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FURTHER INEQUALITIES FOR THE GENERALIZED k-g-FRACTIONAL INTEGRALS OF FUNCTIONS WITH BOUNDED VARIATION

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ABSTRACT. Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). For the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$, we define the *k-g-left-sided fractional integral* of f by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ x\in(a,b]$$

and the *k-g-right-sided fractional integral* of f by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ x\in[a,b),$$

where the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval.

In this paper we establish some new inequalities for the k-g-fractional integrals of functions of bounded variation.Examples for the *generalized left- and right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g and a general exponential fractional integral are also provided.

1. INTRODUCTION

Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. We define the function $K:[0,\infty)\to\mathbb{C}$ by

$$K(t):=\begin{cases}\int_0^t k(s)\,ds \text{ if } 0<t,\\ \\ 0 \text{ if } t=0.\end{cases}$$

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As a simple example, if $k(t)=t^{\alpha-1}$ then for $\alpha\in(0,1)$ the function k is defined on $(0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$. If $\alpha\geq 1$, then k is defined on $[0,\infty)$ and $K(t):=\frac{1}{\alpha}t^{\alpha}$ for $t\in[0,\infty)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). For the Lebesgue integrable function $f:(a,b)\rightarrow\mathbb{C}$, we define the *$k$-$g$-left-sided fractional integral* of f by

$$S_{k,g,a+}f(x)=\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt,\ \ x\in(a,b] \tag{1}$$

and the *k-g-right-sided fractional integral* of f by

$$S_{k,g,b-}f(x)=\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt,\ \ x\in[a,b). \tag{2}$$

If we take $k(t)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, where Γ is the *Gamma function*, then

$$\begin{aligned}S_{k,g,a+}f(x)&=\frac{1}{\Gamma(\alpha)}\int_a^x[g(x)-g(t)]^{\alpha-1}g'(t)\,f(t)\,dt\\&=:I_{a+,g}^{\alpha}f(x),\ \ a<x\leq b\end{aligned} \tag{3}$$

and

$$\begin{aligned}S_{k,g,b-}f(x)&=\frac{1}{\Gamma(\alpha)}\int_x^b[g(t)-g(x)]^{\alpha-1}g'(t)\,f(t)\,dt\\&=:I_{b-,g}^{\alpha}f(x),\ \ a\leq x<b,\end{aligned} \tag{4}$$

which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ as defined in [23, p. 100].

For $g(t)=t$ in (4) we have the classical *Riemann-Liouville fractional integrals* while for the logarithmic function $g(t)=\ln t$ we have the *Hadamard fractional integrals* [23, p. 111]

$$H_{a+}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\left[\ln\left(\frac{x}{t}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ \ 0\leq a<x\leq b \tag{5}$$

and

$$H_{b-}^{\alpha}f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b\left[\ln\left(\frac{t}{x}\right)\right]^{\alpha-1}\frac{f(t)\,dt}{t},\ \ 0\leq a<x<b. \tag{6}$$

One can consider the function $g(t)=-t^{-1}$ and define the *"Harmonic fractional integrals"* by

$$R_{a+}^{\alpha}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_a^x\frac{f(t)\,dt}{(x-t)^{1-\alpha}t^{\alpha+1}},\ \ 0\leq a<x\leq b \tag{7}$$

and

$$R_{b-}^{\alpha}f(x):=\frac{x^{1-\alpha}}{\Gamma(\alpha)}\int_x^b\frac{f(t)\,dt}{(t-x)^{1-\alpha}t^{\alpha+1}},\ \ 0\leq a<x<b. \tag{8}$$

Also, for $g(t) = \exp(\beta t)$, $\beta > 0$, we can consider the *"β-Exponential fractional integrals"*

$$E_{a+,\beta}^{\alpha} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_a^x [\exp(\beta x) - \exp(\beta t)]^{\alpha-1} \exp(\beta t) f(t)\, dt, \tag{9}$$

for $a < x \leq b$ and

$$E_{b-,\beta}^{\alpha} f(x) := \frac{\beta}{\Gamma(\alpha)} \int_x^b [\exp(\beta t) - \exp(\beta x)]^{\alpha-1} \exp(\beta t) f(t)\, dt, \tag{10}$$

for $a \leq x < b$.

If we take $g(t) = t$ in (1) and (2), then we can consider the following *k-fractional integrals*

$$S_{k,a+} f(x) = \int_a^x k(x-t) f(t)\, dt, \; x \in (a,b] \tag{11}$$

and

$$S_{k,b-} f(x) = \int_x^b k(t-x) f(t)\, dt, \; x \in [a,b). \tag{12}$$

In [26], Raina studied a class of functions defined formally by

$$\mathcal{F}_{\rho,\lambda}^{\sigma}(x) := \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k, \; |x| < R, \text{ with } R > 0 \tag{13}$$

for $\rho, \lambda > 0$ where the coefficients $\sigma(k)$ generate a bounded sequence of positive real numbers. With the help of (13), Raina defined the following left-sided fractional integral operator

$$\mathcal{J}_{\rho,\lambda,a+;w}^{\sigma} f(x) := \int_a^x (x-t)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}(w(x-t)^{\rho}) f(t)\, dt, \; x > a \tag{14}$$

where $\rho, \lambda > 0$, $w \in \mathbb{R}$ and f is such that the integral on the right side exists.

In [1], the right-sided fractional operator was also introduced as

$$\mathcal{J}_{\rho,\lambda,b-;w}^{\sigma} f(x) := \int_x^b (t-x)^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}(w(t-x)^{\rho}) f(t)\, dt, \; x < b \tag{15}$$

where $\rho, \lambda > 0$, $w \in \mathbb{R}$ and f is such that the integral on the right side exists. Several Ostrowski type inequalities were also established.

We observe that for $k(t) = t^{\lambda-1} \mathcal{F}_{\rho,\lambda}^{\sigma}(wt^{\rho})$ we re-obtain the definitions of (14) and (15) from (11) and (12).

In [24], Kirane and Torebek introduced the following *exponential fractional integrals*

$$\mathcal{T}_{a+}^{\alpha} f(x) := \frac{1}{\alpha} \int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(x-t)\right\} f(t)\, dt, \; x > a \tag{16}$$

and

$$\mathcal{T}_{b-}^{\alpha} f(x) := \frac{1}{\alpha} \int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(t-x)\right\} f(t)\, dt, \; x < b \tag{17}$$

where $\alpha \in (0,1)$.

We observe that for $k(t) = \frac{1}{\alpha} \exp\left(-\frac{1-\alpha}{\alpha} t\right)$, $t \in \mathbb{R}$ we re-obtain the definitions of (16) and (17) from (11) and (12).

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). We can define the more general exponential fractional integrals

$$\mathcal{T}^{\alpha}_{g,a+} f(x) := \frac{1}{\alpha} \int_a^x \exp\left\{-\frac{1-\alpha}{\alpha}(g(x) - g(t))\right\} g'(t) f(t)\, dt,\ x > a \tag{18}$$

and

$$\mathcal{T}^{\alpha}_{g,b-} f(x) := \frac{1}{\alpha} \int_x^b \exp\left\{-\frac{1-\alpha}{\alpha}(g(t) - g(x))\right\} g'(t) f(t)\, dt,\ x < b \tag{19}$$

where $\alpha \in (0,1)$.

Let g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Assume that $\alpha > 0$. We can also define the *logarithmic fractional integrals*

$$\mathcal{L}^{\alpha}_{g,a+} f(x) := \int_a^x (g(x) - g(t))^{\alpha-1} \ln(g(x) - g(t))\, g'(t) f(t)\, dt, \tag{20}$$

for $0 < a < x \le b$ and

$$\mathcal{L}^{\alpha}_{g,b-} f(x) := \int_x^b (g(t) - g(x))^{\alpha-1} \ln(g(t) - g(x))\, g'(t) f(t)\, dt, \tag{21}$$

for $0 < a \le x < b$, where $\alpha > 0$. These are obtained from (11) and (12) for the kernel $k(t) = t^{\alpha-1} \ln t$, $t > 0$.

For $\alpha = 1$ we get

$$\mathcal{L}_{g,a+} f(x) := \int_a^x \ln(g(x) - g(t))\, g'(t) f(t)\, dt,\ 0 < a < x \le b \tag{22}$$

and

$$\mathcal{L}_{g,b-} f(x) := \int_x^b \ln(g(t) - g(x))\, g'(t) f(t)\, dt,\ 0 < a \le x < b. \tag{23}$$

For $g(t) = t$, we have the simple forms

$$\mathcal{L}^{\alpha}_{a+} f(x) := \int_a^x (x-t)^{\alpha-1} \ln(x-t) f(t)\, dt,\ 0 < a < x \le b, \tag{24}$$

$$\mathcal{L}^{\alpha}_{b-} f(x) := \int_x^b (t-x)^{\alpha-1} \ln(t-x) f(t)\, dt,\ 0 < a \le x < b, \tag{25}$$

$$\mathcal{L}_{a+} f(x) := \int_a^x \ln(x-t) f(t)\, dt,\ 0 < a < x \le b \tag{26}$$

and

$$\mathcal{L}_{b-} f(x) := \int_x^b \ln(t-x) f(t)\, dt,\ 0 < a \le x < b. \tag{27}$$

For several Ostrowski type inequalities for Riemann-Liouville fractional integrals see [2]-[17], [21]-[34] and the references therein.

For k and g as at the beginning of Introduction, we consider the mixed operator

$$\begin{aligned}&S_{k,g,a+,b-}f(x) \qquad (28)\\&:=\frac{1}{2}\left[S_{k,g,a+}f(x)+S_{k,g,b-}f(x)\right]\\&=\frac{1}{2}\left[\int_a^x k(g(x)-g(t))\,g'(t)\,f(t)\,dt+\int_x^b k(g(t)-g(x))\,g'(t)\,f(t)\,dt\right]\end{aligned}$$

for the Lebesgue integrable function $f:(a,b)\to\mathbb{C}$ and $x\in(a,b)$.

We also define the function $\mathbf{K}:[0,\infty)\to[0,\infty)$ by

$$\mathbf{K}(t):=\begin{cases}\int_0^t |k(s)|\,ds \text{ if } 0<t,\\ \\ 0 \text{ if } t=0.\end{cases}$$

In the recent paper [19] we obtained the following result for functions of bounded variation:

Theorem 1. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then we have the Ostrowski type inequality*

$$\begin{aligned}&\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))+K(g(x)-g(a))\right]f(x)\right|\\&\le\frac{1}{2}\left[\int_x^b |k(g(t)-g(x))|\bigvee_x^t(f)\,g'(t)\,dt+\int_a^x |k(g(x)-g(t))|\bigvee_t^x(f)\,g'(t)\,dt\right]\\&\le\frac{1}{2}\left[\mathbf{K}(g(b)-g(x))\bigvee_x^b(f)+\mathbf{K}(g(x)-g(a))\bigvee_a^x(f)\right]\\&\le\frac{1}{2}\begin{cases}\max\{\mathbf{K}(g(b)-g(x)),\mathbf{K}(g(x)-g(a))\}\bigvee_a^b(f);\\ \\ \left[\mathbf{K}^p(g(b)-g(x))+\mathbf{K}^p(g(x)-g(a))\right]^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \left[\mathbf{K}(g(b)-g(x))+\mathbf{K}(g(x)-g(a))\right]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases}\end{aligned} \qquad (29)$$

and the trapezoid type inequality

$$\left|S_{k,g,a+,b-}f(x)-\frac{1}{2}\left[K(g(b)-g(x))\,f(b)+K(g(x)-g(a))\,f(a)\right]\right|$$

$$\leq \frac{1}{2}\left[\int_a^x |k(g(x)-g(t))| \bigvee_a^t (f)\, g'(t)\, dt + \int_x^b |k(g(t)-g(x))| \bigvee_t^b (f)\, g'(t)\, dt\right]$$

$$\leq \frac{1}{2}\left[\mathbf{K}(g(b)-g(x)) \bigvee_x^b (f) + \mathbf{K}(g(x)-g(a)) \bigvee_a^x (f)\right]$$

$$\leq \frac{1}{2}\begin{cases} \max\{\mathbf{K}(g(b)-g(x)), \mathbf{K}(g(x)-g(a))\} \bigvee_a^b (f); \\ \\ [\mathbf{K}^p(g(b)-g(x)) + \mathbf{K}^p(g(x)-g(a))]^{1/p} \\ \times \left(\left(\bigvee_a^x (f)\right)^q + \left(\bigvee_x^b (f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ [\mathbf{K}(g(b)-g(x)) + \mathbf{K}(g(x)-g(a))] \\ \times \left[\frac{1}{2}\bigvee_a^b (f) + \frac{1}{2}\left|\bigvee_a^x (f) - \bigvee_x^b (f)\right|\right] \end{cases} \tag{30}$$

for any $x \in (a,b)$, *where* $\bigvee_c^d (f)$ *denoted the total variation on the interval* $[c,d]$.

Observe that

$$S_{k,g,x+} f(b) = \int_x^b k(g(b)-g(t))\, g'(t)\, f(t)\, dt,\ x \in [a,b) \tag{31}$$

and

$$S_{k,g,x-} f(a) = \int_a^x k(g(t)-g(a))\, g'(t)\, f(t)\, dt,\ x \in (a,b]. \tag{32}$$

We can define also the mixed operator

$$\begin{aligned} &\breve{S}_{k,g,a+,b-} f(x) \\ &:= \frac{1}{2}\left[S_{k,g,x+} f(b) + S_{k,g,x-} f(a)\right] \\ &= \frac{1}{2}\left[\int_x^b k(g(b)-g(t))\, g'(t)\, f(t)\, dt + \int_a^x k(g(t)-g(a))\, g'(t)\, f(t)\, dt\right] \end{aligned} \tag{33}$$

for any $x \in (a,b)$.

In this paper we establish some inequalities for the k-g-fractional integrals of functions with bounded variation $f : [a,b] \to \mathbb{C}$ that provide error bounds in approximating the composite operators $S_{k,g,a+,b-} f$ and $\breve{S}_{k,g,a+,b-} f$ in terms of the *double trapezoid rule*

$$\frac{1}{2}\left[\frac{f(x)+f(b)}{2} K(g(b)-g(x)) + \frac{f(a)+f(x)}{2} K(g(x)-g(a))\right],\ x \in (a,b).$$

Examples for the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g and a general exponential fractional integral are also provided.

2. Further Inequalities for Functions of BV

The following two parameters representation for the operators $S_{k,g,a+,b-}$ and $\breve{S}_{k,g,a+,b-}$ hold [20]:

Lemma 2. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f:[a,b]\to\mathbb{C}$ be an integrable function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then*

$$\begin{aligned} S_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[\gamma K(g(b)-g(x))+\lambda K(g(x)-g(a))\right] \\ &+\frac{1}{2}\int_a^x k(g(x)-g(t))\,g'(t)\left[f(t)-\lambda\right]dt \\ &+\frac{1}{2}\int_x^b k(g(t)-g(x))\,g'(t)\left[f(t)-\gamma\right]dt \end{aligned} \tag{34}$$

and

$$\begin{aligned} \breve{S}_{k,g,a+,b-}f(x) &= \frac{1}{2}\left[\lambda K(g(b)-g(x))+\gamma K(g(x)-g(a))\right] \\ &+\frac{1}{2}\int_a^x k(g(t)-g(a))\,g'(t)\left[f(t)-\gamma\right]dt \\ &+\frac{1}{2}\int_x^b k(g(b)-g(t))\,g'(t)\left[f(t)-\lambda\right]dt \end{aligned} \tag{35}$$

for $x\in(a,b)$ and for any $\lambda,\ \gamma\in\mathbb{C}$.

Proof. We have, by taking the derivative over t and using the chain rule, that

$$[K(g(x)-g(t))]' = K'(g(x)-g(t))(g(x)-g(t))' = -k(g(x)-g(t))\,g'(t)$$

for $t\in(a,x)$ and

$$[K(g(t)-g(x))]' = K'(g(t)-g(x))(g(t)-g(x))' = k(g(t)-g(x))\,g'(t)$$

for $t\in(x,b)$.

Therefore, for any $\lambda,\ \gamma\in\mathbb{C}$ we have

$$\begin{aligned} &\int_a^x k(g(x)-g(t))\,g'(t)\left[f(t)-\lambda\right]dt \\ &= \int_a^x k(g(x)-g(t))\,g'(t)f(t)\,dt-\lambda\int_a^x k(g(x)-g(t))\,g'(t)\,dt \\ &= S_{k,g,a+}f(x)+\lambda\int_a^x [K(g(x)-g(t))]'\,dt \\ &= S_{k,g,a+}f(x)+\lambda\left[K(g(x)-g(t))\right]\Big|_a^x = S_{k,g,a+}f(x)-\lambda K(g(x)-g(a)) \end{aligned} \tag{36}$$

and

$$\int_x^b k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)\left[f\left(t\right)-\gamma\right]dt \tag{37}$$
$$=\int_x^b k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)f\left(t\right)dt-\gamma\int_x^b k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)dt$$
$$=S_{k,g,b-}f\left(x\right)-\gamma\int_x^b\left[K\left(g\left(t\right)-g\left(x\right)\right)\right]'dt$$
$$=S_{k,g,b-}f\left(x\right)-\gamma\left[K\left(g\left(t\right)-g\left(x\right)\right)\right]\Big|_x^b=S_{k,g,b-}f\left(x\right)-\gamma K\left(g\left(b\right)-g\left(x\right)\right)$$

for $x\in\left(a,b\right)$.

If we add the equalities (36) and (37) and divide by 2 then we get the desired result (34).

Moreover, by taking the derivative over t and using the chain rule, we have that

$$\left[K\left(g\left(b\right)-g\left(t\right)\right)\right]'=K'\left(g\left(b\right)-g\left(t\right)\right)\left(g\left(b\right)-g\left(t\right)\right)'=-k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)$$

for $t\in\left(x,b\right)$ and

$$\left[K\left(g\left(t\right)-g\left(a\right)\right)\right]'=K'\left(g\left(t\right)-g\left(a\right)\right)\left(g\left(t\right)-g\left(a\right)\right)'=k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)$$

for $t\in\left(a,x\right)$.

For any $\lambda,\ \gamma\in\mathbb{C}$ we have

$$\int_x^b k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-\lambda\right]dt \tag{38}$$
$$=\int_x^b k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)f\left(t\right)dt-\lambda\int_x^b k\left(g\left(b\right)-g\left(t\right)\right)g'\left(t\right)dt$$
$$=S_{k,g,x+}f\left(b\right)+\lambda\int_x^b\left[K\left(g\left(b\right)-g\left(t\right)\right)\right]'dt$$
$$=S_{k,g,x+}f\left(b\right)-\lambda K\left(g\left(b\right)-g\left(x\right)\right)$$

and

$$\int_a^x k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)\left[f\left(t\right)-\gamma\right]dt \tag{39}$$
$$=\int_a^x k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)f\left(t\right)dt-\gamma\int_a^x k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)dt$$
$$=\int_a^x k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)f\left(t\right)dt-\gamma\int_a^x\left[K\left(g\left(t\right)-g\left(a\right)\right)\right]'dt$$
$$=\int_a^x k\left(g\left(t\right)-g\left(a\right)\right)g'\left(t\right)f\left(t\right)dt-\gamma K\left(g\left(x\right)-g\left(a\right)\right)$$

for $x\in\left(a,b\right)$.

If we add the equalities (38) and (39) and divide by 2 then we get the desired result (35). □

If g is a function which maps an interval I of the real line to the real numbers, and is both continuous and injective then we can define the *g-mean of two numbers* a, $b \in I$ as

$$M_g(a,b) := g^{-1}\left(\frac{g(a)+g(b)}{2}\right).$$

If $I = \mathbb{R}$ and $g(t) = t$ is the *identity function*, then $M_g(a,b) = A(a,b) := \frac{a+b}{2}$, the *arithmetic mean*. If $I = (0,\infty)$ and $g(t) = \ln t$, then $M_g(a,b) = G(a,b) := \sqrt{ab}$, the *geometric mean*. If $I = (0,\infty)$ and $g(t) = \frac{1}{t}$, then $M_g(a,b) = H(a,b) := \frac{2ab}{a+b}$, the *harmonic mean*. If $I = (0,\infty)$ and $g(t) = t^p$, $p \neq 0$, then $M_g(a,b) = M_p(a,b) := \left(\frac{a^p+b^p}{2}\right)^{1/p}$, the *power mean with exponent* p. Finally, if $I = \mathbb{R}$ and $g(t) = \exp t$, then

$$M_g(a,b) = LME(a,b) := \ln\left(\frac{\exp a + \exp b}{2}\right),$$

the *LogMeanExp function*.

Using the g-mean of two numbers we can introduce

$$\begin{aligned} P_{k,g,a+,b-}f &:= S_{k,g,a+,b-}f\left(M_g(a,b)\right) \\ &= \frac{1}{2}\int_a^{M_g(a,b)} k\left(\frac{g(a)+g(b)}{2} - g(t)\right) g'(t) f(t)\,dt \\ &+ \frac{1}{2}\int_{M_g(a,b)}^b k\left(g(t) - \frac{g(a)+g(b)}{2}\right) g'(t) f(t)\,dt. \end{aligned} \tag{40}$$

Using the representation (34) we have

$$\begin{aligned} P_{k,g,a+,b-}f &= K\left(\frac{g(b)-g(a)}{2}\right)\frac{\gamma+\lambda}{2} \\ &+ \frac{1}{2}\int_a^{M_g(a,b)} k\left(\frac{g(a)+g(b)}{2} - g(t)\right) g'(t)\left[f(t)-\lambda\right] dt \\ &+ \frac{1}{2}\int_{M_g(a,b)}^b k\left(g(t) - \frac{g(a)+g(b)}{2}\right) g'(t)\left[f(t)-\gamma\right] dt \end{aligned} \tag{41}$$

for any λ, $\gamma \in \mathbb{C}$.

Also, if

$$\begin{aligned} \breve{P}_{k,g,a+,b-}f &:= \breve{S}_{k,g,a+,b-}f\left(M_g(a,b)\right) \\ &= \frac{1}{2}\int_{M_g(a,b)}^b k\left(g(b) - g(t)\right) g'(t) f(t)\,dt \\ &+ \frac{1}{2}\int_a^{M_g(a,b)} k\left(g(t) - g(a)\right) g'(t) f(t)\,dt. \end{aligned} \tag{42}$$

then by (35) we get

$$\breve{P}_{k,g,a+,b-}f = K\left(\frac{g(b)-g(a)}{2}\right)\frac{\gamma+\lambda}{2} + \frac{1}{2}\int_a^{M_g(a,b)} k(g(t)-g(a))\,g'(t)\,[f(t)-\gamma]\,dt + \frac{1}{2}\int_{M_g(a,b)}^b k(g(b)-g(t))\,g'(t)\,[f(t)-\lambda]\,dt \tag{43}$$

for any λ, $\gamma \in \mathbb{C}$.

Theorem 3. *Assume that the kernel k is defined either on $(0,\infty)$ or on $[0,\infty)$ with complex values and integrable on any finite subinterval. Let $f : [a,b] \to \mathbb{C}$ be a function of bounded variation on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Then we have the double trapezoid inequalities*

$$\begin{aligned}
&\Bigg| S_{k,g,a+,b-}f(x) - \frac{1}{2}\left[\frac{f(x)+f(b)}{2}K(g(b)-g(x)) + \frac{f(a)+f(x)}{2}K(g(x)-g(a))\right]\Bigg| \\
&\leq \frac{1}{4}\left[\mathbf{K}(g(x)-g(a))\bigvee_a^x(f) + \mathbf{K}(g(b)-g(x))\bigvee_x^b(f)\right] \\
&\leq \frac{1}{4}\begin{cases}
\max\left\{\mathbf{K}(g(b)-g(x)), \mathbf{K}(g(x)-g(a))\right\}\bigvee_a^b(f); \\
\\
\left[\mathbf{K}^p(g(b)-g(x)) + \mathbf{K}^p(g(x)-g(a))\right]^{1/p}\left(\left(\bigvee_a^x(f)\right)^q + \left(\bigvee_x^b(f)\right)^q\right)^{1/q} \\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\
\\
\left[\mathbf{K}(g(b)-g(x)) + \mathbf{K}(g(x)-g(a))\right]\left[\frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^x(f) - \bigvee_x^b(f)\right|\right]
\end{cases}
\end{aligned} \tag{44}$$

and

$$\begin{aligned}
&\Bigg| \breve{S}_{k,g,a+,b-}f(x) - \frac{1}{2}\left[\frac{f(x)+f(b)}{2}K(g(b)-g(x)) + \frac{f(a)+f(x)}{2}K(g(x)-g(a))\right]\Bigg| \\
&\leq \frac{1}{4}\left[\mathbf{K}(g(x)-g(a))\bigvee_a^x(f) + \mathbf{K}(g(b)-g(x))\bigvee_x^b(f)\right]
\end{aligned}$$

$$\leq \frac{1}{4}\begin{cases} \max\left\{\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right),\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)\right\}\bigvee_{a}^{b}\left(f\right); \\ \\ \left[\mathbf{K}^{p}\left(g\left(b\right)-g\left(x\right)\right)+\mathbf{K}^{p}\left(g\left(x\right)-g\left(a\right)\right)\right]^{1/p}\left(\left(\bigvee_{a}^{x}\left(f\right)\right)^{q}+\left(\bigvee_{x}^{b}\left(f\right)\right)^{q}\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \\ \left[\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right)+\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)\right]\left[\frac{1}{2}\bigvee_{a}^{b}\left(f\right)+\frac{1}{2}\left|\bigvee_{a}^{x}\left(f\right)-\bigvee_{x}^{b}\left(f\right)\right|\right] \end{cases} \tag{45}$$

for $x\in(a,b)$.

Proof. Using the identity (34) for $\lambda=\frac{f(a)+f(x)}{2}$ and $\gamma=\frac{f(x)+f(b)}{2}$ we have

$$\begin{aligned} & S_{k,g,a+,b-}f\left(x\right) \\ & =\frac{1}{2}\left[\frac{f\left(x\right)+f\left(b\right)}{2}K\left(g\left(b\right)-g\left(x\right)\right)+\frac{f\left(a\right)+f\left(x\right)}{2}K\left(g\left(x\right)-g\left(a\right)\right)\right] \\ & +\frac{1}{2}\int_{a}^{x}k\left(g\left(x\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-\frac{f\left(a\right)+f\left(x\right)}{2}\right]dt \\ & +\frac{1}{2}\int_{x}^{b}k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)\left[f\left(t\right)-\frac{f\left(x\right)+f\left(b\right)}{2}\right]dt \end{aligned} \tag{46}$$

for $x\in(a,b)$.

Since f is of bounded variation, then

$$\begin{aligned} \left|f\left(t\right)-\frac{f\left(a\right)+f\left(x\right)}{2}\right| & =\left|\frac{f\left(t\right)-f\left(a\right)+f\left(t\right)-f\left(x\right)}{2}\right| \\ & \leq\frac{1}{2}\left[\left|f\left(t\right)-f\left(a\right)\right|+\left|f\left(x\right)-f\left(t\right)\right|\right]\leq\frac{1}{2}\bigvee_{a}^{x}\left(f\right) \end{aligned}$$

and

$$\begin{aligned} \left|f\left(t\right)-\frac{f\left(x\right)+f\left(b\right)}{2}\right| & =\left|\frac{f\left(t\right)-f\left(x\right)+f\left(t\right)-f\left(b\right)}{2}\right| \\ & \leq\frac{1}{2}\left[\left|f\left(t\right)-f\left(x\right)\right|+\left|f\left(b\right)-f\left(t\right)\right|\right]\leq\frac{1}{2}\bigvee_{x}^{b}\left(f\right) \end{aligned}$$

for $x\in(a,b)$.

Using the equality (46) we have

$$\begin{aligned} & \left|S_{k,g,a+,b-}f\left(x\right)\right. \\ & \qquad\left.-\frac{1}{2}\left[\frac{f\left(x\right)+f\left(b\right)}{2}K\left(g\left(b\right)-g\left(x\right)\right)+\frac{f\left(a\right)+f\left(x\right)}{2}K\left(g\left(x\right)-g\left(a\right)\right)\right]\right| \\ & \qquad\quad\leq\frac{1}{2}\left|\int_{a}^{x}k\left(g\left(x\right)-g\left(t\right)\right)g'\left(t\right)\left[f\left(t\right)-\frac{f\left(a\right)+f\left(x\right)}{2}\right]dt\right| \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\left|\int_x^b k\left(g\left(t\right)-g\left(x\right)\right)g'\left(t\right)\left[f\left(t\right)-\frac{f\left(x\right)+f\left(b\right)}{2}\right]dt\right| \\
&\leq \frac{1}{2}\int_a^x \left|k\left(g\left(x\right)-g\left(t\right)\right)\right|\left|f\left(t\right)-\frac{f\left(a\right)+f\left(x\right)}{2}\right|g'\left(t\right)dt \\
&+\frac{1}{2}\int_x^b \left|k\left(g\left(t\right)-g\left(x\right)\right)\right|\left|f\left(t\right)-\frac{f\left(x\right)+f\left(b\right)}{2}\right|g'\left(t\right)dt \\
&\leq \frac{1}{4}\left[\bigvee_a^x\left(f\right)\int_a^x\left|k\left(g\left(x\right)-g\left(t\right)\right)\right|g'\left(t\right)dt+\bigvee_x^b\left(f\right)\int_x^b\left|k\left(g\left(t\right)-g\left(x\right)\right)\right|g'\left(t\right)dt\right] \\
&=: B\left(x\right) \quad (47)
\end{aligned}
$$

for $x\in(a,b)$.

We have, by taking the derivative over t and using the chain rule, that

$$
\left[\mathbf{K}\left(g\left(x\right)-g\left(t\right)\right)\right]'=\mathbf{K}'\left(g\left(x\right)-g\left(t\right)\right)\left(g\left(x\right)-g\left(t\right)\right)'=-\left|k\left(g\left(x\right)-g\left(t\right)\right)\right|g'\left(t\right)
$$

for $t\in(a,x)$ and

$$
\left[\mathbf{K}\left(g\left(t\right)-g\left(x\right)\right)\right]'=\mathbf{K}'\left(g\left(t\right)-g\left(x\right)\right)\left(g\left(t\right)-g\left(x\right)\right)'=\left|k\left(g\left(t\right)-g\left(x\right)\right)\right|g'\left(t\right)
$$

for $t\in(x,b)$.

Then

$$
\int_a^x\left|k\left(g\left(x\right)-g\left(t\right)\right)\right|g'\left(t\right)dt=-\int_a^x\left[\mathbf{K}\left(g\left(x\right)-g\left(t\right)\right)\right]'dt=\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)
$$

and

$$
\int_x^b\left|k\left(g\left(t\right)-g\left(x\right)\right)\right|g'\left(t\right)dt=\int_x^b\left[\mathbf{K}\left(g\left(t\right)-g\left(x\right)\right)\right]'dt=\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right).
$$

Therefore

$$
\begin{aligned}
B\left(x\right)&=\frac{1}{4}\left[\bigvee_a^x\left(f\right)\int_a^x\left|k\left(g\left(x\right)-g\left(t\right)\right)\right|g'\left(t\right)dt+\bigvee_x^b\left(f\right)\int_x^b\left|k\left(g\left(t\right)-g\left(x\right)\right)\right|g'\left(t\right)dt\right] \\
&=\frac{1}{4}\left[\mathbf{K}\left(g\left(x\right)-g\left(a\right)\right)\bigvee_a^x\left(f\right)+\mathbf{K}\left(g\left(b\right)-g\left(x\right)\right)\bigvee_x^b\left(f\right)\right].
\end{aligned}
$$

The last part of (44) is obvious by making use of the elementary Hölder type inequalities for positive real numbers c, d, m, $n\geq 0$

$$
mc+nd\leq\begin{cases}\max\left\{m,n\right\}\left(c+d\right); \\ \\ \left(m^p+n^p\right)^{1/p}\left(c^q+d^q\right)^{1/q} \text{ with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1.\end{cases}
$$

Using the identity (35) for $\lambda=\frac{f(x)+f(b)}{2}$ and $\gamma=\frac{f(x)+f(a)}{2}$ we also have

$$
\left|\breve{S}_{k,g,a+,b-}f\left(x\right)\right.
$$

$$
\begin{aligned}
&-\frac{1}{2}\left[\frac{f(x)+f(b)}{2}K(g(b)-g(x))+\frac{f(x)+f(a)}{2}K(g(x)-g(a))\right]\Bigg| \\
&\qquad\le \frac{1}{2}\int_a^x |k(g(t)-g(a))|\left|f(t)-\frac{f(x)+f(a)}{2}\right| g'(t)\,dt \\
&\qquad+\frac{1}{2}\int_x^b |k(g(b)-g(t))|\left|f(t)-\frac{f(x)+f(b)}{2}\right| g'(t)\,dt \\
&\le \frac{1}{4}\bigvee_a^x (f)\int_a^x |k(g(t)-g(a))|\,g'(t)\,dt+\frac{1}{4}\bigvee_x^b (f)\int_x^b |k(g(b)-g(t))|\,g'(t)\,dt \\
&\qquad\qquad =: C(x).
\end{aligned}
$$

We also have, by taking the derivative over t and using the chain rule, that

$$
[\mathbf{K}(g(b)-g(t))]'=\mathbf{K}'(g(b)-g(t))(g(b)-g(t))'=-|k(g(b)-g(t))|\,g'(t)
$$

for $t\in(x,b)$ and

$$
[\mathbf{K}(g(t)-g(a))]'=\mathbf{K}'(g(t)-g(a))(g(t)-g(a))'=|k(g(t)-g(a))|\,g'(t)
$$

for $t\in(a,x)$.

Therefore

$$
\int_a^x |k(g(t)-g(a))|\,g'(t)\,dt=\mathbf{K}(g(x)-g(a))
$$

and

$$
\int_x^b |k(g(b)-g(t))|\,g'(t)\,dt=\mathbf{K}(g(b)-g(x))
$$

giving that

$$
C(x)=\frac{1}{4}\bigvee_a^x (f)\,\mathbf{K}(g(x)-g(a))+\frac{1}{4}\bigvee_x^b (f)\,\mathbf{K}(g(b)-g(x))
$$

for $x\in(a,b)$, and the inequality (45) is thus proved. □

Corollary 4. *With the assumptions of Theorem 3 we have*

$$
\begin{aligned}
&\left|P_{k,g,a+,b-}f-\frac{1}{2}K\left(\frac{g(b)-g(a)}{2}\right)\left[f(M_g(a,b))+\frac{f(a)+f(b)}{2}\right]\right| \qquad (48)\\
&\le \frac{1}{4}\mathbf{K}\left(\frac{g(b)-g(a)}{2}\right)\bigvee_a^b (f)
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|\breve{P}_{k,g,a+,b-}f-\frac{1}{2}K\left(\frac{g(b)-g(a)}{2}\right)\left[f(M_g(a,b))+\frac{f(a)+f(b)}{2}\right]\right| \qquad (49)\\
&\le \frac{1}{4}\mathbf{K}\left(\frac{g(b)-g(a)}{2}\right)\bigvee_a^b (f).
\end{aligned}
$$

If we take $x=\frac{a+b}{2}$ in (44) and (45), then we get

$$\begin{aligned}
&\left|S_{k,g,a+,b-}f\left(\frac{a+b}{2}\right)-\frac{f\left(\frac{a+b}{2}\right)+f(b)}{4}K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)\right.\\
&\qquad\left.-\frac{f(a)+f\left(\frac{a+b}{2}\right)}{4}K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right|\\
&\leq\frac{1}{4}\left[\mathbf{K}\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\bigvee_{a}^{\frac{a+b}{2}}(f)+\mathbf{K}\left(g(b)-g\left(\frac{a+b}{2}\right)\right)\bigvee_{\frac{a+b}{2}}^{b}(f)\right]\\
&\leq\frac{1}{4}\begin{cases}\max\left\{\mathbf{K}\left(g(b)-g\left(\frac{a+b}{2}\right)\right),\mathbf{K}\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right\}\bigvee_{a}^{b}(f);\\[2mm]
\left[\mathbf{K}^{p}\left(g(b)-g\left(\frac{a+b}{2}\right)\right)+\mathbf{K}^{p}\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right]^{1/p}\\
\left(\left(\bigvee_{a}^{\frac{a+b}{2}}(f)\right)^{q}+\left(\bigvee_{\frac{a+b}{2}}^{b}(f)\right)^{q}\right)^{1/q}\\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2mm]
\left[\mathbf{K}\left(g(b)-g\left(\frac{a+b}{2}\right)\right)+\mathbf{K}\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right]\\
\left[\frac{1}{2}\bigvee_{a}^{b}(f)+\frac{1}{2}\left|\bigvee_{a}^{\frac{a+b}{2}}(f)-\bigvee_{\frac{a+b}{2}}^{b}(f)\right|\right]\end{cases}
\end{aligned}\tag{50}$$

and

$$\begin{aligned}
&\left|\breve{S}_{k,g,a+,b-}f\left(\frac{a+b}{2}\right)-\frac{f\left(\frac{a+b}{2}\right)+f(b)}{4}K\left(g(b)-g\left(\frac{a+b}{2}\right)\right)\right.\\
&\qquad\left.-\frac{f(a)+f\left(\frac{a+b}{2}\right)}{4}K\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right|\\
&\leq\frac{1}{4}\left[\mathbf{K}\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\bigvee_{a}^{\frac{a+b}{2}}(f)+\mathbf{K}\left(g(b)-g\left(\frac{a+b}{2}\right)\right)\bigvee_{x}^{b}(f)\right]
\end{aligned}$$

$$\leq \frac{1}{4}\begin{cases} \max\left\{\mathbf{K}\left(g(b)-g\left(\frac{a+b}{2}\right)\right),\mathbf{K}\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right\}\bigvee_a^b(f); \\ \left[\mathbf{K}^p\left(g(b)-g\left(\frac{a+b}{2}\right)\right)+\mathbf{K}^p\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right]^{1/p} \\ \left(\left(\bigvee_a^{\frac{a+b}{2}}(f)\right)^q+\left(\bigvee_{\frac{a+b}{2}}^b(f)\right)^q\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\mathbf{K}\left(g(b)-g\left(\frac{a+b}{2}\right)\right)+\mathbf{K}\left(g\left(\frac{a+b}{2}\right)-g(a)\right)\right] \\ \left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^{\frac{a+b}{2}}(f)-\bigvee_{\frac{a+b}{2}}^b(f)\right|\right] \end{cases} \tag{51}$$

for $x\in(a,b)$.

We use the classical Lebesgue p-norms defined as

$$\|h\|_{[c,d],\infty}:=\operatorname*{essup}_{s\in[c,d]}|h(s)|$$

and

$$\|h\|_{[c,d],p}:=\left(\int_c^d|h(s)|^p\,ds\right)^{1/p},\ p\geq 1.$$

Using Hölder's integral inequality we have for $t>0$ that

$$K(t)=\int_0^t|k(s)|\,ds\leq\begin{cases} t\,\|k\|_{[0,t],\infty} \text{ if } k\in L_\infty[0,t] \\ t^{1/p}\,\|k\|_{[0,t],q} \text{ if } k\in L_q[0,t],\ p,q>1,\ \frac{1}{p}+\frac{1}{q}=1. \end{cases}$$

Therefore by the first inequality in (44) and (45) we get for $p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1$

$$\begin{aligned} &\Bigg|S_{k,g,a+,b-}f(x) \\ &\quad-\frac{1}{2}\left[\frac{f(x)+f(b)}{2}K(g(b)-g(x))+\frac{f(a)+f(x)}{2}K(g(x)-g(a))\right]\Bigg| \\ &\qquad\leq\frac{1}{4}\bigvee_a^x(f)\begin{cases}(g(x)-g(a))\,\|k\|_{[0,g(x)-g(a)],\infty} \\ (g(x)-g(a))^{1/p}\,\|k\|_{[0,g(x)-g(a)],q}\end{cases} \\ &\qquad\quad+\frac{1}{4}\bigvee_x^b(f)\begin{cases}(g(b)-g(x))\,\|k\|_{[0,g(b)-g(x)],\infty} \\ (g(b)-g(x))^{1/p}\,\|k\|_{[0,g(b)-g(x)],q}\end{cases} \end{aligned} \tag{52}$$

and

$$\begin{aligned} &\Bigg|\breve{S}_{k,g,a+,b-}f(x) \\ &\quad-\frac{1}{2}\left[\frac{f(x)+f(b)}{2}K(g(b)-g(x))+\frac{f(a)+f(x)}{2}K(g(x)-g(a))\right]\Bigg| \end{aligned}$$

$$\leq \frac{1}{4}\bigvee_a^x (f) \begin{cases} (g(x)-g(a))\|k\|_{[0,g(x)-g(a)],\infty} \\ \\ (g(x)-g(a))^{1/p}\|k\|_{[0,g(x)-g(a)],q} \end{cases}$$

$$+\frac{1}{4}\bigvee_x^b (f) \begin{cases} (g(b)-g(x))\|k\|_{[0,g(b)-g(x)],\infty} \\ \\ (g(b)-g(x))^{1/p}\|k\|_{[0,g(b)-g(x)],q} \end{cases} \tag{53}$$

for $x \in (a,b)$.

From (48) and (49) we also have for $p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1$ that

$$\left|P_{k,g,a+,b-}f - \frac{1}{2}K\left(\frac{g(b)-g(a)}{2}\right)\left[f(M_g(a,b))+\frac{f(a)+f(b)}{2}\right]\right|$$
$$\leq \frac{1}{4}\bigvee_a^b (f) \begin{cases} \left(\frac{g(b)-g(a)}{2}\right)\|k\|_{\left[0,\frac{g(b)-g(a)}{2}\right],\infty} \\ \\ \left(\frac{g(b)-g(a)}{2}\right)^{1/p}\|k\|_{\left[0,\frac{g(b)-g(a)}{2}\right],q} \end{cases} \tag{54}$$

and

$$\left|\breve{P}_{k,g,a+,b-}f - \frac{1}{2}K\left(\frac{g(b)-g(a)}{2}\right)\left[f(M_g(a,b))+\frac{f(a)+f(b)}{2}\right]\right|$$
$$\leq \frac{1}{4}\bigvee_a^b (f) \begin{cases} \left(\frac{g(b)-g(a)}{2}\right)\|k\|_{\left[0,\frac{g(b)-g(a)}{2}\right],\infty} \\ \\ \left(\frac{g(b)-g(a)}{2}\right)^{1/p}\|k\|_{\left[0,\frac{g(b)-g(a)}{2}\right],q}. \end{cases} \tag{55}$$

3. Applications for Generalized Riemann-Liouville Fractional Integrals

If we take $k(t)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}$, where Γ is the *Gamma function*, then

$$S_{k,g,a+}f(x) = I^{\alpha}_{a+,g}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x [g(x)-g(t)]^{\alpha-1} g'(t) f(t)\,dt$$

for $a<x\leq b$ and

$$S_{k,g,b-}f(x) = I^{\alpha}_{b-,g}f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b [g(t)-g(x)]^{\alpha-1} g'(t) f(t)\,dt$$

for $a\leq x<b$, which are the *generalized left-* and *right-sided Riemann-Liouville fractional integrals* of a function f with respect to another function g on $[a,b]$ as defined in [23, p. 100].

We consider the mixed operators

$$I^{\alpha}_{g,a+,b-}f(x) := \frac{1}{2}\left[I^{\alpha}_{a+,g}f(x)+I^{\alpha}_{b-,g}f(x)\right] \tag{56}$$

and

$$\breve{I}_{g,a+,b-}^{\alpha}f\left(x\right):=\frac{1}{2}\left[I_{x+,g}^{\alpha}f\left(b\right)+I_{x-,g}^{\alpha}f(a)\right] \tag{57}$$

for $x\in\left(a,b\right)$.

We observe that for $\alpha>0$ we have

$$K\left(t\right)=\frac{1}{\Gamma\left(\alpha\right)}\int_{0}^{t}s^{\alpha-1}ds=\frac{t^{\alpha}}{\alpha\Gamma\left(\alpha\right)}=\frac{t^{\alpha}}{\Gamma\left(\alpha+1\right)},\ t\geq0.$$

If we use the inequalities (44) and (45) we get

$$\begin{aligned}
&\Big|I_{g,a+,b-}^{\alpha}f\left(x\right)\\
&\quad-\frac{1}{2\Gamma\left(\alpha+1\right)}\left[\frac{f\left(x\right)+f\left(b\right)}{2}\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}+\frac{f\left(a\right)+f\left(x\right)}{2}\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}\right]\Bigg|\\
&\qquad\leq\frac{1}{4\Gamma\left(\alpha+1\right)}\left[\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}\bigvee_{a}^{x}\left(f\right)+\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}\bigvee_{x}^{b}\left(f\right)\right]\\
&\qquad\leq\frac{1}{4\Gamma\left(\alpha+1\right)}\\
&\times\begin{cases}
\left[\frac{g(b)-g(a)}{2}+\left|g\left(x\right)-\frac{g(b)+g(a)}{2}\right|\right]^{\alpha}\bigvee_{a}^{b}\left(f\right);\\[2ex]
\left[\left(g\left(b\right)-g\left(x\right)\right)^{p\alpha}+\left(g\left(x\right)-g\left(a\right)\right)^{p\alpha}\right]^{1/p}\left(\left(\bigvee_{a}^{x}\left(f\right)\right)^{q}+\left(\bigvee_{x}^{b}\left(f\right)\right)^{q}\right)^{1/q}\\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]
\left[\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}+\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}\right]\left[\frac{1}{2}\bigvee_{a}^{b}\left(f\right)+\frac{1}{2}\left|\bigvee_{a}^{x}\left(f\right)-\bigvee_{x}^{b}\left(f\right)\right|\right]
\end{cases}
\end{aligned} \tag{58}$$

and

$$\begin{aligned}
&\Big|\breve{I}_{g,a+,b-}^{\alpha}f\left(x\right)\\
&\quad-\frac{1}{2\Gamma\left(\alpha+1\right)}\left[\frac{f\left(x\right)+f\left(b\right)}{2}\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}+\frac{f\left(a\right)+f\left(x\right)}{2}\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}\right]\Bigg|\\
&\qquad\leq\frac{1}{4\Gamma\left(\alpha+1\right)}\left[\left(g\left(x\right)-g\left(a\right)\right)^{\alpha}\bigvee_{a}^{x}\left(f\right)+\left(g\left(b\right)-g\left(x\right)\right)^{\alpha}\bigvee_{x}^{b}\left(f\right)\right]\\
&\qquad\leq\frac{1}{4\Gamma\left(\alpha+1\right)}
\end{aligned}$$

$$\times\begin{cases}\left[\frac{g(b)-g(a)}{2}+\left|g(x)-\frac{g(b)+g(a)}{2}\right|\right]^{\alpha}\bigvee_a^b(f);\\ \\ \left[(g(b)-g(x))^{p\alpha}+(g(x)-g(a))^{p\alpha}\right]^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \left[(g(b)-g(x))^{\alpha}+(g(x)-g(a))^{\alpha}\right]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases} \tag{59}$$

for $x\in(a,b)$.

From (48) and (49) we get

$$\left|I_{g,a+,b-}^{\alpha}f(M_g(a,b))-\frac{(g(b)-g(a))^{\alpha}}{2^{\alpha+1}\Gamma(\alpha+1)}\left[f(M_g(a,b))+\frac{f(a)+f(b)}{2}\right]\right| \leq\frac{1}{2^{\alpha+2}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\bigvee_a^b(f) \tag{60}$$

and

$$\left|\breve{I}_{g,a+,b-}^{\alpha}f(M_g(a,b))-\frac{(g(b)-g(a))^{\alpha}}{2^{\alpha+1}\Gamma(\alpha+1)}\left[f(M_g(a,b))+\frac{f(a)+f(b)}{2}\right]\right| \leq\frac{1}{2^{\alpha+2}\Gamma(\alpha+1)}(g(b)-g(a))^{\alpha}\bigvee_a^b(f). \tag{61}$$

4. Example for an Exponential Kernel

For α, $\beta\in\mathbb{R}$ we consider the kernel $k(t):=\exp[(\alpha+\beta i)t]$, $t\in\mathbb{R}$. We have

$$K(t)=\frac{\exp[(\alpha+\beta i)t]-1}{(\alpha+\beta i)},\ \text{if } t\in\mathbb{R}$$

for α, $\beta\neq 0$.

Also, we have

$$|k(s)|:=|\exp[(\alpha+\beta i)s]|=\exp(\alpha s)\ \text{ for } s\in\mathbb{R}$$

and

$$\mathbf{K}(t)=\int_0^t\exp(\alpha s)\,ds=\frac{\exp(\alpha t)-1}{\alpha}\ \text{if } 0<t,$$

for $\alpha\neq 0$.

Let $f:[a,b]\to\mathbb{C}$ be a function of bounded variation on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). We consider the operator

$$\mathcal{H}_{g,a+,b-}^{\alpha+\beta i}f(x):=\frac{1}{2}\int_a^x\exp[(\alpha+\beta i)(g(x)-g(t))]g'(t)f(t)\,dt \tag{62}$$

$$+\frac{1}{2}\int_x^b \exp\left[(\alpha+\beta i)\left(g(t)-g(x)\right)\right] g'(t) f(t)\,dt$$

for $x\in(a,b)$.

If $g=\ln h$ where $h:[a,b]\to(0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we can consider the following operator as well

$$\begin{aligned}&\kappa_{h,a+,b-}^{\alpha+\beta i} f(x) \quad (63)\\ &:=\mathcal{H}_{\ln h,a+,b-}^{\alpha+\beta i} f(x)\\ &=\frac{1}{2}\left[\int_a^x \left(\frac{h(x)}{h(t)}\right)^{\alpha+\beta i}\frac{h'(t)}{h(t)} f(t)\,dt+\int_x^b \left(\frac{h(t)}{h(x)}\right)^{\alpha+\beta i}\frac{h'(t)}{h(t)} f(t)\,dt\right],\end{aligned}$$

for $x\in(a,b)$.

Using the inequality (44) we have for $x\in(a,b)$

$$\begin{aligned}&\left|\mathcal{H}_{g,a+,b-}^{\alpha+\beta i} f(x)-\frac{1}{2}\frac{f(x)+f(b)}{2}\frac{\exp\left[(\alpha+\beta i)\left(g(b)-g(x)\right)\right]-1}{(\alpha+\beta i)}\right.\\ &\qquad\left.-\frac{f(a)+f(x)}{2}\frac{\exp\left[(\alpha+\beta i)\left(g(x)-g(a)\right)\right]-1}{(\alpha+\beta i)}\right|\\ &\le\frac{1}{4}\left[\frac{\exp\left(\alpha\left(g(x)-g(a)\right)\right)-1}{\alpha}\bigvee_a^x(f)+\frac{\exp\left(\alpha\left(g(b)-g(x)\right)\right)-1}{\alpha}\bigvee_x^b(f)\right]\\ &\le\frac{1}{4}\begin{cases}\max\left\{\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha},\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\right\}\bigvee_a^b(f);\\ \\ \left[\left(\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right)^p+\left(\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\right)^p\right]^{1/p}\\ \times\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \left[\frac{\exp(\alpha(g(x)-g(a)))+\exp(\alpha(g(b)-g(x)))-2}{\alpha}\right]\\ \times\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]\end{cases}\quad (64)\end{aligned}$$

and if we take $g=\ln h$ where $h:[a,b]\to(0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we get

$$\left|\kappa_{h,a+,b-}^{\alpha+\beta i} f(x)-\frac{1}{2}\left[\frac{f(x)+f(b)}{2}\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha+\beta i}-1}{(\alpha+\beta i)}\right.\right.$$

$$-\frac{f(a)+f(x)}{2}\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha+\beta i}-1}{(\alpha+\beta i)}\Bigg]\Bigg|$$

$$\leq\frac{1}{4}\left[\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha}\bigvee_a^x(f)+\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\bigvee_x^b(f)\right]$$

$$\leq\frac{1}{4}\begin{cases}\max\left\{\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha},\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\right\}\bigvee_a^b(f);\\ \\ \left[\left(\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha}\right)^p+\left(\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\right)^p\right]^{1/p}\left(\left(\bigvee_a^x(f)\right)^q+\left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\ \\ \left[\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}+\left(\frac{h(b)}{h(x)}\right)^{\alpha}-2}{\alpha}\right]\left[\frac{1}{2}\bigvee_a^b(f)+\frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right].\end{cases}\quad(65)$$

If we take if we take $x_h:=h^{-1}\left(\sqrt{h(a)h(b)}\right)=h^{-1}(G(h(a),h(b)))\in(a,b)$, where G is the geometric mean, then from (65) we get

$$\left|\bar{\kappa}_{h,a+,b-}^{\alpha+\beta i}f-\frac{\left(\frac{h(b)}{h(a)}\right)^{\frac{\alpha+\beta i}{2}}-1}{2(\alpha+\beta i)}\left[f\left(h^{-1}(G(h(a),h(b)))\right)+\frac{f(a)+f(b)}{2}\right]\right|$$
$$\leq\frac{1}{4}\frac{\left(\frac{h(b)}{h(a)}\right)^{\frac{\alpha}{2}}-1}{\alpha}\bigvee_a^b(f),\quad(66)$$

where $\bar{\kappa}_{h,a+,b-}^{\alpha+\beta i}f=\kappa_{h,a+,b-}^{\alpha+\beta i}f(x_h)$.

Let $f:[a,b]\to\mathbb{C}$ be an integrable function on $[a,b]$ and g be a strictly increasing function on (a,b), having a continuous derivative g' on (a,b). Also define

$$\begin{aligned}&\breve{\mathcal{H}}_{g,a+,b-}^{\alpha}f(x)\quad(67)\\&:=\frac{1}{2}\int_x^b\exp[\alpha(g(b)-g(t))]g'(t)f(t)\,dt\\&+\frac{1}{2}\int_a^x\exp[\alpha(g(t)-g(a))]g'(t)f(t)\,dt\end{aligned}$$

for any $x\in(a,b)$.

If $g=\ln h$ where $h:[a,b]\to(0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we can consider the following operator as well

$$
\begin{aligned}
&\breve{\kappa}_{h,a+,b-}^{\alpha} f(x) \qquad (68)\\
&:= \breve{\mathcal{H}}_{\ln h,a+,b-}^{\alpha} f(x)\\
&= \frac{1}{2}\left[\int_x^b \left(\frac{h(b)}{h(t)}\right)^{\alpha} \frac{h'(t)}{h(t)} f(t)\,dt + \int_a^x \left(\frac{h(t)}{h(a)}\right)^{\alpha} \frac{h'(t)}{h(t)} f(t)\,dt\right],
\end{aligned}
$$

for any $x \in (a,b)$.

Using the inequality (45) we have for $x \in (a,b)$ that

$$
\begin{aligned}
&\left|\breve{\mathcal{H}}_{g,a+,b-}^{\alpha+\beta i} f(x) - \frac{1}{2}\left[\frac{f(x)+f(b)}{2}\frac{\exp[(\alpha+\beta i)(g(b)-g(x))]-1}{(\alpha+\beta i)}\right.\right.\\
&\qquad\left.\left. - \frac{f(a)+f(x)}{2}\frac{\exp[(\alpha+\beta i)(g(x)-g(a))]-1}{(\alpha+\beta i)}\right]\right|\\
&\le \frac{1}{4}\left[\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\bigvee_a^x(f) + \frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\bigvee_x^b(f)\right]\\
&\le \frac{1}{4}\begin{cases}
\max\left\{\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}, \frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\right\}\bigvee_a^b(f);\\[2ex]
\left[\left(\frac{\exp(\alpha(g(x)-g(a)))-1}{\alpha}\right)^p + \left(\frac{\exp(\alpha(g(b)-g(x)))-1}{\alpha}\right)^p\right]^{1/p}\\
\times\left(\left(\bigvee_a^x(f)\right)^q + \left(\bigvee_x^b(f)\right)^q\right)^{1/q}\\
\text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1;\\[2ex]
\left[\frac{\exp(\alpha(g(x)-g(a)))+\exp(\alpha(g(b)-g(x)))-2}{\alpha}\right]\\
\times\left[\frac{1}{2}\bigvee_a^b(f) + \frac{1}{2}\left|\bigvee_a^x(f)-\bigvee_x^b(f)\right|\right]
\end{cases} \qquad (69)
\end{aligned}
$$

and if we take $g = \ln h$ where $h : [a,b] \to (0,\infty)$ is a strictly increasing function on (a,b), having a continuous derivative h' on (a,b), then we get

$$
\begin{aligned}
&\left|\breve{\kappa}_{h,a+,b-}^{\alpha+\beta i} f(x) - \frac{1}{2}\left[\frac{f(x)+f(b)}{2}\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha+\beta i}-1}{(\alpha+\beta i)}\right.\right.\\
&\qquad\left.\left. - \frac{f(a)+f(x)}{2}\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha+\beta i}-1}{(\alpha+\beta i)}\right]\right|\\
&\le \frac{1}{4}\left[\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha}\bigvee_a^x(f) + \frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\bigvee_x^b(f)\right]
\end{aligned}
$$

$$\leq \frac{1}{4}\begin{cases} \max\left\{\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha}, \frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\right\}\bigvee_{a}^{b}(f); \\ \left[\left(\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}-1}{\alpha}\right)^{p}+\left(\frac{\left(\frac{h(b)}{h(x)}\right)^{\alpha}-1}{\alpha}\right)^{p}\right]^{1/p}\left(\left(\bigvee_{a}^{x}(f)\right)^{q}+\left(\bigvee_{x}^{b}(f)\right)^{q}\right)^{1/q} \\ \text{with } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1; \\ \left[\frac{\left(\frac{h(x)}{h(a)}\right)^{\alpha}+\left(\frac{h(b)}{h(x)}\right)^{\alpha}-2}{\alpha}\right]\left[\frac{1}{2}\bigvee_{a}^{b}(f)+\frac{1}{2}\left|\bigvee_{a}^{x}(f)-\bigvee_{x}^{b}(f)\right|\right]. \end{cases} \quad (70)$$

If we take if we take $x_h = h^{-1}(G(h(a), h(b))) \in (a, b)$, where G is the geometric mean, then from (65) we get

$$\left|\bar{\ell}_{h,a+,b-}^{\alpha+\beta i} f - \frac{\left(\frac{h(b)}{h(a)}\right)^{\frac{\alpha+\beta i}{2}}-1}{2(\alpha+\beta i)}\left[f\left(h^{-1}(G(h(a), h(b)))\right)+\frac{f(a)+f(b)}{2}\right]\right| \leq \frac{1}{4}\frac{\left(\frac{h(b)}{h(a)}\right)^{\frac{\alpha}{2}}-1}{\alpha}\bigvee_{a}^{b}(f), \quad (71)$$

where $\bar{\ell}_{h,a+,b-}^{\alpha+\beta i} f = \breve{\kappa}_{h,a+,b-}^{\alpha+\beta i} f(x_h)$.

References

[1] Agarwal, R. P., Luo, M.-J. and Raina, R. K., On Ostrowski type inequalities, *Fasc. Math.* **56** (2016), 5-27.

[2] Aljinović, A. Aglić, Montgomery identity and Ostrowski type inequalities for Riemann-Liouville fractional integral. *J. Math.* **2014**, Art. ID 503195, 6 pp.

[3] Apostol, T. M., *Mathematical Analysis,* Second Edition, Addison-Wesley Publishing Company, 1975.

[4] Akdemir, A. O., Inequalities of Ostrowski's type for m- and (α, m)-logarithmically convex functions via Riemann-Liouville fractional integrals. *J. Comput. Anal. Appl.* **16** (2014), no. 2, 375–383

[5] Anastassiou, G. A., Fractional representation formulae under initial conditions and fractional Ostrowski type inequalities. *Demonstr. Math.* **48** (2015), no. 3, 357–378

[6] Anastassiou, G. A., The reduction method in fractional calculus and fractional Ostrowski type inequalities. *Indian J. Math.* **56** (2014), no. 3, 333–357.

[7] Budak, H., Sarikaya, M. Z. and Set, E., Generalized Ostrowski type inequalities for functions whose local fractional derivatives are generalized s-convex in the second sense. *J. Appl. Math. Comput. Mech.* **15** (2016), no. 4, 11–21.

[8] Cerone, P. and Dragomir, S. S., Midpoint-type rules from an inequalities point of view. *Handbook of analytic-computational methods in applied mathematics*, 135–200, Chapman & Hall/CRC, Boca Raton, FL, 2000.

[9] Dragomir, S. S., The Ostrowski's integral inequality for Lipschitzian mappings and applications. *Comput. Math. Appl.* **38** (1999), no. 11-12, 33–37.

[10] Dragomir, S. S., The Ostrowski integral inequality for mappings of bounded variation. *Bull. Austral. Math. Soc.* **60** (1999), No. 3, 495–508.

[11] Dragomir, S. S., On the midpoint quadrature formula for mappings with bounded variation and applications. *Kragujevac J. Math.* **22** (2000), 13–19.

[12] Dragomir, S. S., On the Ostrowski's integral inequality for mappings with bounded variation and applications, *Math. Ineq. Appl.* **4** (2001), No. 1, 59-66. Preprint: *RGMIA Res. Rep. Coll.* **2** (1999), Art. 7, [Online: http://rgmia.org/papers/v2n1/v2n1-7.pdf]

[13] Dragomir, S. S., Refinements of the generalised trapezoid and Ostrowski inequalities for functions of bounded variation. *Arch. Math.* (Basel) **91** (2008), no. 5, 450–460.

[14] Dragomir, S. S., Refinements of the Ostrowski inequality in terms of the cumulative variation and applications, *Analysis* (Berlin) **34** (2014), No. 2, 223–240. Preprint: *RGMIA Res. Rep. Coll.* **16** (2013), Art. 29 [Online:http://rgmia.org/papers/v16/v16a29.pdf].

[15] Dragomir, S. S., Ostrowski type inequalities for Lebesgue integral: a survey of recent results, *Australian J. Math. Anal. Appl.*, Volume **14**, Issue 1, Article 1, pp. 1-287, 2017. [Online http://ajmaa.org/cgi-bin/paper.pl?string=v14n1/V14I1P1.tex].

[16] Dragomir, S. S., Ostrowski type inequalities for Riemann-Liouville fractional integrals of bounded variation, Hölder and Lipschitzian functions, Preprint *RGMIA Res. Rep. Coll.* **20** (2017), Art. 48. [Online http://rgmia.org/papers/v20/v20a48.pdf].

[17] Dragomir, S. S., Ostrowski type inequalities for generalized Riemann-Liouville fractional integrals of functions with bounded variation, *RGMIA Res. Rep. Coll.* **20** (2017), Art. 58. [Online http://rgmia.org/papers/v20/v20a58.pdf].

[18] Dragomir, S. S., Further Ostrowski and trapezoid type inequalities for the generalized Riemann-Liouville fractional integrals of functions with bounded variation, *RGMIA Res. Rep. Coll.* **20** (2017), Art. 84. [Online http://rgmia.org/papers/v20/v20a84.pdf].

[19] Dragomir, S. S., Ostrowski and trapezoid type inequalities for the generalized k-g-fractional integrals of functions with bounded variation, *RGMIA Res. Rep. Coll.* **20** (2017), Art. 111. [Online http://rgmia.org/papers/v20/v20a111.pdf].

[20] Dragomir, S. S., Some inequalities for the generalized k-g-fractional integrals of functions under complex boundedness conditions, *RGMIA Res. Rep. Coll.* **20** (2017), Art. 119. [Online http://rgmia.org/papers/v20/v20a119.pdf].

[21] Guezane-Lakoud, A. and Aissaoui, F., New fractional inequalities of Ostrowski type. *Transylv. J. Math. Mech.* **5** (2013), no. 2, 103–106

[22] Kashuri, A. and Liko, R., Ostrowski type fractional integral inequalities for generalized (s, m, φ)-preinvex functions. *Aust. J. Math. Anal. Appl.* **13** (2016), no. 1, Art. 16, 11 pp.

[23] Kilbas, A., Srivastava, H. M. and Trujillo, J. J., *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006. xvi+523 pp. ISBN: 978-0-444-51832-3; 0-444-51832-0.

[24] Kirane, M. and Torebek, B. T., Hermite-Hadamard, Hermite-Hadamard-Fejer, Dragomir-Agarwal and Pachpatte type Inequalities for convex functions via fractional integrals, Preprint arXiv:1701.00092.

[25] Noor, M. A., Noor, K. I. and Iftikhar, S., Fractional Ostrowski inequalities for harmonic h-preinvex functions. *Facta Univ. Ser. Math. Inform.* **31** (2016), no. 2, 417–445

[26] Raina, R. K., On generalized Wright's hypergeometric functions and fractional calculus operators, *East Asian Math. J.*, **21**(2)(2005), 191-203.

[27] Sarikaya, M. Z. and Filiz, H., Note on the Ostrowski type inequalities for fractional integrals. *Vietnam J. Math.* **42** (2014), no. 2, 187–190

[28] Sarikaya, M. Z. and Budak, H., Generalized Ostrowski type inequalities for local fractional integrals. *Proc. Amer. Math. Soc.* **145** (2017), no. 4, 1527–1538.

[29] Set, E., New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. *Comput. Math. Appl.* 63 (2012), no. 7, 1147–1154.

[30] Tunç, M., On new inequalities for h-convex functions via Riemann-Liouville fractional integration, *Filomat* **27**:4 (2013), 559–565.

[31] Tunç, M., Ostrowski type inequalities for m- and (α, m)-geometrically convex functions via Riemann-Louville fractional integrals. *Afr. Mat.* **27** (2016), no. 5-6, 841–850.

[32] Yildirim, H. and Kirtay, Z., Ostrowski inequality for generalized fractional integral and related inequalities, *Malaya J. Mat.*, **2**(3)(2014), 322-329.

[33] Yildiz, C., Özdemir, E and Muhamet, Z. S., New generalizations of Ostrowski-like type inequalities for fractional integrals. *Kyungpook Math. J.* **56** (2016), no. 1, 161–172.

[34] Yue, H., Ostrowski inequality for fractional integrals and related fractional inequalities. *Transylv. J. Math. Mech.* **5** (2013), no. 1, 85–89.

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