category of distributions is that, when the parent distribution G be exponential, the proposed model reduced to *NH* distribution.

The cumulative hazard rate function is a prominent concept in topics of survival analysis and reliability engineering and plays an important role in this area of science. Suppose that X be a random variable with density function f and cumulative distribution function F , then hazard rate and cumulative hazard rate functions are defined;

$$h(x) = \frac{f(x)}{R(x)}$$

and

$$H(x) = -\log R(x) = e^{-\int_0^x h(t)dt},$$

respectively, where $R(x) = 1 - F(x)$ denotes the survival function of X (Barlow and Proschan (1975)).

In the next, we first obtain the fundamental and statistical properties of *NLCH* – G in general setting and then we propose an especial case of *NLCH* – G model by considering Weibull distribution instead of the parent distribution G . It is referred as *NLCH* – *Weibull* (or *NLCH* – W) distribution. We provide a comprehensive discussion about statistical and reliability properties of new *NLCH* – W model. Furthermore, we consider Maximum likelihood, Bayesian and bootstrap estimation procedures in order to estimate the unknown parameters of the new model for complete, right-censored and progressively type-I interval-censored data sets. In the Bayesian discussion, we consider different types of symmetric and asymmetric loss functions such as squared error, absolute value, Linear Exponential (*LINEX*) and generalized entropy to estimate three unknown parameters of *NLCH* – W model. Since the parameter space for all three parameters is positive, we use gamma priors distributions. Bayesian %95 credible and highest posterior density (*HPD*) intervals (see Chen et al. (1999)) are provided for each parameter of proposed

model. In addition, the asymptotic confidence intervals and parametric and non-parametric bootstrap confidence intervals are calculated in order to compare with corresponding Bayesian intervals.

The rest of the paper organized as follows. In the section 2, a new category of lifetime distributions is introduced based on the fundamental quantity $H(x)$ and then the main statistical and reliability properties are obtained in general setting. In section 3, by considering the Weibull distribution as the base distribution, a new model is presented according to the general model discussed in section 1 and its prominent characteristics are studied. This new model refer as $NLCH - W$ distribution. In section 4, we examine the inferential procedures for estimation unknown parameters of the $NLCH - W$ model. In this Section, we provide discussions about three important estimation methods maximum likelihood, Bayesian and bootstrap. Here we use four well-known loss functions like squared error, absolute value, LINEX and generalized entropy. Application and numerical analysis of three real data sets (complete, right-censored and progressively type-I interval-censored) are presented in section 5. Finally, in section 6 the paper is concluded.

2. NEW MODEL AND PROPERTIES

In this section, first we introduce a new category of lifetime distributions and then we obtain main statistical and reliability properties of the proposed family in general setting.

Definition 2.1. *A random variable X is said to have $NLCH - G$ distribution if its probability distribution function (pdf) is given by*

$$f(x; \alpha, \gamma) = \alpha h(x) (\gamma + H(x))^{\alpha-1} e^{\gamma - (\gamma + H(x))^\alpha}, \quad x > 0, \alpha > 0, \gamma > 0, \quad (1)$$

and its cumulative distribution function (cdf) is given by

$$F(x; \alpha, \gamma) = 1 - e^{\gamma - (\gamma + H(x))^\alpha}, \quad x > 0, \alpha > 0, \gamma > 0 \quad (2)$$

where, $H(x)$ is cumulative hazard function of baseline distribution $G(x)$ and $h(x) = \frac{\partial H(x)}{\partial x}$.

The corresponding survival function of (1) is given as

$$R(x; \alpha, \gamma) = e^{\gamma - (\gamma + H(x))^\alpha}, \quad x > 0, \alpha > 0, \gamma > 0. \quad (3)$$

Remark 2.2. *Let $\alpha = 1$, then we get $F(x; \alpha = 1, \gamma) = G(x)$.*

In the following theorem we investigate the connection between NH and $NLCH - G$ models.

Theorem 2.3. *Suppose that the random variable X be a continuous random variable with cumulative hazard rate function $H(x)$, and the random variable Y has NH distribution with parameter α and λ . Then the transformed variable $Z = H^{-1}(\lambda Y)$ has a density with pdf (1) as parameter $\gamma = 1$. $H^{-1}(\cdot)$ is inverse function of $H(\cdot)$.*

Proof. Using the method of distribution function we have;

$$\begin{aligned}
 F_Z(z) &= P(Z \leq z) \\
 &= P(H^{-1}(\lambda Y) \leq z) \\
 &= P\left(Y \leq \frac{1}{\lambda} H(z)\right) \\
 &= 1 - e^{1 - (1 + H(z))^\alpha},
 \end{aligned}$$

so the proof is completed. \square

Following this section, we get some fundamental properties of proposed model such as hazard rate function, survival function, quantile function and order statistic distribution. It is seen that all of these measures have closed expression in terms of quantity $H(x)$.

2.1. Hazard Rate Function. The hazard rate is a key concept in analysis of reliability and measuring the aging process. Knowing shape and behavior of the hazard rate in reliability theory, risk analysis, and so on, is very important. The hazard rate function of the $NLCH - G$ distribution is given as

$$\begin{aligned}
 h_F(x, \alpha, \gamma) &= \frac{f(x; \alpha, \gamma)}{R(x; \alpha, \gamma)} \\
 &= \alpha h(x) (\gamma + H(x))^{\alpha-1}.
 \end{aligned} \tag{4}$$

Remark 2.4. *In fact the hazard rate function of new model is a weighted version of baseline hazard with weight $w(x) = \alpha(\gamma + H(x))^{\alpha-1}$.*

Lemma 2.5. *By considering (4), we have*

- if $r(x)$ is increasing and $\alpha \geq 1$ then $r_F(x, \alpha, \gamma)$ is increasing.
- if $r(x)$ is decreasing and $\alpha \leq 1$ then $r_F(x, \alpha, \gamma)$ is decreasing.

Proof. The proof is straightforward. \square

In the following lemma we provide a result about stochastic order in hazard function to compare proposed model and baseline distribution. First we recall the following definition. The random variable X is said to be less than variable Y in hazard rate order, $X \leq_{hr} Y$, if $h_X(x) \geq h_Y(x)$, for all x in the union of supports of X and Y , where $h_X(x)$ ($h_Y(x)$) is the hazard rate of X (Y). For more details see Shaked and Shanthikumar (2007).

Lemma 2.6. *Let X_F and X_G be two random variables corresponding with proposed model (1) and distribution G respectively, then under the condition $\gamma \geq 1$*

- if $\alpha > 1$ then $X_F \leq_{hr} X_G$.
- if $\alpha < 1$ then $X_G \leq_{hr} X_F$.

Proof. The proof is straightforward. \square

2.2. Random variate generation. One important quantity for each probabilistic model is to have the data generator function based on an explicit formula, because the simulation studies researchers are more satisfied with the data generator functions of a given form. For generating random variables from the $NLCH - G$ distribution, we use the inverse transformation method. The quantile of order p of the $NLCH - G$ distribution is

$$x_p = F^{-1}(p; \alpha, \gamma) = H^{-1}((\gamma - \log(1 - p))^{\frac{1}{\alpha}} - \gamma). \quad (5)$$

where $H^{-1}(x)$ is inverse function of quantity $H(x)$. Let U be a random variable generated from a uniform distribution on $(0, 1)$, then

$$X = H^{-1}((\gamma - \log(1 - U))^{\frac{1}{\alpha}} - \gamma) \quad (6)$$

is a random variable generated from the $NLCH - G$ distribution by the probability integral transform.

2.3. Order statistics. Order statistics have applications in various directions such as statistical inference, reliability engineering, quality control and etc. Let X_1, X_2, \dots, X_n be a random sample from $NLCH - G$ distribution. Let $X_{i:n}$ denote the i th order statistic. Then the *pdf* of $X_{i:n}$ is given by

$$\begin{aligned} g_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!} g(x) [G(x)]^{i-1} [\bar{G}(x)]^{n-i} \\ &= \frac{n! e^{n-i+1}}{(i-1)!(n-i)!} \alpha h(x) (\gamma + H(x))^{\alpha-1} e^{(n-i+1)(\gamma+H(x))\alpha} \\ &\quad \times (1 - e^{\gamma - (\gamma+H(x))\alpha})^{i-1} \end{aligned}$$

3. NLCH-WEIBULL (NLCH-W) MODEL

Without loss of generality let parameter $\gamma = 1$ and consider the Weibull distribution as a parent distribution with *cdf* function $F(x; \beta, \lambda) = 1 - e^{-\lambda x^\beta}$, $x > 0, \beta > 0, \lambda > 0$. By replacing this model in relation (3), the *pdf* of the $NLCH - W$ is given as

$$f(x; \alpha, \beta, \lambda) = \alpha \lambda \beta x^{\beta-1} (1 + \lambda x^\beta)^{\alpha-1} e^{1 - (1 + \lambda x^\beta)^\alpha} \quad (7)$$

and its *cdf* is given by

$$F(x; \alpha, \lambda) = 1 - e^{1 - (1 + \lambda x^\beta)^\alpha}. \quad (8)$$

Remark 3.1. If $\alpha = 1$, we attain the *pdf* of Weibull distribution and If $\beta = 1$, we get *NH* distribution respectively.

3.1. Density shape. It is easy to investigate that the shape of $NLCH - W$ is unimodal and

- if $\beta > 1$ then $\lim_{x \rightarrow 0} f(x) = 0$,
- if $\beta < 1$ then $\lim_{x \rightarrow 0} f(x) = \infty$,

and

$$\lim_{x \rightarrow \infty} f(x) = 0. \quad (9)$$

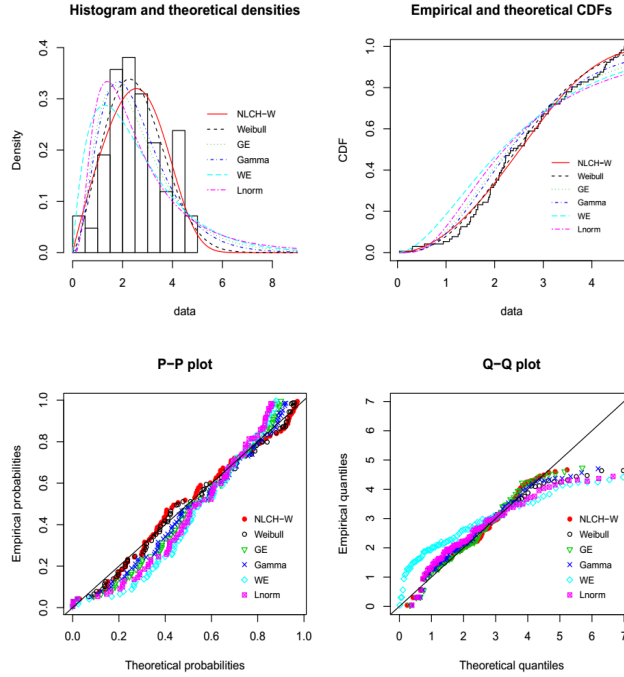


FIGURE 1. The graphs of pdf (a) and hazard rate function (b, c and d) of the $NLCH - W$ distribution for some selected values of parameters.

In the next section, we consider the hazard shape of $NLCH - W$ distribution.

3.2. Hazard rate function of NLCH-W distribution. The hazard rate function of $NLCH - W$ distribution is

$$\begin{aligned} h(x) &= \frac{f(x)}{1 - F(x)} \\ &= \alpha\lambda\beta x^{\beta-1}(1 + \lambda x^\beta)^{\alpha-1}. \end{aligned}$$

Determining the behavior of the hazard rate is very important in various applications, especially in reliability theory. It can easily be shown that the proposed model (7) has a variety of hazard shapes. The hazard rate function allows for constant, monotonically increasing, monotonically decreasing, unimodal and bathtub shaped hazard rates. In summary, different types of hazard rates are as follows.

- if $\beta > 1$ and $\alpha\beta > 1$ then $h(x)$ is monotonically increases with $h(0) = 0$.
- if $\beta < 1$ and $\alpha\beta < 1$ then $h(x)$ is monotonically decreases with $h(0) = \infty$.
- if $\beta > 1$ and $\alpha\beta < 1$ then $h(x)$ is bathtub shape.
- if $\beta < 1$ and $\alpha\beta > 1$ then $h(t)$ is upside down bathtub shape.
- if $\beta = 1$ and $\alpha = 1$ then $h(t)$ is constant.

Some shapes of *pdf* and hazard function for the selected values of parameters is given in Figure 1.

4. ESTIMATION PROCEDURES

Nowadays, three methods of maximum likelihood estimation, Bayesian and bootstrap procedures are of particular importance in the theory of statistical inference undoubtedly. In this section, we describe each of these methods separately for estimating the parameters α , β and λ of the *NLCH* – *W* distribution. For all methods we consider the case when all three parameters are unknown.

4.1. Maximum likelihood estimation. The maximum likelihood procedure is one of the most common methods for obtaining an estimator for an unknown parameter in classic statistical inference. The likelihood function is a function that written based on the mechanism of the observations occurrence. Here, the structure of the likelihood function is expressed for three modes of observations including complete data, right-censored and progressive interval-censored data sets.

4.1.1. Maximum likelihood estimation for complete data set. Let X_1, X_2, \dots, X_n be a random sample of size n from *NLCH* – *W* distribution. The likelihood function is given for equation (7) by

$$L(\underline{x}, \alpha, \beta, \lambda) = \prod_{i=1}^n \alpha \lambda \beta x_i^{\beta-1} (1 + \lambda x_i^\beta)^{\alpha-1} e^{1-(1+\lambda x_i^\beta)^\alpha}. \quad (10)$$

So, the log-likelihood function is written as

$$\begin{aligned} \ell(\underline{x}, \alpha, \beta, \lambda) &= \log L(\underline{x}, \alpha, \beta, \lambda) = n[\log \alpha + \log \lambda + \log \beta] + (\beta - 1) \sum_{i=1}^n \log x_i \\ &+ (\alpha - 1) \sum_{i=1}^n \log(1 + \lambda x_i^\beta) + n - \sum_{i=1}^n (1 + \lambda x_i^\beta)^\alpha. \end{aligned}$$

The normal equations are derived by differentiation of the log-likelihood function with respect to parameters α, β and λ .

$$\begin{aligned}\frac{\partial \ell}{\partial \alpha} &= \frac{n}{\alpha} + \sum_{i=1}^n \log(1 + \lambda x_i^\beta) - \sum_{i=1}^n (1 + \lambda x_i^\beta) \log(1 + \lambda x_i^\beta), \\ \frac{\partial \ell}{\partial \beta} &= \frac{n}{\beta} + \sum_{i=1}^n (1 + \lambda x_i^\beta) \log x_i + (\alpha - 1) \sum_{i=1}^n \frac{\lambda x_i^\beta \log x_i}{1 + \lambda x_i^\beta} - \sum_{i=1}^n \lambda x_i^\beta (1 + \lambda x_i^\beta)^\alpha \log x_i, \\ \frac{\partial \ell}{\partial \lambda} &= \frac{n}{\lambda} + \sum_{i=1}^n \frac{x_i^\beta}{1 + \lambda x_i^\beta} - \alpha \sum_{i=1}^n x_i^\beta (1 + \lambda x_i^\beta)^{\alpha-1}.\end{aligned}$$

Setting these differentiations equal to zero and solving for α, β and λ , then we can obtain the maximum likelihood estimator *MLE* of parameters α, β and λ .

4.1.2. *Maximum likelihood estimation for right-censored data set.* Let $(X_1, \delta_1), (X_2, \delta_2), \dots, (X_n, \delta_n)$ be a right-censored random sample of size n from *NLCH* – *W* distribution. Where δ_i is a censoring indicator variable, that is, $\delta_i = 1$ for an observed survival time and $\delta_i = 0$ for a right-censored survival time. In the case *NLCH* – *W* distribution the likelihood function and the corresponding log-likelihood are given as

$$L(\underline{x}, \underline{\delta}, \alpha, \beta, \lambda) = \prod_{i=1}^n \left(\alpha \lambda \beta x_i^{\beta-1} (1 + \lambda x_i^\beta)^{\alpha-1} e^{1-(1+\lambda x_i^\beta)^\alpha} \right)^{\delta_i} \left(e^{1-(1+\lambda x_i^\beta)^\alpha} \right)^{1-\delta_i} \quad (11)$$

and

$$\begin{aligned}\ell(\underline{x}, \underline{\delta}, \alpha, \beta, \lambda) &= \log L(\underline{x}, \underline{\delta}, \alpha, \beta, \lambda) \\ &= [\log \alpha + \log \lambda + \log \beta] \sum_{i=1}^n \delta_i + (\beta - 1) \sum_{i=1}^n \delta_i \log x_i \\ &\quad + (\alpha - 1) \sum_{i=1}^n \delta_i \log(1 + \lambda x_i^\beta) + n - \sum_{i=1}^n (1 + \lambda x_i^\beta)^\alpha,\end{aligned}$$

respectively. Analogous above results the normal equations can be derived in the case right-censored sample data.

4.1.3. *Maximum likelihood estimation for progressively type-I interval-censored data set.* Let n items to be applied on a life testing simultaneously at time $t = 0$ and suppose that m pre-specified times $t_1 < t_2 < \dots < t_m$, where t_m is scheduled time to terminate the experiment, be determined. At the i th inspection time, t_i , the number, X_i , of failures within $(t_i - 1, t_i]$ is recorded and R_i surviving items are randomly removed from the life testing for $i = 1, 2, \dots, m$. Therefore, a progressively

type-I interval-censored sample can be denoted as $S = (X_i, R_i, t_i)$ and sample size is $n = \sum_{i=1}^m (X_i + R_i)$. The likelihood function of density (1) based on progressively type-I interval-censored sample

$$S = (X_i, R_i, t_i), i = 1, 2, \dots, n$$

is given as

$$L(S, \alpha, \beta, \lambda) = \prod_{i=1}^m \left[e^{1-(1+\lambda t_{i-1})^\alpha} - e^{1-(1+\lambda t_i)^\alpha} \right]^{X_i} \left[1 - e^{1-(1+\lambda t_i)^\alpha} \right]^{R_i}. \quad (12)$$

The log-likelihood function is given as;

$$\begin{aligned} \ell(S, \alpha, \beta, \lambda) &= \log L(S, \alpha, \beta, \lambda) = \sum_{i=1}^n X_i \log \left[e^{1-(1+\lambda t_{i-1})^\alpha} - e^{1-(1+\lambda t_i)^\alpha} \right] \\ &+ \sum_{i=1}^n R_i \log \left[1 - e^{1-(1+\lambda t_i)^\alpha} \right]. \end{aligned}$$

Also, here we can derive normal equations for corresponding log-likelihood function similar complete and right-censored samples data. In practical due to the non-linearity of corresponding normal equations in three cases that discussed above, we use numerical algorithms to extract MLEs estimators.

4.2. Bootstrap estimation. The uncertainty in parameters of the fitted distribution can be estimated by parametric (re-sampling from the fitted distribution) or non-parametric (re-sampling with replacement from the original data set) bootstraps re-sampling Efron and Tibshirani (1994). These two parametric and non-parametric bootstrap procedures for complete data set are described as follows.

Parametric bootstrap procedure:

- (1) Estimate θ (vector of unknown parameters), say $\hat{\theta}$, by using the *MLE* procedure based on a random sample.
- (2) Generate a bootstrap sample $\{X_1^*, \dots, X_m^*\}$, using θ and obtain the bootstrap estimate of θ , say $\hat{\theta}^*$, from the bootstrap sample based on the *MLE* procedure.
- (3) Repeat step 2 *NBOOT* times.
- (4) Order $\hat{\theta}_1^*, \dots, \hat{\theta}_{NBOOT}^*$ as $\hat{\theta}_{(1)}^*, \dots, \hat{\theta}_{(NBOOT)}^*$. Then obtain γ -quantiles and $100(1 - \alpha)\%$ confidence intervals of parameters.

In case of the *NLCH - W* distribution, the parametric bootstrap estimators (PBs) of α, β and λ , say $\hat{\alpha}_{PB}, \hat{\beta}_{PB}$ and $\hat{\lambda}_{PB}$, respectively.

Non-parametric bootstrap procedure:

- (1) Generate a bootstrap sample $\{X_1^*, \dots, X_m^*\}$, with replacement from original data set. Obtain the bootstrap estimate of θ with MLE procedure, say $\hat{\theta}^*$ using the bootstrap sample.
- (2) Repeat step 2 $NBOOT$ times.
- (3) Order $\hat{\theta}_1^*, \dots, \hat{\theta}_{NBOOT}^*$ as $\hat{\theta}_{(1)}^*, \dots, \hat{\theta}_{(NBOOT)}^*$. Then obtain γ -quantiles and $100(1 - \alpha)\%$ confidence intervals of parameters.

In case of the $NLCH - W$ distribution, the non-parametric bootstrap estimators (NPBs) of α, β and λ , say $\hat{\alpha}_{NPB}, \hat{\beta}_{NPB}$ and $\hat{\lambda}_{NPB}$, respectively. Analogous algorithms can be expressed for bootstrap estimation of right-censored sample data.

4.3. Bayesian estimation. Bayesian inference procedure for censored data have been taken into consideration by many statistical researchers, especially researchers in the field of survival analysis and reliability engineering. In this section, a complete sample data and two widely used types of censored observations, right-censored and progressively type-I interval-censored observations are analyzed through Bayesian point of view. We assume that the parameters α, β and λ of $NLCH - W$ distribution have independent prior distributions as

$$\alpha \sim Gamma(a, b), \beta \sim Gamma(c, d), \lambda \sim Gamma(e, f),$$

where a, b, c, d, e and f are positive. Hence, the joint prior density function is formulated as follow:

$$\pi(\alpha, \beta, \lambda) = \frac{b^a d^c f^e}{\Gamma(a)\Gamma(c)\Gamma(e)} \alpha^{a-1} \beta^{c-1} \lambda^{e-1} e^{-(b\alpha + d\beta + f\lambda)}. \quad (13)$$

In the Bayesian estimation, according to that we do not know the actual value of the parameter, we may be adversely affected by loss when we choose an estimator. This loss can be measured by a function of the parameter and corresponding estimator. Four well-known loss functions and associated Bayesian estimators are presented as:

- Squared error loss function and Bayesian estimator

$$L(\gamma(\theta), d(\underline{x})) = (\gamma(\theta) - d(\underline{x}))^2$$

$$d_B(\underline{x}) = E(\gamma(\theta)|d(\underline{x}))$$

- Absolute value loss function and Bayesian estimator

$$L(\gamma(\theta), d(\underline{x})) = |\gamma(\theta) - d(\underline{x})|$$

$$d_B(\underline{x}) = Median(\gamma(\theta)|d(\underline{x}))$$

- LINEX loss function and Bayesian estimator

$$L(\gamma(\theta), d(\underline{x})) = \left[e^{c(\gamma(\theta) - d(\underline{x}))} - (\gamma(\theta) - d(\underline{x})) - 1 \right]$$

$$d_B(\underline{x}) = -\frac{\log E(e^{-c\gamma(\theta)}|d(\underline{x}))}{c}$$

- Generalized entropy loss function and Bayesian estimator

$$L(\gamma(\theta), d(\underline{x})) = \left[\left(\frac{\gamma(\theta)}{d(\underline{x})} \right)^c - \log \left(\frac{\gamma(\theta)}{d(\underline{x})} \right) - 1 \right]$$

$$d_B(\underline{x}) = \left(E(\gamma^{-c}(\theta) | \underline{x}) \right)^{-\frac{1}{c}}.$$

For more details see Calabria and Pulcini (1996).

In the next, we provide the posterior probability distributions in three modes: complete, right-censored and progressively type-I interval-censored data sets. Let we define the function φ as

$$\varphi(\alpha, \beta, \lambda) = \alpha^{\alpha-1} \beta^{c-1} \lambda^{e-1} e^{-(b\alpha+d\beta+f\lambda)}, \quad \alpha > 0, \quad \beta > 0, \quad \lambda > 0.$$

The joint posterior distribution in terms of a given likelihood function $L(data)$ and joint prior distribution $\pi(\alpha, \beta, \lambda)$ defined as

$$\pi^*(\alpha, \beta, \lambda | data) \propto \pi(\alpha, \beta, \lambda) L(data). \quad (14)$$

Hence, we get joint posterior density of parameters α , β and λ for complete sample data by combining the likelihood function (10) and joint prior density (13). Therefore, the joint posterior density function is given by

$$\pi^*(\alpha, \beta, \lambda | \underline{x}) = K \varphi(\alpha, \beta, \lambda) \prod_{i=1}^n \alpha \beta \lambda x_i^{\beta-1} (1 + \lambda x_i^\beta)^{\alpha-1} e^{-(1+\lambda x_i^\beta)^\alpha} \quad (15)$$

where K is given as

$$K^{-1} = \int_0^\infty \int_0^\infty \int_0^\infty \varphi(\alpha, \beta, \lambda) \prod_{i=1}^n \alpha \beta \lambda x_i^{\beta-1} (1 + \lambda x_i^\beta)^{\alpha-1} e^{-(1+\lambda x_i^\beta)^\alpha} d\alpha d\beta d\lambda.$$

Furthermore, by using likelihood functions (11), (12) and joint prior distribution (13) the joint posterior probability distribution functions for right-censored $(\underline{x}, \underline{\delta})$ and progressively type-I interval-censored sample data $(S = (X_i, R_i, t_i), i = 1, 2, \dots, n)$ presented respectively with

$$\pi^*(\alpha, \beta, \lambda | \underline{x}, \underline{\delta}) = M \varphi(\alpha, \beta, \lambda) \prod_{i=1}^n \left(\alpha \lambda \beta x_i^{\beta-1} (1 + \lambda x_i^\beta)^{\alpha-1} \right)^{\delta_i} e^{-(1+\lambda x_i^\beta)^\alpha}$$

and

$$\pi^*(\alpha, \beta, \lambda | S) = Z \varphi(\alpha, \beta, \lambda) \prod_{i=1}^n \left[e^{1-(1+\lambda t_{i-1}^\beta)^\alpha} - e^{1-(1+\lambda t_i^\beta)^\alpha} \right]^{X_i} \left[1 - e^{1-(1+\lambda t_i^\beta)^\alpha} \right]^{R_i},$$

where M is given as

$$M^{-1} = \int_0^\infty \int_0^\infty \int_0^\infty \varphi(\alpha, \beta, \lambda) \prod_{i=1}^n \left(\alpha \lambda \beta x_i^{\beta-1} (1 + \lambda x_i^\beta)^{\alpha-1} \right)^{\delta_i} e^{-(1+\lambda x_i^\beta)^\alpha} d\alpha d\beta d\lambda.$$

and Z is given as

$$Z^{-1} = \int_0^\infty \int_0^\infty \int_0^\infty \varphi(\alpha, \beta, \lambda) \times \prod_{i=1}^n \left[e^{1-(1+\lambda t_{i-1}^\beta)^\alpha} - e^{1-(1+\lambda t_i^\beta)^\alpha} \right]^{X_i} \left[1 - e^{1-(1+\lambda t_i^\beta)^\alpha} \right]^{R_i} d\alpha d\beta d\lambda.$$

Here we interested in obtaining Bayesian estimators for three sample data sets (complete, right-censored and type-I progressive interval-censored data sets) under the four loss functions described above. As it is observed, there are no closed forms for the Bayes estimators. It is possible to simulated posterior sample data sets by using Gibbs sampling method and Metropolis-Hasting algorithm. Thus, by applying MCMC algorithm the corresponding Bayes estimators, Bayesian credible and HPD intervals are calculated.

5. APPLICATION OF NLCH-W DISTRIBUTION ON THE REAL DATASETS

This section aims to show applications of the *NLCH* – *W* model under the methods discussed in the section 4 via real data examples. In order to achieve this target, we consider three real data sets to illustrate the application of proposed distribution in real world and the superiority of this model to some other useful classic models. Furthermore, in this section, we provide Bayesian and bootstrap analysis of parameter estimation of *NLCH* – *W* model for three real data sets. The following data sets contain three modes of real world observations: complete, right-censored and progressively type-I interval-censored.

Complete data set: Failure times of 84 Aircraft Windshield

We consider the data of service times for a particular model windshield. These data were recently studied by Ramos et al. (2013). The data consist of 84 observations.

0.040 1.866 2.385 3.443 0.301 1.876 2.481 3.467 0.309 1.899 2.610 3.478 0.557 1.911
2.625 3.578 0.943 1.912 2.632 3.595 1.070 1.914 2.646 3.699 1.124 1.981 2.661 3.779
1.248 2.010 2.688 3.924 1.281 2.038 2.82 3 4.035 1.281 2.085 2.890 4.121 1.303 2.089
2.902 4.167 1.432 2.097 2.934 4.240 1.480 2.135 2.962 4.255 1.505 2.154 2.964 4.278
1.506 2.190 3.000 4.305 1.568 2.194 3.103 4.376 1.615 2.223 3.114 4.449 1.619 2.224
3.117 4.485 1.652 2.229 3.166 4.570 1.652 2.300 3.344 4.602 1.757 2.324 3.376 4.663

Right-censored data set: Lifetimes of 30 devices

Meeker and Escobar (2014) represented observed lifetimes of 30 devices that includes eight censored observations. 2 10 13 23 23 28 30 65 80 88 106 143 147 173 181 212 245 247 261 266 275 293 300+ 300+ 300+ 300+ 300+ 300+ 300+ 300+ The + sign indicates right-ensored observations.

Progressively type-I interval-censored data set: 112 patients with plasma cell myeloma

Table 1 contains a typical progressively type-I interval-censored data that devoted to 112 patients with plasma cell myeloma treated at the National Cancer Institute (see Carbone et al. (1967)).

TABLE 1. Progressively type-I interval-censored data set

Interval in months	Number at risk	Number of withdrawals
[0,5.5)	112	1
[5.5,10.5)	93	1
[10.5,15.5)	76	3
[15.5,20.5)	55	0
[20.5,25.5)	45	0
[25.5,30.5)	34	1
[30.5, 40.5)	25	2
[40.5,50.5)	10	3
[50.5,60,6)	3	2
[60.5, ∞)	0	0

5.1. MLE, bootstrap and Bayesian estimation of NLCH-W model and comparing with other models in case complete data set. First, we fit the proposed distribution to the complete real data set by *MLE* method and compare the results with the gamma, Weibull, log-normal (*Lnorm*), generalized exponential (*GE*) and weighted exponential (*WE*) distributions with respective densities

$$f_{gamma}(x) = \frac{1}{\Gamma(\alpha)} \lambda^\alpha x^{\alpha-1} e^{-\lambda x}$$

$$f_{Weibull}(x) = \frac{\beta}{\lambda^\beta} x^{\beta-1} e^{-\left(\frac{x}{\lambda}\right)^\beta}$$

$$f_{Lnorm}(x) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\log x - \mu)^2}{2\sigma^2}}$$

$$f_{GE}(x) = \alpha \lambda e^{-\lambda x} (1 - e^{-\lambda x})^{\alpha-1}$$

$$f_{WE}(x) = \frac{\alpha + 1}{\alpha} \lambda e^{-\lambda x} (1 - e^{-\alpha \lambda x}).$$

Table 2 includes the MLEs of parameters, log-likelihood and Akaike information criterion (AIC) for $NLCH-W$ distribution and the mentioned above distributions in the case complete real data set. The results of Table 2 shows that, the $NLCH-W$

Table 2: The $MLEs$ of parameters for complete data set.

Model	Estimation	Log-likelihood	AIC
NLCH-W	$(\hat{\alpha}, \hat{\beta}, \hat{\lambda}) = (3.874, 1.938, 0.024)$	-128.052	262.105
gamma	$(\hat{\alpha}, \hat{\lambda}) = (3.492, 1.365)$	-136.937	277.874
Weibull	$(\hat{\beta}, \hat{\lambda}) = (2.374, 2.863)$	-130.053	264.107
Lnorm	$(\hat{\mu}, \hat{\sigma}) = (0.789, 0.687)$	-153.920	311.840
WE	$(\hat{\alpha}, \hat{\lambda}) = (0.002, 0.781)$	-143.025	290.049
GE	$(\hat{\alpha}, \hat{\lambda}) = (3.562, 0.758)$	-139.841	283.681

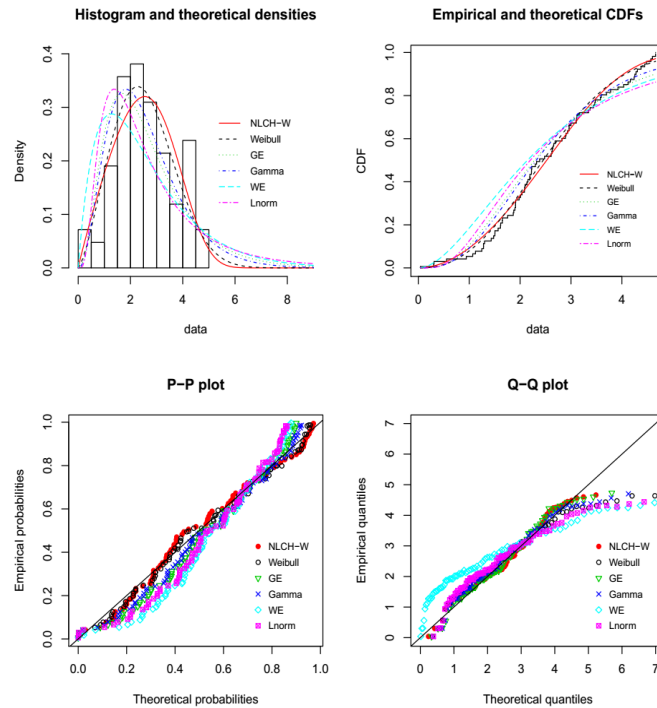


FIGURE 2. Histogram and fitted density plots, the plots of empirical and fitted $cdfs$, $P - P$ plots and $Q - Q$ plots for the complete data set.

distribution provides the best fit for the complete data set as it has lower AIC statistic than the other competitor models. The histogram of data set, fitted pdf

of the $NLCH - W$ distribution and fitted $pdfs$ of other competitor distributions for the real data set are plotted in Figure 2. Also, the plots of empirical and fitted $cdfs$ functions, $P - P$ plots and $Q - Q$ plots for the $NLCH - W$ and other fitted distributions are displayed in Figure 2. These plots also support the results in Table 2.

In the rest of this subsection, we provide Bayesian and Bootstrap estimation results. It is clear from the equation (15) that there is no closed form for the Bayesian estimators under the four loss functions described in subsection 4.3, so we suggest using an MCMC procedure based on 1000 replicates to compute Bayesian estimators. The corresponding Bayesian point and interval estimation provided in Table 3. The posterior samples extracted by using Gibbs sampling technique. Moreover, we provide the posterior summary plots. These plots confirm that the sampling process is of prime quality and convergence is occurred.

Also, here we obtain point and %95 confidence interval estimation of parameters of the $NLCH - W$ distribution by parametric and non-parametric bootstrap methods for complete real data set. We provide results of bootstrap estimation based on 10000 bootstrap replicates in Table 3. It is interesting to look at the joint distribution of the bootstrapped values in a scatter plot in order to understand the potential structural correlation between parameters (see Figures 3 and 4).

Table 3: Bayesian and bootstrap estimation of parameters of $NLCH - W$ for complete data set.

Estimation procedure	Bootstrap estimation of parameters		
Parametric Bootstrap	$\hat{\alpha}_{PB}$	$\hat{\beta}_{PB}$	$\hat{\lambda}_{PB}$
Point estimation	4.517	1.981	0.019
Confidence interval	(0.74, 37.944)	(1.584, 2.973)	(0.003, 0.105)
Non-Parametric Bootstrap	$\hat{\alpha}_{NPB}$	$\hat{\beta}_{NPB}$	$\hat{\lambda}_{NPB}$
point estimation	21.481	1.859	0.005
confidence interval	(0.589, 49.932)	(1.507, 3.260)	(0.0022, 0.088)
Bayesian procedure	Bayesian estimation of parameters		
Loss function	$\hat{\alpha}_B$	$\hat{\beta}_B$	$\hat{\lambda}_B$
Squared error	3.934	1.952	0.022
Absolute value	3.905	1.955	0.022
LINEX ($c = -0.5$)	4.059	1.959	0.021
Generalized entropy ($c = -0.5$)	3.904	1.949	0.022
Bayesian Interval	$\hat{\alpha}_B$	$\hat{\beta}_B$	$\hat{\lambda}_B$
Credible interval	(3.444, 4.386)	(1.845, 2.052)	(0.019, 0.024)
HPD	(2.550, 5.266)	(1.624, 2.247)	(0.015, 0.029)

By analyzing the results of the present table, we can see that the estimated values of parameters are similar for both Bayesian and bootstrap procedures in terms of point and interval (quantile bootstrap, %95 credible and HPD intervals) estimation. In addition, by comparing this results with $MLEs$ estimation of parameters

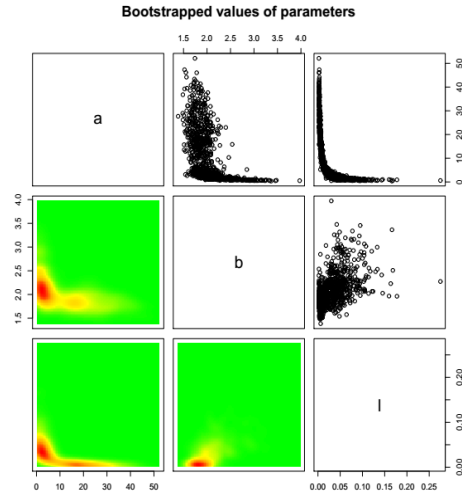


FIGURE 3. Parametric bootstrapped values of parameters of the $NLCH - W$ distribution for the complete data.

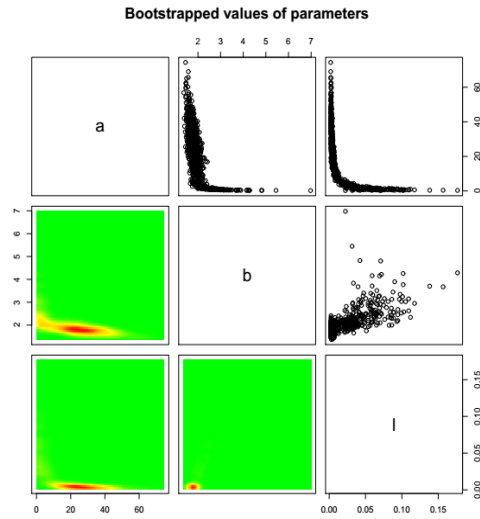


FIGURE 4. Non-parametric bootstrapped values of parameters of the $NLCH - W$ distribution for the complete data.

of $NLCH - W$ in Table 2, it can be seen that, in general the estimation results are

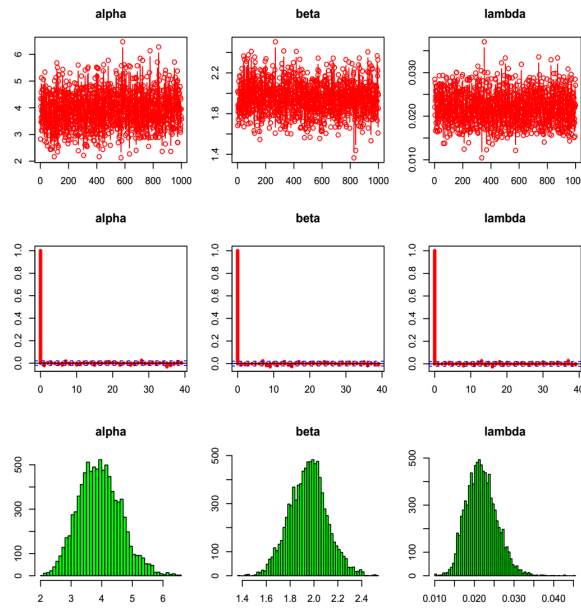


FIGURE 5. Plots of Bayesian analysis and performance of Gibbs sampling for complete data set. Top panel: trace plots; Middle panel: autocorrelation plots; Bottom panel: histograms of each parameter of $NLCH - W$ distribution.

similar under three estimation procedures that described in section 5. Figures 3 and 4 relate to the parametric and non-parametric bootstrap estimation of parameters α , β and λ . Also, Figure 5 relates to the Bayesian analysis process, including history (Trace plot), autocorrelation function (acf) and histogram of three parameters samples down from posterior distribution (15). These plots show that convergence was reached, no autocorrelation problems were encountered and the density of the posterior is extracted.

5.2. MLE, bootstrap and Bayesian estimation in case right-censored data set. Here, we provide the MLE, non-parametric bootstrap and Bayesian estimation of α , β and λ , the parameters of $NLCH - W$ distribution for right-censored data set that given at the beginning of section 5. In order to compare different estimation results, we also provide interval estimation (%95 asymptotic confidence, quantile bootstrap, %95 credible and HPD intervals) of parameters under the three estimation procedures that considered in section 4. Table 4 shows the corresponding results for right-censored data set. In addition, the plots of empirical and theoretical cdf s and diagrams of the Bayesian analysis process are provided in Figure 6 and

Figure 7, respectively. Associated Bayesian procedure plots show that convergence was reached and no autocorrelation problems there exist. Also, Figure 6 represent the estimated lower and upper bound of cumulative probability.

Table 4: MLE, Bayesian and bootstrap estimation of parameters of $NLCH - W$ distribution for right-censored data set.

Estimation procedure	Maximum likelihood estimation		
MLE's	\hat{a}	$\hat{\beta}$	$\hat{\lambda}$
Point estimation	3.344	0.835	0.002
Confidence interval	(0, 9.603)	(0.477, 1.192)	(0.001, 0.0034)
LL	-142.259		
AIC	290.517		
Estimation procedure	Bootstrap estimation of parameters		
Non-Parametric Bootstrap	\hat{a}_{NPB}	$\hat{\beta}_{NPB}$	$\hat{\lambda}_{NPB}$
point estimation	1.070	0.835	0.007
confidence interval	(0.116, 9.822)	(0.579, 1.831)	(0.002, 0.054)
Bayesian procedure	Bayesian estimation of parameters		
Loss function	\hat{a}_B	$\hat{\beta}_B$	$\hat{\lambda}_B$
Squared error	3.296	0.876	0.0022
Absolute value	3.129	0.873	0.002
LINEX ($c = -0.5$)	3.790	0.876	0.0022
Generalized entropy ($c = -0.5$)	3.175	0.872	0.0021
Bayesian Interval	\hat{a}_B	$\hat{\beta}_B$	$\hat{\lambda}_B$
Credible interval	(2.335, 4.027)	(0.800, 0.949)	(0.001, 0.0028)
HPD	(1.145, 5.937)	(0.677, 1.089)	(0.0005, 0.004)

5.3. MLE and Bayesian estimation in the case progressively type-I interval-censored data set. Analogous two previous subsections, here we provide a summary of numerical analysis of progressively type-I interval-censored data set based on the Bayesian and maximum likelihood methods described in section 5. Table 5 is devoted to estimation of parameters. This table provides the Bayesian estimators and %95 credible and HPD intervals for each parameter of proposed $NLCH - W$ model. In addition, the maximum likelihood estimators are calculated in order to compare with corresponding Bayesian estimators under the different loss functions. Plots of history, acf plots and histogram of posterior samples of each parameter of proposed distribution provided in Figures 8. These figures show that the simulation processes of Gibbs algorithm has good quality and convergence is occurred.

6. CONCLUSION

In this article, a new model of lifetime distributions is introduced and main properties of it are obtained. One of the interesting and important properties of proposed family is that, it results the Nadarajah and Haghghi (2011) famous

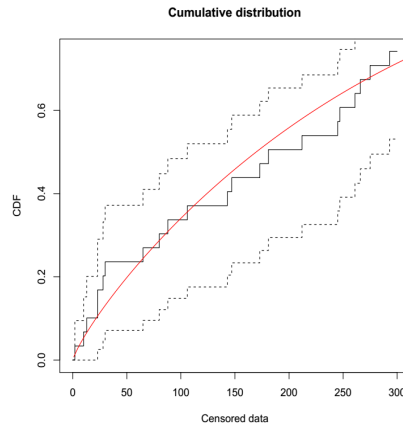


FIGURE 6. Plots of *cdf* of *NLCH – W* distribution for right-censored data set.

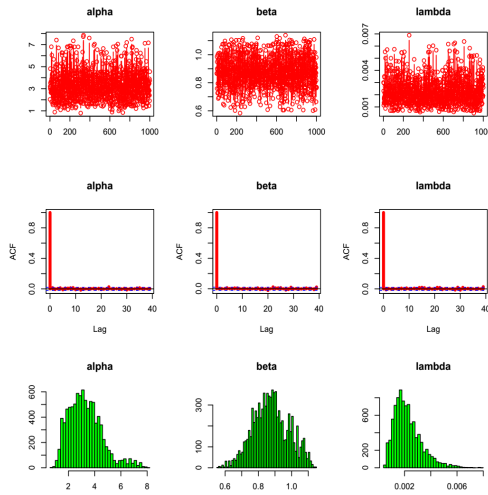


FIGURE 7. Plots of Bayesian analysis and performance of Gibbs sampling for right-censored data set. Top panel: trace plots; Middle panel: autocorrelation plots; Bottom panel: histograms of each parameter of *NLCH – W* distribution.

distribution, as an especial case, when the parent distribution is exponential. An especial example of this family is introduced by considering Weibull distribution as the base distribution. We also show that the proposed distribution has variability

Table 5: Bayesian estimation of parameters of $NLCH - W$ for progressively type-I interval-censored data set.

Estimation method			
Maximum likelihood estimation	\hat{a}	$\hat{\beta}$	$\hat{\lambda}$
MLE's	0.996	1.228	0.019
LL	230.340		
AIC	466.681		
Bayesian estimation			
Loss function	\hat{a}_B	$\hat{\beta}_B$	$\hat{\lambda}_B$
Squared error	1.005	1.333	0.019
Absolute value	0.939	1.322	0.018
LINEX ($c = 0.5$)	0.976	1.328	0.019
Generalized entropy ($c = 0.5$)	0.924	1.321	0.018
Bayesian Interval			
Credible interval	$(0.758, 1.181)$	$(1.226, 1.427)$	$(0.015, 0.022)$
HPD	$(0.429, 1.728)$	$(1.048, 1.623)$	$(0.009, 0.029)$

of hazard rate shapes such as increasing, decreasing, bathtub shape and upside-down bathtub shapes. Classic and Bayesian inferences for three cases of real data such as complete, right-censored and progressively type-I interval-censored data sets are investigated. Bayesian estimators under the four well-known loss functions are presented. Numerical results of maximum likelihood, Bayesian and bootstrap procedures for each set of real data are presented in separate tables. From a practical point of view, the distribution introduced in this study was shown to be better than some common statistical distributions for some real data sets applied as an example.

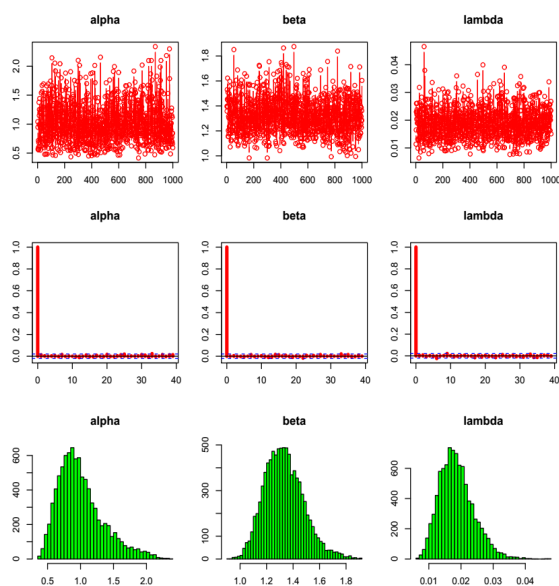


FIGURE 8. Plots of Bayesian analysis and performance of Gibbs sampling for progressively type-I interval-censored data set. Top panel: trace plots; Middle panel: autocorrelation plots; Bottom panel: histograms of each parameter of $NLCH - W$ distribution.

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