INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 12 NO. 2 PAGE 144–156 (2019)

On a Class of Submanifolds in a Tangent Bundle with a g-Natural Metric - Normal lift

Stanisław Ewert-Krzemieniewski*

(Communicated by Murat Tosun)

ABSTRACT

An isometric immersion of a Riemannian manifold M into a Riemannian manifold N gives rise in a natural way to variety of immersions into the tangent bundle TN with a non-degenerate g-natural metric G. In the paper we introduce and study an immersion into TN defined by the immersion $f : M \longrightarrow N$ itself and the normal bundle.

Keywords: *Riemannian manifold; tangent bundle; g-natural metric; submanifold; isometric immersion; totally geodesic distribution; non-degenerate metric.*
AMS Subject Classification (2010): *Primary: 53B20; 53C07; Secondary: 53B21; 55C25.*

1. Introduction

An isometric immersion of a Riemannian manifold M into a Riemannian manifold N gives rise in a natural way to variety of immersions into the tangent bundle TN with a non-degenerate g-natural metric G. The isometric immersion defined by the tangent bundle of the submanifold was introduced by the author in [11], [12]. In the present paper we introduce and study an immersion $\widetilde{f} : LM \longrightarrow TN$ defined by the immersion $f : M \longrightarrow N$ itself and the normal bundle.

In Preliminaries we recall basic facts on the decomposition of the tangent bundle and g-natural metrics. We also present basic notions on submanifolds and give short resumé on van der Waerden-Bortolotti covariant derivative. In Section 3 basic equations are presented. The main results are given in Section 4. We give the condition sufficient for LM being totally geodesic submanifold of TN.

Throughout the paper all manifolds under consideration are Hausdorff and smooth. The metrics on the base manifolds are Riemannian and the metrics on tangent spaces are non-degenerate. We adopt the Einstein summation convention.

2. Preliminaries

2.1. *Decomposition of the tangent space*

Let $\pi : TN \longrightarrow N$ be the tangent bundle of a Riemannian manifold N with the Levi-Civita connection ∇ on N, π being the projection. Then at each point $(x, u) \in TN$ the tangent space $T_{(x,u)}TN$ splits into direct sum of two isomorphic spaces $V_{(x,u)}TN$ and $H_{(x,u)}TN$, where

$$V_{(x,u)}TN = Ker(d\pi|_{(x,u)}), \quad H_{(x,u)}TN = Ker(K|_{(x,u)})$$

and K is the connection map [7], see also [15].

More precisely, if $Z = \left(Z^r \frac{\partial}{\partial x^r} + \overline{Z}^r \frac{\partial}{\partial u^r}\right)|_{(x,u)} \in T_{(x,u)}TN,\ r = 1, ..., n,$ then the vertical and horizontal projections of Z on T_xN are given by

$$(d\pi)_{(x,u)} Z = Z^r \frac{\partial}{\partial x^r}|_x, \quad K_{(x,u)}(Z) = \left(\overline{Z}^r + u^s Z^t \Gamma^r_{st}\right) \frac{\partial}{\partial x^r}|_x,$$

Received : *23-April-2018,* ***Accepted :*** *21-February-2019*
**** Corresponding author***

where Γ^r_{st} are components of the Levi-Civita connection on N.

On the other hand, to each vector field X on N there correspond uniquely determined vector fields X^v and X^h on TN such that

$$\begin{aligned} d\pi|_{(x,u)}(X^v) &= 0, \quad K|_{(x,u)}(X^v) = X, \\ K|_{(x,u)}(X^h) &= 0, \quad d\pi|_{(x,u)}(X^h) = X. \end{aligned}$$

X^v and X^h are called the vertical lift and the horizontal lift of a given X to TN, respectively.

In local coordinates $((x^r),(u^r))$, $r = 1,..,n$, on TN, the horizontal and vertical lifts of a vector field $X = X^r \frac{\partial}{\partial x^r}$ on N to TN are vector fields given respectively by

$$X^h = X^r \frac{\partial}{\partial x^r} - u^s X^t \Gamma^r_{st} \frac{\partial}{\partial u^r}, \quad X^v = X^r \frac{\partial}{\partial u^r}. \tag{2.1}$$

Recall that for a given isometric immersion $f : M \longrightarrow N$, we have two tangent bundles $\pi_N : TN \longrightarrow N$ and $\pi_M : TM \longrightarrow M$, where the latter is the subbundle of the former. Let M, N be two Riemannian manifolds with metrics g_M and g_N and Levi-Civita connections ∇_M and ∇_N respectively. Then T_pTM and T_pTN have at a common point p their own decompositions into vertical and horizontal parts, i.e.

$$T_pTM = V_pTM \oplus H_pTM = V_M \oplus H_M$$

and

$$T_pTN = V_pTN \oplus H_pTN = V_N \oplus H_N,$$

but neither $V_M \subset V_N$ nor $H_M \subset H_N$ need to hold along TM. See also [15]

Remark also that totally geodesic submanifolds of tangent bundle with g-natural metric are also studied in [1] and [10].

2.2. *Preliminaries on g-natural metrics*

In [13] the class of g-natural metrics was defined. We have

Lemma 2.1. *([13], [2], [3]) Let (M, g) be a Riemannian manifold and G be a g-natural metric on TM. There exist functions $a_j, b_j :< 0, \infty) \longrightarrow R$, $j = 1, 2, 3$, such that for every $X, Y, u \in T_xM$*

$$G_{(x,u)}(X^h, Y^h) = (a_1 + a_3)(r^2) g_x(X, Y) + (b_1 + b_3)(r^2) g_x(X, u) g_x(Y, u),$$
$$G_{(x,u)}(X^h, Y^v) = G_{(x,u)}(X^v, Y^h) = a_2(r^2) g_x(X, Y) + b_2(r^2) g_x(X, u) g_x(Y, u),$$
$$G_{(x,u)}(X^v, Y^v) = a_1(r^2) g_x(X, Y) + b_1(r^2) g_x(X, u) g_x(Y, u),$$

where $r^2 = g_x(u, u)$. For $\dim M = 1$ the same holds for $b_j = 0$, $j = 1, 2, 3$.

Setting $a_1 = 1$, $a_2 = a_3 = b_j = 0$ we obtain the Sasaki metric, while setting $a_1 = b_1 = \frac{1}{1+r^2}$, $a_2 = b_2 = 0 = 0$, $a_1 + a_3 = 1$, $b_1 + b_3 = 1$ we get the Cheeger-Gromoll one.

Following [2] we put

1. $a(t) = a_1(t)\left(a_1(t) + a_3(t)\right) - a_2^2(t)$,
2. $F_j(t) = a_j(t) + t b_j(t)$,
3. $F(t) = F_1(t)\left[F_1(t) + F_3(t)\right] - F_2^2(t)$
   for all $t \in< 0, \infty)$.

We shall often abbreviate: $A = a_1 + a_3$, $B = b_1 + b_3$.

Lemma 2.2. *([2], Proposition 2.7) The necessary and sufficient conditions for a g-natural metric G on the tangent bundle of a Riemannian manifold (M, g) to be non-degenerate are*

$$a(t) \neq 0, \quad F(t) \neq 0$$

for all $t \in< 0, \infty)$. If $\dim M = 1$, this is equivalent to $a(t) \neq 0$ for all $t \in< 0, \infty)$.

Moreover, (TM, G) is Riemannian one if and only if

$$a(t) > 0, \quad F(t) > 0, \quad a_1(t) > 0, \quad F_1(t) > 0$$

hold for all $t \in< 0, \infty)$.

We also have

Proposition 2.1. *([4], [5]) Let (N, g) be a Riemannian manifold, ∇ its Levi-Civita connection and R its Riemann curvature tensor. If G is a non-degenerate g-natural metric on TN, then the Levi-Civita connection $\widetilde{\nabla}$ of (TN, G) is given at a point $(x, u) \in TN$ by*

$$\left(\widetilde{\nabla}_{X^h} Y^h\right)_{(x,u)} = (\nabla_X Y)^h_{(x,u)} + h\left\{\mathbf{A}(u, X_x, Y_x)\right\} + v\left\{\mathbf{B}(u, X_x, Y_x)\right\},$$

$$\left(\widetilde{\nabla}_{X^h} Y^v\right)_{(x,u)} = (\nabla_X Y)^v_{(x,u)} + h\left\{\mathbf{C}(u, X_x, Y_x)\right\} + v\left\{\mathbf{D}(u, X_x, Y_x)\right\},$$

$$\left(\widetilde{\nabla}_{X^v} Y^h\right)_{(x,u)} = h\left\{\mathbf{C}(u, Y_x, X_x)\right\} + v\left\{\mathbf{D}(u, Y_x, X_x)\right\},$$

$$\left(\widetilde{\nabla}_{X^v} Y^v\right)_{(x,u)} = h\left\{\mathbf{E}(u, X_x, Y_x)\right\} + v\left\{\mathbf{F}(u, X_x, Y_x)\right\},$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$ are some F-tensors defined on the product $TN \otimes TN \otimes TN$.

Remark 2.1. Expressions for $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$, $\mathbf{E}$, $\mathbf{F}$ were presented for the first time in the original papers ([2], [3]). Unfortunately, they contain some misprints and omissions. Therefore, for the correct form, we refer the reader to ([4], [5]), see also ([8], [9]).

2.3. *Submanifolds*

Let M be a manifold isometrically immersed in a pseudo-Riemannian manifold N with metric g. Denote by $\widetilde{\nabla}$ and ∇ the Levi-Civita connections of the metric g on N and that of the induced metric on M and by $D^\perp$ the connection induced in the normal bundle $T^\perp M$. Then the Gauss and Weingarten equations

$$\widetilde{\nabla}_X Y = \nabla_X Y + H(X, Y),$$

$$\widetilde{\nabla}_X \eta = -A_\eta X + D^\perp_X Y,$$

hold for all vectors fields X, Y tangent to M and all vector fields η normal to M. Here $H(X, Y)$ is the second fundamental form which is symmetric and takes values in $T^\perp M$ while $A_\eta X$ is the shape operator taking values in TM. It is well known that A_η and H are related by

$$g(A_\eta X, Y) = g(\eta, H(X, Y)).$$

M is said to be totally geodesic if $H(X, Y) = 0$ for all $X, Y \in TM$.

For the local immersion $x^r = x^r(y^a)$, $r = 1, ..., n$, $a = 1, ..., m$, the components of the Levi-Civita connection ∇ of the induced metric $g_{ab} = g_{rs} B^r_a B^s_b$, $B^r_a = \frac{\partial x^r}{\partial y^a}$, are

$$\Gamma^c_{ab} = \left[B^r_{a.b} + \Gamma^r_{st} B^s_a B^t_b\right] B^c_r, \quad B^c_r = g^{cd} B^t_d g_{tr},$$

where the dot denotes partial derivative with respect to y^b.

Similarly, the components of the connection $D^\perp$ are

$$\Gamma^x_{ay} = \left[N^r_{y.a} + \Gamma^r_{st} B^s_a N^t_y\right] N^x_r, \quad N^x_r = g^{xy} N^t_y g_{tr},$$

where $\eta_z = N^r_z \frac{\partial}{\partial x^r}$, $z = m+1, ..., n$ are unit vector fields normal to M.

2.3.1. *Van der Waerden-Bortolotti covariant derivative* Van der Waerden-Bortolotti covariant derivative $\overline{\nabla}$ is a covariant differentiation of tensor fields of mixed types defined along a submanifold M isometrically immersed in a pseudo-Riemannian manifold (N, g) and can be considered as a direct sum $\widetilde{\nabla} \oplus \nabla \oplus \nabla^\perp$ of the Levi-Civita connections of the metric g on N, the one induced on M and of the metric induced in normal bundle $T^\perp M$.

If $\widetilde{X}$, X and η are vector fields, respectively, tangent to N, tangent to M, normal to M and $\widetilde{X}^*$, X^*, η^* are respective 1-forms, then,

$$\begin{aligned}(\overline{\nabla}_Y T)\left(\widetilde{X}, X, \eta, \widetilde{X}^*, X^*, \eta^*\right) = Y\left(T\left(\widetilde{X}, X, \eta, \widetilde{X}^*, X^*, \eta^*\right)\right) - \\ T\left(\widetilde{\nabla}_Y \widetilde{X}, X, \eta, \widetilde{X}^*, X^*, \eta^*\right) - T\left(\widetilde{X}, X, \eta, \widetilde{\nabla}_Y \widetilde{X}^*, X^*, \eta^*\right) - \\ T\left(\widetilde{X}, \nabla_Y X, \eta, \widetilde{X}^*, X^*, \eta^*\right) - T\left(\widetilde{X}, X, \eta, \widetilde{X}^*, \nabla_Y X^*, \eta^*\right) - \\ T\left(\widetilde{X}, X, \nabla^{\perp}_Y \eta, \widetilde{X}^*, X^*, \eta^*\right) - T\left(\widetilde{X}, X, \eta, \widetilde{X}^*, X^*, \nabla^{\perp}_Y \eta^*\right)\end{aligned}$$

for any vector field Y tangent to M and tensor field T of mixed type $(3,3)$.

Let $x^k = x^k(y^a)$ be the local expression of the immersion, $B^k_a = \frac{\partial x^k}{\partial y^a}$. Let $\eta_x = N^r_x \frac{\partial}{\partial x^r}$, $x = m+1, ..., n$, be an orthonormal set of vectors normal to M.

For the local coordinate vector fields $\frac{\partial}{\partial x^k}$ tangent to N, $\frac{\partial}{\partial y^a}$ tangent to M, $\frac{\partial}{\partial v^x}$ normal to M and the respective 1-forms dx^k, dy^a, dv^x, denote by Γ^l_{hk}, Γ^c_{ab}, Γ^z_{ay} components of the connections $\widetilde{\nabla}$, ∇ and $\nabla^{\perp}$. If

$$T = T^{kby}_{hax} \frac{\partial}{\partial x^k} \otimes dx^h \otimes \frac{\partial}{\partial y^b} \otimes dy^a \otimes \frac{\partial}{\partial v^y} \otimes dv^x,$$

then

$$\begin{aligned}\nabla_c T^{kby}_{hax} = \partial_b T^{kby}_{hax} - \Gamma^s_{hr} B^r_c T^{kby}_{sax} + \Gamma^k_{rs} B^r_c T^{sby}_{hax} - \\ \Gamma^d_{ca} T^{kby}_{hdx} + \Gamma^b_{cd} T^{kdy}_{hax} - \Gamma^z_{cx} T^{kby}_{haz} + \Gamma^y_{cz} T^{kbz}_{hax},\end{aligned}$$

where $h,k,r,s = 1,...,n$, $a,b,c,d = 1,...,m$, $m < n$, and $x,y,z = m+1,...,n$.

In particular, $\overline{\nabla}_a B^r_b$ and $\overline{\nabla}_a N^r_x$ give rise to the components of the second fundamental form and the shape operator:

$$\begin{aligned}\overline{\nabla}_b B^r_a \partial_r &= \left(B^r_{a.b} + \Gamma^r_{st} B^s_a B^t_b - \Gamma^c_{ab} B^r_c\right)\partial_r = h^z_{ab} N^r_z \partial_r, \\ \overline{\nabla}_a N^r_x \partial_r &= \left(\partial_a N^r_x + \Gamma^r_{st} B^s_a N^t_x - \Gamma^y_{ax} N^r_y\right)\partial_r. \qquad (2.2)\end{aligned}$$

In a free of coordinate notation we have respectively:

$$\overline{\nabla}_X Y = \widetilde{\nabla}_X Y - \nabla_X Y,$$

$$\overline{\nabla}_X \eta = \widetilde{\nabla}_X \eta - D^{\perp}_X \eta.$$

See also [14] and [16].

2.3.2. Isometric immersion defined by normal bundle Let $f : M \longrightarrow N$ be an isometric immersion of a Riemannian manifold M into a Riemannian manifold N. Suppose that the following diagram commute

$$\begin{array}{ccc} (\pi_N^{-1}(U), (x^r, u^r))\ TN & \longleftarrow - \quad \widetilde{f} \quad -- & \left(\left(\pi^{\perp}_M\right)^{-1}(V), (y^a, v^a)\right)\ T^{\perp}M \\ \big\downarrow \pi_N & & \big\downarrow \pi^{\perp}_M \\ (U, (x^r))\ N & \longleftarrow - \quad f \quad -- & (V, (y^a))\ M \end{array},$$

where $(U, (x^r))$ and $(V, (y^a))$ are coordinate neighbourhoods on N and M respectively, while the local expression for f is:

$$f : x^r = x^r(y^a).$$

Let

$$\eta_x = N^r_x \frac{\partial}{\partial x^r}, \qquad x = m+1, ..., n$$

be a set of orthonormal vectors normal to M.

The coordinate neighbourhoods on TN and normal bundle $T^{\perp}M$ are defined respectively by

$$((x^r),(u^r)), \qquad r=1,...,n,$$

$$((y^a),(v^x)), \qquad x=m+1,...,n, \quad a=1,...,m,$$

where $(u^r)_{r=1,\dots,n}$, are components of the vector u tangent to N at a point with coordinates (x^r) and $(v^x)_{x=m+1,\dots,n}$ are components of the vector normal to M at a point $x^r=x^r(y^a)$.

If

$$f: M \longrightarrow N; \qquad (y^a)\mapsto x^r(y^a)$$

then

$$\widetilde{f}: x^r = x^r(y^a), \quad u^r = v^x N^r_x \tag{2.3}$$

defines locally an immersion into TN.

2.3.3. Vectors tangent to LM The coordinate vector fields tangent to $LM=\widetilde{f}(TM^{\perp})$ are

$$\frac{\partial}{\partial v^x} = N^r_x\frac{\partial}{\partial u^r} = (\eta_x)^v,$$

$$\frac{\partial}{\partial y^a} = B^r_a\frac{\partial}{\partial x^r} + v^z\partial_a N^r_z\frac{\partial}{\partial u^r} \overset{(2.2)}{=} B^r_a\left(\frac{\partial}{\partial x^r} - v^z N^t_z\Gamma^s_{tr}\frac{\partial}{\partial u^s}\right) + v^z\overline{\nabla}_a N^r_z\frac{\partial}{\partial u^r} + v^z\Gamma^y_{az}N^r_y\frac{\partial}{\partial u^r} \overset{(2.3),(2.1)}{=} \left(\frac{\delta}{\delta y^a}\right)^h + M^r_a\left(\frac{\partial}{\partial x^r}\right)^v + N^r_a\left(\frac{\partial}{\partial x^r}\right)^v = \left(\frac{\delta}{\delta y^a}\right)^h + (M_a)^v + (N_a)^v,$$

where $M_a = v^z\overline{\nabla}_a N^r_z\frac{\partial}{\partial x^r} = v^z M_{az}$ are tangent to M and $N_a = v^z\Gamma^y_{az}N^r_y\frac{\partial}{\partial x^r} = N^y_a N^r_y\frac{\partial}{\partial x^r}$ are normal to M.

Along M we also have

$$\nabla_{\delta_a}u = \nabla_{\delta_a}(v^y\eta_y) = \nabla_{\delta_a}\left(v^y N^r_y\frac{\partial}{\partial x^r}\right) = M_a + N_a.$$

3. Basic equations

In this section we derive, using the equations of Gauss and Weingarten and the formulas for the Levi-Civita connection on (TN,G) with a non-degenerate g-natural metric G, the basic equations for the immersion given by (2.3) to be used throughout the paper. H^h+V^v is a unique decomposition of a vector field normal to LM into its horizontal and vertical parts, where H and V are vector fields along M, not necessary tangent to M. ∇ and $\widetilde{\nabla}$ denote the Levi-Civita connections of the metric g and g-natural non-degenerate metric G, respectively. $\widetilde{H}$ is the second fundamental form of the immersion (2.3). Finally, R stands for the Riemann curvature tensor of g. The computations in this section were performed and checked with Mathematica software. In virtue of Proposition 4.1, the pairs of equations in each subsection must satisfy

$$G(\widetilde{H}(\partial_x,\partial_y), H^h+V^v) - G(\widetilde{A}_{H^h+V^v}\partial_x,\partial_y) = 0. \tag{3.1}$$

It also can be used to verify the correctness of computations.

Equation 1

$$
\begin{aligned}
G(\widetilde{\nabla}_{\partial_x}\partial_y, H^h + V^v) = G(\widetilde{H}(\partial_x, \partial_y), H^h + V^v) &= \\
G(\widetilde{\nabla}_{\eta_x^v}\eta_y^v, H^h + V^v) &= \\
G(h\{\mathbf{E}(u,\eta_x,\eta_y)\} + v\{\mathbf{F}(u,\eta_x,\eta_y)\}, H^h + V^v) &=
\end{aligned}
$$

$$
\begin{aligned}
& b_2 g(\eta_x,\eta_y)g(u,H) + (b_1 - a_1')g(\eta_x,\eta_y)g(u,V) + \\
& a_1' g(\eta_x, V)g(u,\eta_y) + a_1' g(\eta_y, V) g(u,\eta_x) + \\
& \left(a_2' + \frac{b_2}{2}\right)(g(\eta_x, H)g(u,\eta_y) + g(\eta_y, H)g(u,\eta_x)) + \\
& g(u,\eta_x)g(u,\eta_y)g(u, b_1'V + 2b_2'H). \qquad (3.2)
\end{aligned}
$$

$$
\begin{aligned}
& G(\widetilde{\nabla}_{\partial_x}(H^h + V^v), \partial_y) = G(-\widetilde{A}_{H^h+V^v}\partial_x, \partial_y) = \\
& G(h\{\mathbf{C}(u,H,\eta_x)\} + v\{\mathbf{D}(u,H,\eta_x)\} + h\{\mathbf{E}(u,\eta_x,V)\} + v\{\mathbf{F}(u,\eta_x,V)\}, \eta_y^v) =
\end{aligned}
$$

$$
\begin{aligned}
& (b_1 - a_1')g(\eta_x, V)g(u,\eta_y) + a_1' g(\eta_y, V)g(u,\eta_x) + a_1' g(\eta_x,\eta_y)g(u,V) + \\
& \left(a_2' - \frac{b_2}{2}\right)(g(\eta_y, H)g(u,\eta_x) - g(\eta_x, H)g(u,\eta_y)) + \\
& b_1' g(u,\eta_x)g(u,\eta_y)g(u,V).
\end{aligned}
$$

In virtue of the equality (3.1) the above two equations yield

$$g(u,\eta_x)g(\eta_y,T) - g(u,\eta_y)g(\eta_x,T) = 0,$$

where $T = (b_1' - 2a_1')V + (b_2' - 2a_2')H.$

Equation 2

$$
\begin{aligned}
G(\widetilde{\nabla}_{\partial_x}\partial_a, H^h + V^v) = G(\widetilde{H}(\partial_x, \partial_a), H^h + V^v) &= \\
G(\widetilde{\nabla}_{\eta_x^v}(\delta_a^h + M_a^v + N_a^v), H^h + V^v) &= \\
G(h\{\mathbf{C}(u,\delta_a,\eta_x)\} + v\{\mathbf{D}(u,\delta_a,\eta_x)\}, H^h + V^v) &+ \\
G(h\{\mathbf{E}(u,\eta_x, M_a + N_a)\} + v\{\mathbf{F}(u,\eta_x, M_a + N_a)\}, H^h + V^v) &=
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{2}a_1 R(H,\delta_a,u,\eta_x) + A' g(H,\delta_a)g(u,\eta_x) + \left(a_2' - \frac{b_2}{2}\right) g(V,\delta_a)g(u,\eta_x) + \\
& (b_1 - a_1')g(u,V)g(N_a,\eta_x) + b_2 g(u,H)g(N_a,\eta_x) + \\
& a_1' g(u,\eta_x)g(V, M_a + N_a) + a_1' g(u, N_a)g(V,\eta_x) + \\
& \left(a_2' + \frac{b_2}{2}\right)[g(H, M_a + N_a)g(u,\eta_x) + g(H,\eta_x)g(u,N_a)] + \\
& g(u, b_1'V + 2b_2'H)g(u,\eta_x)g(u,N_a). \qquad (3.3)
\end{aligned}
$$

$$
\begin{aligned}
G(\widetilde{\nabla}_{\partial_x}(H^h + V^v), \partial_a) = G(-\widetilde{A}_{H^h+V^v}\partial_x, \partial_a) &= \\
G(\widetilde{\nabla}_{\partial_x}(H^h + V^v), \delta_a^h + M_a^v + N_a^v) &= \\
G(h\{\mathbf{C}(u,H,\eta_x)\} + v\{\mathbf{D}(u,H,\eta_x)\}, \delta_a^h + M_a^v + N_a^v) &+ \\
G(h\{\mathbf{E}(u,\eta_x,V)\} + v\{\mathbf{F}(u,\eta_x,V)\}, \delta_a^h + M_a^v + N_a^v) &=
\end{aligned}
$$

$$\frac{1}{2}a_1R(H,\delta_a,u,\eta_x)+(b_1-a_1')g(u,N_a)g(V,\eta_x)+a_1'g(u,\eta_x)g(V,M_a+N_a)+ a_1'g(u,V)g(N_a,\eta_x)+\left(a_2'+\frac{b_2}{2}\right)g(u,\eta_x)g(V,\delta_a)+ \left(a_2'-\frac{b_2}{2}\right)[g(H,M_a+N_a)g(u,\eta_x)-g(H,\eta_x)g(u,N_a)]+ A'g(H,\delta_a)g(u,\eta_x)+b_1'g(u,\eta_x)g(u,N_a)g(u,V). \tag{3.4}$$

Hence, in virtue of (3.1), we get

$$g(u,b_1V+b_2H)g(N_a,\eta_x)+2g(u,\eta_x)g(A'H+a_2'V,\delta_a)+ 2g(u,\eta_x)g(a_1'V+a_2'H,M_a+N_a)+ g(u,N_a)\,[g(b_1V+b_2H,\eta_x)+2g(u,\eta_x)g(b_1'V+b_2'H,u)]=0.$$

Equation 3

$$G(\widetilde{\nabla}_{\partial_a}\partial_x,H^h+V^v)=G(\widetilde{H}(\partial_a,\partial_x),H^h+V^v)= G(\widetilde{\nabla}_{(\delta_a^h+M_a^v+N_a^v)}(\eta_x)^v,H^h+V^v)=$$

$$G((\nabla_{\delta_a}\eta_x)^v+h\{\mathbf{C}(u,\delta_a,\eta_x)\}+v\{\mathbf{D}(u,\delta_a,\eta_x)\},H^h+V^v)+ G(h\{\mathbf{E}(u,\eta_x,M_a+N_a)\}+v\{\mathbf{F}(u,\eta_x,M_a+N_a)\},H^h+V^v).$$

Since $\widetilde{H}(\partial_a,\partial_x)$ is symmetric, comparing the last equation with (3.3), we obtain

$$G\left((\nabla_{\delta_a}\eta_x)^v,H^h+V^v\right)=0. \tag{3.5}$$

$$G(\widetilde{\nabla}_{\partial_a}(H^h+V^v),\partial_x)=G(-\widetilde{A}_{H^h+V^v}\partial_a,\partial_x)= G(\widetilde{\nabla}_{(\delta_a^h+M_a^v+N_a^v)}(H^h+V^v),\eta_x^v)= G((\nabla_{\delta_a}H)^h+h\{\mathbf{A}(u,\delta_a,H)\}+v\{\mathbf{B}(u,\delta_a,H)\},\eta_x^v)+ G((\nabla_{\delta_a}V)^v+h\{\mathbf{C}(u,\delta_a,V)\}+v\{\mathbf{D}(u,\delta_a,V)\},\eta_x^v)+ G(h\{\mathbf{C}(u,H,M_a+N_a)\}+v\{\mathbf{D}(u,H,M_a+N_a)\},\eta_x^v)+ G(h\{\mathbf{E}(u,M_a+N_a,V)\}+v\{\mathbf{F}(u,M_a+N_a,V)\},\eta_x^v)=$$

$$\frac{1}{2}a_1R(H,\delta_a,u,\eta_x)+a_1g(\eta_x,\nabla_{\delta_a}V)+b_1g(u,\eta_x)g(u,\nabla_{\delta_a}V)+ a_2g(\eta_x,\nabla_{\delta_a}H)+b_2g(u,\eta_x)g(u,\nabla_{\delta_a}H)-A'g(u,\eta_x)g(H,\delta_a)+ (b_1-a_1')g(u,\eta_x)g(V,M_a+N_a)+a_1'g(u,N_a)g(V,\eta_x)+a_1'g(u,V)g(N_a,\eta_x)+ \left(a_2'-\frac{b_2}{2}\right)[g(H,\eta_x)g(u,N_a)-g(H,M_a+N_a)g(u,\eta_x)-g(u,\eta_x)g(V,\delta_a)]+ b_1'g(u,V)g(u,N_a)g(u,\eta_x). \tag{3.6}$$

Equation 4 Body Math

$$G(\widetilde{\nabla}_{\partial_a}\partial_b,H^h+V^v)=G(\widetilde{H}(\partial_a,\partial_b),H^h+V^v)= G(\widetilde{\nabla}_{(\delta_a^h+M_a^v+N_a^v)}(\delta_b^h+M_b^v+N_b^v),H^h+V^v)= G((\nabla_{\delta_a}\delta_b)^h+h\left\{\mathbf{A}(u,\delta_a,\delta_b)\right\}+v\left\{\mathbf{B}(u,\delta_a,\delta_b)\right\}+ (\nabla_{\delta_a}(M_b+N_b))^v+h\left\{\mathbf{C}(u,\delta_a,M_b+N_b)\right\}+v\left\{\mathbf{D}(u,\delta_a,M_b+N_b)\right\}+ h\left\{\mathbf{C}(u,\delta_b,M_a+N_a)\right\}+v\left\{\mathbf{D}(u,\delta_b,M_a+N_a)\right\}+ h\left\{\mathbf{E}(u,M_a+N_a,M_b+N_b)\right\}+v\left\{\mathbf{F}(u,M_a+N_a,M_b+N_b)\right\},H^h+V^v)=$$

$$
\begin{aligned}
&- a_2 R(H,\delta_b,u,\delta_a)- \\
&\frac{1}{2}a_1R(H,\delta_a,u,M_b+N_b)-\frac{1}{2}a_1R(H,\delta_b,u,M_a+N_a)-\frac{1}{2}a_1R(u,V,\delta_a,\delta_b)+ \\
&g(AH+a_2V,\nabla_{\delta_a}\delta_b)+g(BH+b_2V,u)g(u,\nabla_{\delta_a}\delta_b)+ \\
&\frac{1}{2}Bg(H,u)\left(g(M_a,\delta_b)+g(M_b,\delta_a)\right)+ \\
&g(a_1V+a_2H,\nabla_{\delta_a}(M_b+N_b))+g(b_2H+b_1V,u)g(u,\nabla_{\delta_a}(M_b+N_b))+ \\
&+g(b_2H+(b_1-a_1')V,u)(g(M_a,M_b)+g(N_a,N_b))+ \\
&A'\left[g(H,\delta_a)g(u,N_b)+g(H,\delta_b)g(u,N_a)-g(V,u)g(\delta_a,\delta_b)\right]+ \\
&a_1'g(u,N_a)g(V,M_b+N_b)+a_1'g(u,N_b)g(V,M_a+N_a)+ \\
&\left(a_2'-\frac{b_2}{2}\right)\left\{g(u,N_b)g(V,\delta_a)+g(u,N_a)g(V,\delta_b)-g(u,V)\left[g(M_b,\delta_a)+g(M_a,\delta_b)\right]\right\}+ \\
&\left(a_2'+\frac{b_2}{2}\right)\left(g(H,M_a+N_a)g(u,N_b)+g(H,M_b+N_b)g(u,N_a)\right)+ \\
&g(u,b_1'V+2b_2'H)g(u,N_a)g(u,N_b).
\end{aligned}
\tag{3.7}
$$

Body Math

$$
\begin{aligned}
&G(\widetilde{\nabla}_{\partial_a}(H^h+V^v),\partial_b)=G(-\widetilde{A}_{H^h+V^v}\partial_a,\partial_b)= \\
&G(\widetilde{\nabla}_{(\delta_a^h+M_a^v+N_a^v)}(H^h+V^v),\eta_b^v)= \\
&a_2R(H,\delta_b,u,\delta_a)+ \\
&\frac{1}{2}a_1R(H,\delta_a,u,M_b+N_b)+\frac{1}{2}a_1R(H,\delta_b,u,M_a+N_a)+\frac{1}{2}a_1R(u,V,\delta_a,\delta_b)+ \\
&Ag(\delta_b,\nabla_{\delta_a}H)+a_2g(\delta_b,\nabla_{\delta_a}V)+a_2g(M_b+N_b,\nabla_{\delta_a}H)+a_1g(M_b+N_b,\nabla_{\delta_a}V)+ \\
&\frac{1}{2}Bg(H,u)\left(g(M_a,\delta_b)-g(M_b,\delta_a)\right)+ \\
&g(u,N_b)(b_2g(u,\nabla_{\delta_a}H)+b_1g(u,\nabla_{\delta_a}V))+ \\
&A'(-g(H,\delta_a)g(u,N_b)+g(H,\delta_b)g(u,N_a)+g(V,u)g(\delta_a,\delta_b))+ \\
&(b_1-a_1')g(u,N_b)g(V,M_a+N_a)+a_1'g(u,N_a)g(V,M_b+N_b)+ \\
&a_1'g(V,u)\left[g(M_a,M_b)+g(N_a,N_b)\right]+ \\
&\left(a_2'-\frac{b_2}{2}\right)(g(u,V)g(M_b,\delta_a)-g(u,N_b)g(V,\delta_a))+ \\
&\left(a_2'+\frac{b_2}{2}\right)(g(u,V)g(M_a,\delta_b)+g(u,N_a)g(V,\delta_b))+ \\
&\left(a_2'-\frac{b_2}{2}\right)\left[g(-H,M_a+N_a)g(u,N_b)+g(H,M_b+N_b)g(u,N_a)\right]+ \\
&b_1'g(u,V)g(u,N_a)g(u,N_b).
\end{aligned}
$$

Applying (3.1), we find

$$
\begin{aligned}
&0=g(AH+a_2V,\nabla_{\delta_a}\delta_b)+ \\
&g(BH+b_2V,u)g(u,\nabla_{\delta_a}\delta_b)+g(a_2H+a_1V,\nabla_{\delta_a}(M_b+N_b))+ \\
&a_2g(M_b+N_b,\nabla_{\delta_a}H)+a_1g(M_b+N_b,\nabla_{\delta_a}V)+ \\
&Ag(\delta_b,\nabla_{\delta_a}H)+a_2g(\delta_b,\nabla_{\delta_a}V)+g(BH+b_2V,u)g(M_a,\delta_b)+ \\
&g(u,b_2H+b_1V)g(u,\nabla_{\delta_a}(M_b+N_b)+ \\
&g(u,N_b)\left[b_1g(u,\nabla_{\delta_a}V)+b_2g(u,\nabla_{\delta_a}H)+g(b_2H+b_1V,M_a+N_a)\right]+ \\
&g(u,b_2H+b_1V)\left[g(M_a,M_b)+g(N_a,N_b)\right]+2g(u,N_a)\times \\
&\left[g(A'H+a_2'V,\delta_b)+g(a_2'H+a_1'V,M_b+N_b)+g(u,N_b)g(b_2'H+b_1'V,u)\right].
\end{aligned}
$$

4. Main results

The first proposition of this section establishes a number of various relations that allow us to show that the right hand sides of the pairs of equations in each subsection of the former section satisfy (3.1). The results are presented in Proposition 4.1. Theorem 4.1 states the condition sufficient for the space normal to LM being spanned by lifts of vectors tangent to M. The main results are presented in Theorems 4.2 and 4.3.

Proposition 4.1. *Let $\widetilde{f}$ be the immersion given by (2.3) defined by the isometric immersion $f : M \to (N, g)$ into a Riemannian manifold. Suppose, moreover, that TN is endowed with non-degenerate g-natural metric G. Then in the notation as above the following identities are satisfied.*

1.
$$g(\eta_x, S) = 0, \tag{4.1}$$
where $S = a_2H + a_1V + g(u, b_2H + b_1V)u$.

2.
$$g(u, S) = g(N_a, S) = 0. \tag{4.2}$$

3.
$$g(\nabla_{\delta_a}\eta_x, S) = 0. \tag{4.3}$$

4.
$$g(\eta_x, \nabla_{\delta_a}S) = 0. \tag{4.4}$$

5.
$$g(u, \nabla_{\delta_a}S) = g(N_b, \nabla_{\delta_a}S) = 0. \tag{4.5}$$

6.
$$g(\nabla_{\delta_a}u, S) = g(\nabla_{\delta_a}N_b, S) = 0. \tag{4.6}$$

7.
$$g(\delta_a, AH + a_2V) = g(M_a, a_2H + a_1V) = 0. \tag{4.7}$$

8.
$$g(\delta_a, A'H + a_2'V) = g(M_a, a_2'H + a_1'V) = 0. \tag{4.8}$$

9.
$$g(M_b, M_a) + g(u, \nabla_{\delta_a}M_b) = 0. \tag{4.9}$$

10.
$$g(M_b, \delta_a) + g(u, \nabla_{\delta_b}\delta_a) = 0. \tag{4.10}$$

11.
$$g(\nabla_{\delta_a}\eta_x, \delta_b) + g(\eta_x, \nabla_{\delta_a}\delta_b) = 0. \tag{4.11}$$

Moreover, if M is not a hypersurface of N, then

12.
$$X_u = g(u, b_2H + b_1V) = 0. \tag{4.12}$$

13.
$$X_{\eta_x} = g(\eta_x, b_2H + b_1V) = 0. \tag{4.13}$$

14.
$$Y_{\eta_x} = g(\eta_x, b_2'H + b_1'V) = 0. \tag{4.14}$$

15.
$$Y_u = g(u, b_2'H + b_1'V) = 0. \tag{4.15}$$

16.

$$Z_{\eta_x} = g(\eta_x, a_2'H + a_1'V) = 0, \quad Z_u = g(u, a_2'H + a_1'V) = 0. \tag{4.16}$$

17.

$$g(\eta_x, a_2H + a_1V) = 0. \tag{4.17}$$

Finaly

18.

$$S = a_2H + a_1V.$$

Proof. (4.1) results from

$$G(\partial_x, H^h + V^v) = G(\eta_x^v, H^h + V^v) = 0.$$

Then (4.2) is obvious since $u = v^y N_y^r \partial_r = v^y \eta_y$ and $N_a = N_a^y \eta_y$ are normal to M. Now (4.3) is a consequence of (3.5), whence, by (4.1), (4.4) results.

Once again, by orthogonality of u and N_a with respect to M, we have (4.5). Consequently, in virtue of (4.2), we obtain (4.6).

Observe that the identity

$$(3.4) - (3.6) + g(\eta_x, \nabla_{\delta_a} S) - \sum_y \frac{\partial}{\partial v^y} g(\eta_x, S) N_a^y = 0 \tag{4.18}$$

gives

$$g(\delta_a, A'H + a_2'V) + g(M_a, a_2'H + a_1'V) = 0.$$

On the other hand, relations

$$G(\partial_a, H^h + V^v) = G(\delta_a^h + M_a^v + N_a^v, H^h + V^v) = 0$$

and (4.2) yield

$$g(\delta_a, AH + a_2V) + g(M_a, a_2H + a_1V) = 0. \tag{4.19}$$

Differentiating (4.19) with respect to v^x and using (4.18) we find

$$g(M_{ax}, a_2H + a_1V) = 0,$$

where $M_{ax} = \nabla_{\delta_a} N_x^r \partial_r$. Consequently, (4.19) yields (4.7). Hence, by differentiating with respect to v^x, (4.8) results.

Since M_a, δ_a are tangent to M and u, η_x are normal, by covariant differentiation of $g(u, M_a) = 0$, $g(u, \delta_a) = 0$, $g(\eta_x, \delta_a) = 0$ we get (4.9) - (4.11).

Differentiating (4.1) with respect to v^y we get

$$g(\eta_x, \eta_y) X_u + 2g(\eta_x, u) g(u, \eta_y) Y_u + g(\eta_x, u) X_{\eta_y} + 2g(u, \eta_y) Z_{\eta_x} = 0,$$

where $X_u = g(u, b_2H + b_1V)$, $X_{\eta_x} = g(\eta_x, b_2H + b_1V)$, $Y_u = g(u, b_2'H + b_1'V)$, $Z_u = g(u, a_2'H + a_1'V)$ and $Z_{\eta_x} = g(\eta_x, a_2'H + a_1'V)$.

Transvecting in turn with v^x, v^y, $v^x v^y$ and, finally, contracting wit g^{xy} we get for each $x = m+1, ..., n$ a system of four equations:

$$\begin{aligned} v_x X_u + 2r^2 v_x Y_u + r^2 X_{\eta_x} + 2v_x Z_u &= 0, \\ v_x X_u + 2r^2 v_x Y_u + v_x X_u + 2r^2 Z_{\eta_x} &= 0, \\ r^2 (X_u + r^2 Y_u + Z_u) &= 0, \\ (n - m + 1) X_u + 2r^2 Y_u + 2Z_u &= 0, \end{aligned} \tag{4.20}$$

where $v_x = g(u, \eta_x)$. Solving it with respect to X, X_{η_x}, Y, Z_{η_x} we obtain

$$X_u = X_{\eta_x} = 0, \quad Y_u = -\frac{Z_u}{r^2}, \quad Z_{\eta_x} = -\frac{v_x Z_u}{r^2} \tag{4.21}$$

for any $u = v^x \eta_x \neq 0$. By continuity, $X_u = X_{\eta_x} = 0$ hold for any u. Then $\frac{\partial}{\partial v^z} X_{\eta_x} = Y_{\eta_x} g(\eta_z, u) = 0$ for all $u \neq 0$, whence, in virtue of continuity, $Y_{\eta_x} = 0$ for any u. Consequently, we have $Y_u = 0$. Now, $Z_u = 0$ follows from (4.20) and $Z_{\eta_x} = 0$ results from (4.21). Finally, (4.17) is a consequence of (4.1) and (4.12). Thus the lemma is proved. □

Theorem 4.1. *Let (x, u) be a point of LM immersed in TN. If $codimM > 1$ and*

$$a_2b_1 - a_1b_2 \neq 0 \tag{4.22}$$

at $t = g(u, u)$, then the normal space at $(x, u) \in LM$ is spanned by lifts of vectors tangent to M.

Proof. It results from the identities (4.17) and (4.13). Note that other conditions, similar to that of (4.22) can be deduced in the same way from (4.17), (4.16), (4.13) and (4.14). □

We shall prove that the conditions $codimM > 1$ and (4.22) are essential in that sense that there exist immersion $f : M \longrightarrow N$, M being a hypersurface of N, and a metric G on TN satisfying $a_2b_1 - a_1b_2 = 0$ such that the normal component of at least one of the vectors H, V does not vanish.

Example 4.1. Let $f : S^1 \longrightarrow \left(R^2, Euclid\ metric\right)$ be the immersion given by $f(t) = [\cos t, \sin t]$. The vector tangent to S^1 is $\mathbf{s} = [-\sin t, \cos t]$ and the normal one is $\mathbf{n} = [\cos t, \sin t]$. Then

$$\mathbf{s}^v = [0, 0, -\sin t, \cos t], \qquad \mathbf{s}^h = [-\sin t, \cos t, 0, 0],$$
$$\mathbf{n}^v = [0, 0, \cos t, \sin t], \qquad \mathbf{n}^h = [\cos t, \sin t, 0, 0].$$

The vectors tangent to $L\left(S^1\right)$ are

$$\frac{\partial}{\partial t} = \mathbf{s}^h + v\mathbf{s}^v, \quad \frac{\partial}{\partial v} = \mathbf{n}^v.$$

Suppose

$$H^h + V^v = \alpha \mathbf{s}^h + \beta \mathbf{n}^h + \gamma \mathbf{s}^v + \delta \mathbf{n}^v$$

and consider a non-degenerate g-natural metric on TR^2 such that $B = b_1 = b_2 = 0$.

Then

$$G(\frac{\partial}{\partial v}, H^h + V^v) = G(\mathbf{n}^v, H^h + V^v) = a_2 g(\mathbf{n}, H) + a_1 g(\mathbf{n}, V) = a_2\beta + a_1\delta$$

and

$$G(\frac{\partial}{\partial t}, H^h + V^v) = G(\mathbf{s}^h, H^h + V^v) = Ag(\mathbf{s}, H) + a_2 g(\mathbf{s}, V) = A\alpha + a_2\gamma.$$

We put

$$\alpha = -\frac{a_2 + va_1}{A + va_2}\gamma \neq 0, \qquad \beta = -\frac{a_1}{a_2}\delta \neq 0.$$

The restriction on codimension can be omitted as the next proposition shows.

Proposition 4.2. *If $a_2 = b_2 = 0$ for all $t \in <0, \infty)$, then the normal bundle of LM at a point (x, u), $x \in M$, $u \in T_xM$, is spanned by the vectors η_y^h and $\delta_a^v - \frac{a_1}{A}(\nabla_{\delta_a} u)^h$.*

Proof. Direct calculation. Property (4.10) is applied. □

Applying Proposition 4.1 to (3.2), (3.3) and (3.7) we obtain

Theorem 4.2. *Let M, $codimM > 1$, be a submanifold isometrically immersed in a manifold N. Then along LM we have:*

1.

$$\widetilde{G}\left[H^h + V^v, \widetilde{\nabla}_{\partial_x}\partial_y\right] = \widetilde{G}\left[H^h + V^v, \widetilde{H}(\partial_x, \partial_y)\right] =$$
$$g\left[H, \frac{b_2}{2} g\left(u, \eta_x\right)\eta_y + \frac{b_2}{2} g\left(u, \eta_y\right)\eta_x + a_2' g\left(\eta_x, \eta_y\right) u + b_2' g\left(u, \eta_x\right) g\left(u, \eta_y\right) u\right],$$

2.

$$\widetilde{G}\left[H^h + V^v, \widetilde{\nabla}_{\partial_x}\partial_b\right] = \widetilde{G}\left[H^h + V^v, \widetilde{H}(\partial_x, \partial_a)\right] =$$
$$g\left[H, \frac{1}{2} a_1 R(u, \eta_x)\delta_a\right] +$$
$$g\left[H, \frac{b_2}{2} g(u, N_a)\eta_x + \frac{b_2}{2} g(u, \eta_x)(M_a + N_a) + a_2' g(N_a, \eta_x) u + b_2' g\left(u, \eta_x\right) g\left(u, N_a\right) u\right] -$$
$$g\left[V, \frac{b_2}{2} g(u, \eta_x)\delta_a\right],$$

3.

$$\widetilde{G}\left[H^h+V^v,\widetilde{\nabla}_{\partial_a}\partial_b\right]=\widetilde{G}\left[H^h+V^v,\widetilde{H}(\partial_a,\partial_b)\right]=$$
$$g[AH+a_2V,\nabla_{\delta_a}\delta_b]-a_2R(H,\delta_b,u,\delta_a)-$$
$$\frac{a_1}{2}\left[R(H,\delta_a,u,M_b+N_b)+R(H,\delta_b,u,M_a+N_a)+R(u,V,\delta_a,\delta_b\right]+$$
$$g\left[a_2H+a_1V,\nabla_{\delta_a}(M_b+N_b)\right]-$$
$$g(u,V)\left[A'g(\delta_a,\delta_b)+a_1'\left(g(M_a,M_b)+g(N_a,N_b)\right)+2a_2g(M_a,\delta_b)\right]-$$
$$\frac{1}{2}b_2\left[g(u,N_a)g(V,\delta_b)+g(u,N_b)g(V,\delta_a)\right]+$$
$$g\left[H,\frac{b_2}{2}g(u,N_a)\left(M_b+N_b\right)+\frac{b_2}{2}g(u,N_b)\left(M_a+N_a\right)\right]+$$
$$b_2'g(u,H)g(u,N_a)g(u,N_b)$$

for a nondegenerate g-natural metric $\widetilde{G}$.

Proof. Straightforward computation. □

Remark 4.1. The third equation of the last Proposition can be written as

$$\widetilde{G}\left[H^h+V^v,\widetilde{\nabla}_{\partial_a}\partial_b\right]=\widetilde{G}\left[H^h+V^v,\widetilde{H}(\partial_a,\partial_b)\right]=$$
$$G[H^h+V^v,(\nabla_{\delta_a}\delta_b)^h+(\nabla_{\delta_a}\nabla_{\delta_b}u)^v]-g(BH+b_2V,u)g(u,\nabla_{\delta_a}\nabla_{\delta_b}u)-$$
$$a_2R(H,\delta_b,u,\delta_a)-\frac{a_1}{2}\left[R(H,\delta_a,u,M_b+N_b)+R(H,\delta_b,u,M_a+N_a)+R(u,V,\delta_a,\delta_b\right]+$$
$$g(u,V)\left[A'g(\delta_a,\delta_b)+a_1'\left(g(M_a,M_b)+g(N_a,N_b)\right)+2a_2g(M_a,\delta_b)\right]-$$
$$\frac{1}{2}b_2\left[g(u,N_a)g(V,\delta_b)+g(u,N_b)g(V,\delta_a)\right]+$$
$$g\left[H,\frac{b_2}{2}g(u,N_a)\left(M_b+N_b\right)+\frac{b_2}{2}g(u,N_b)\left(M_a+N_a\right)\right]+$$
$$b_2'g(u,H)g(u,N_a)g(u,N_b).$$

Definition 4.1. A distribution D on a manifold M is said to be totally geodesic if it is invariant with respect to covariant differentiation, i.e. $\nabla_XY\in D$ for all $X,Y\in D$.

Theorem 4.3. *Let M, $codimM>1$, be a submanifold isometrically immersed in a manifold (N,g). Suppose that LM is a submanifold isometrically immersed by (2.3) in TN with non-degenerate g-natural metric G.*

1. *If either the normal bundle of LM is spanned by vectors of the form H^h+V^v, where H and V are tangent to M or $b_2=a_2'=0$ along M, then vector fields $\{\partial_x\}$, $x=m+1,...,n$ define on LM the totally geodesic distribution that is involutive.*
2. *If*

   (a) $a_1=0$, $b_2=0$, $a_2=const\neq 0$ *along M or*

   (b) N is a space of constant curvature and $a_2'=0$, $b_2=0$ along M,

   then LM is mixed totally geodesic.
   Here, along M,
$$\nabla_{\delta_a}u=\nabla_{\delta_a}(v^y\eta_y)=\nabla_{\delta_a}\left(v^yN_y^r\frac{\partial}{\partial x^r}\right)=M_a+N_a.$$

Proof. In virtue of the assumptions, the first equation of Theorem 4.2 yields $\widetilde{G}\left[H^h+V^v,\widetilde{H}(\partial_x,\partial_y)\right]=0$. Hence $\widetilde{\nabla}_{\partial_x}\partial_y$ is tangent to LM. Since $\partial_x=\eta_x^v$ are vertical vector fields, the distribution is involutive. This proves the first point. The proof of the second one is obvious. □

References

[1] Abbassi, M. T. K., Yampolsky, A., Transverse totally geodesic submanifolds of the tangent bundle. *Publ. Math. Debrecen* 64/1-2 (2004), 129-154.
[2] Abbassi, M. T. K., Sarih, Maâti, On natural metrics on tangent bundles of Riemannian manifolds. *Arch. Math. (Brno)* 41 (2005), no. 1, 71–92.
[3] Abbassi, M. T. K., Sarih, Maâti, On some hereditary properties of Riemannian g-natural metrics on tangent bundles of Riemannian manifolds. *Differential Geom. Appl.* 22 (2005), no. 1, 19–47.
[4] Abbassi, M. T. K., Métriques Naturelles Riemanniennes sur la Fibré tangent une variété Riemannienne, Editions Universitaires Européénnes, Saarbrücken, Germany, 2012.
[5] Degla, S., Ezin, J. P., Todjihounde, L., On g-natural metrics of constant sectional curvature on tangent bundles. *Int. Electronic J. Geom.* 2(1) (2009), p. 74-94.
[6] Deshmukh S., Al-Odan, H., Shaman, T. A., Tangent bundle of the hypersurfaces in a Euclidean space. *Acta Math. Acad. Pedagog. Nyíregyháziensis* 23(1) (2007),71-87.
[7] Dombrowski, P., On the Geometry of Tangent Bundle. *J. Reine Angew. Math.* 210 (1962), p. 73-88.
[8] Ewert-Krzemieniewski, S., On a classification of Killing vector fields on a tangent bundle with g-natural metric. arXiv:1305:3817v1.
[9] Ewert-Krzemieniewski, S., On a Killing vector fields on a tangent bundle with g-natural metric Part I. *Note Mat.* 34 no. 2, (2014), 107-133.
[10] Ewert-Krzemieniewski, S., Totally geodesic submanifolds in tangent bundle with g-natural metric. *Int. J. Geom. Methods Mod. Phys.* 11 (2014), no. 9, 1460033 (9 pages).
[11] Ewert-Krzemieniewski, S., On a class of submanifolds in tangent bundle with g-natural metric. arXiv:1411.3274.
[12] Ewert-Krzemieniewski, S., On a class of submanifolds in a tangent bundle with a g-natural metric. *Coll. Math.*, 150 no.1, (2017), 121-133.
[13] Kowalski, O., Sekizawa, M., Natural transformations of Riemannian metrics on manifolds to metrics on tangent bundles, A classification. *Bull. Tokyo Gakugei Univ.* (4) 40 (1988), 1–29.
[14] Yano, K., Kon, M., Structures on Manifolds. World Scientific, 1984.
[15] Yano, K., Ishihara, S., Tangent and cotangent bundles. Marcel Dekker, Inc. New York, 1973.
[16] Yano, K., Submanifolds with parallel mean curvature vector. *J. Diff. Geom.* 6 (1971), 95-118.

Affiliations

STANISŁAW EWERT-KRZEMIENIEWSKI
ADDRESS: West Pomeranian University of Technology Szczecin, School of Mathematics
Al. Piastów 17, 70-310 Szczecin, Poland
E-MAIL: ewert@zut.edu.pl
ORCID ID:0000-0003-0644-6268