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On the Geometry of Some (α, β)-Metrics on the Nilpotent Groups $H(p, r)$

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ABSTRACT

In this paper we study the Riemann-Finsler geometry of the Lie groups $H(p, r)$ which are a generalization of the Heisenberg Lie groups. For a certain Riemannian metric $\langle \cdot, \cdot \rangle$, the Levi-Civita connection and the sectional curvature are given. We classify all left invariant Randers metrics of Douglas type induced by $\langle \cdot, \cdot \rangle$, compute their flag curvatures and show that all of them are non-Berwaldian.

Keywords: *(α, β)-metric; nilpotent Lie group; left invariant Finsler metric; left invariant Riemannian metric; curvature.*

AMS Subject Classification (2010): *Primary: 53C60 ; Secondary: 53B21; 22E25.*

1. Introduction

The Riemannian geometry of the Heisenberg Lie group H_{2p+1}, equipped with a certain left invariant Riemannian metric $\mathbf{a}$, is very important in the study of contact geometry of a special Pfaff equation $\omega = 0$. In fact the group of transformations preserving the codimension 1 distribution $\ker(\omega)$ (the group of contact transformations) coincides on the isometry group of the Riemannian manifold $(H_{2p+1}, \mathbf{a})$ (for more details see [10]). In [9], Goze and Haraguchi defined the notion of $r-$contact system (also see [10]). The Lie groups $H(p, r)$, that generalize the Heisenberg Lie groups H_{2p+1}, are examples of Lie groups that admit a left-invariant $r-$contact system (see [9] and [10]). The Riemannian geometry of $(H(p, r), \mathbf{a})$, where $\mathbf{a}$ is a special left invariant Riemannian metric whose its isometry group preserves the distribution associated to the $r-$contact system, have been studied by Piu and Goze in [10]. In this paper we use the Riemannian metric $\mathbf{a}$ for construction of left invariant $(\alpha, \beta)-$metrics on $H(p, r)$.
(α, β)-metrics establish a rich family of interesting Finsler metrics. These metrics have many applications in physics. In fact some of famous (α, β)-metrics such as Randers metric, Matsumoto metric and Kropina metric were introduced because of their physical applications (see [2]). Let $(M, \mathbf{a})$ be a Riemannian manifold and β be a 1-form on M. Suppose that $\alpha(x, y) = \sqrt{\mathbf{a}(y, y)}$, and $\phi : (-b_0, b_0) \longrightarrow \mathbb{R}^+$ is a smooth map such that

$$\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, \qquad |s| \leq b < b_0. \tag{1.1}$$

If $\|\beta\|_\alpha < b_0$ then the function $F = \alpha\phi(\frac{\beta}{\alpha})$ is a Finsler metric on M which is called a (α, β)-metric (see [5]).
If we put $\phi(s) = 1 + s$, $\phi(s) = \frac{1}{1-s}$ or $\phi(s) = \frac{1}{s}$, then we obtain three important families of (α, β)-metrics respectively called Randers metrics $F = \alpha + \beta$, Matsumoto metrics $F = \frac{\alpha^2}{\alpha - \beta}$ and Kropina metrics $F = \frac{\alpha^2}{\beta}$.
It is easily to see that for an arbitrary 1-form β on a Riemannian manifold $(M, \mathbf{a})$ there exists a unique vector field X on M such that

$$\mathbf{a}(y, X(x)) = \beta(x, y) \text{ for every } x \in M, y \in T_xM. \tag{1.2}$$

This notation is very useful for constructing left invariant (α, β)-metrics on Lie groups. Let G be a Lie group and e be its unit element. Suppose that, for any $x \in G$, l_x denotes the left translation. Then a Finsler metric F on G is called left invariant if

$$F(x, y) = F(e, dl_{x^{-1}}y) \text{ for every } x \in G, y \in T_xG. \tag{1.3}$$

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So in the definition of a (α,β)-metric on a Lie group G if we consider $\mathbf{a}$ is a left invariant Riemannian metric and X is a left invariant vector field on G such that $\|X\|_\alpha = \mathbf{a}(X,X) < b_0$, then the (α,β)-metric is left invariant (see [6] and [7]). In a special case, if $\mathbf{a}$ is a left invariant Riemannian metric and X is a left invariant vector field on a Lie group G such that $\mathbf{a}(X,X) < 1$, then the function

$$F(x,y) = \sqrt{\mathbf{a}\,(y,y)} + \mathbf{a}\,(X\,(x)\,,y)\,, \tag{1.4}$$

is a left invariant Randers metric on G.
For a Finsler manifold (M,F), one can define the notion of the flag curvature as a generalization of the sectional curvature in Riemannian geometry by the following formula:

$$\tilde{K}(P,y) = \frac{g_y\left(\tilde{R}_y\,(u)\,,u\right)}{g_y(y,y)g_y(u,u) - g_y^2(y,u)}, \tag{1.5}$$

where $P = span\{u,y\}$ and $g_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial s \partial t}F^2(y+su+tv)\mid_{s=t=0}$ (see [4, 5]).
We mention that in this formula we have used the Riemann curvature tensor of the Finsler manifold (M,F) which is defined by

$$\tilde{R}_y(u) = \tilde{R}(u,y)y = \tilde{\nabla}_u\tilde{\nabla}_y y - \tilde{\nabla}_y\tilde{\nabla}_u y - \tilde{\nabla}_{[u,y]}y, \tag{1.6}$$

where $\tilde{\nabla}$ denotes the Chern connection of the Finsler manifold (M,F) (for more details see [4, 5]).
Two special types of Finsler metrics are Finsler metrics of Berwald type and Finsler metrics of Douglas type. In fact Finsler metrics of Douglas type are a generalization of Finsler metrics of Berwald type (see [3]). For these two definitions we need to define the spray coefficients. The spray coefficients G^i of a Finsler manifold (M,F) are defined as follows:

$$G^i = \frac{1}{4}\sum_{i,l} g^{il}\Big(\sum_m [F^2]_{x^m y^l}y^m - [F^2]_{x^l}\Big), \tag{1.7}$$

where we have used the standard local coordinate system of TM. If the spray coefficients G^i are of the form

$$G^i = \frac{1}{2}\sum_{j,k}\Gamma^i_{jk}(x)y^j y^k + P(x,y)y^i, \tag{1.8}$$

where $P(x,y)$ is a local positively homogeneous function of degree one on TM, then F is called of Douglas type and if $P(x,y) = 0$ then F is called of Berwald type see [3] and [5]).
There exists a criterion to determine (α,β)-metrics of Berwald type. In fact a (α,β)-metric F is of Berwald type if and only if the 1-form β is parallel with respect to the Levi-Civita connection of $\mathbf{a}$ (see [4]).
If a (α,β)-metric F is of Berwald type then the Finsler metric F and the Riemannian metric $\mathbf{a}$ have the same geodesics.
In this article, the Riemann-Finsler geometry of the Lie groups of the form $H(p,r)$ is studied. We give the Levi-Civita connection and the sectional curvature of the Riemannian metric $\langle\cdot,\cdot\rangle$ which is considered in [10]. We classify all left invariant Randers metrics of Douglas type induced by $\langle\cdot,\cdot\rangle$ and compute their flag curvatures. Also we show that all these Douglas metrics are non-Berwaldian.

2. On the Riemannian geometry of the Lie groups $H(p,r)$

In this section we review some preliminaries about generalized Heisenberg Lie groups $H(p,r)$ and investigate the Riemannian geometry of $(H(p,r),\mathbf{a})$ where $\mathbf{a}$ is a special left invariant Riemannian metric.
The generalized Heisenberg Lie group $H(p,r)$ in the sense of [9] is a Lie group of the form

$$H(p,r) = \{(x,y,z)|x \in \mathcal{M}_{1\times p}(\mathbb{R}), y \in \mathcal{M}_{p\times r}(\mathbb{R}), z \in \mathcal{M}_{1\times r}(\mathbb{R})\}, \tag{2.1}$$

endowed with the multiplication

$$(x,y,z)(x',y',z') = (x+x', y+y', z+z' + \frac{1}{2}(xy' - x'y)) \;\; \forall (x,y,z),(x',y',z') \in H(p,r), \tag{2.2}$$

where $\mathcal{M}_{m\times n}(\mathbb{R})$ denotes the set of all $m\times n$ real matrices.
It is shown that the groups $H(p,r)$ are $(rp+r+p)-$dimensional, two-step nilpotent, connected, and simply

connected real Lie groups with $r-$dimensional center $\mathcal{Z}$ isomorphic to the Abelian group $\mathcal{M}_{1\times r}(\mathbb{R})$ (see [9] and [10]). Also it is proven that the Lie group $H(p,r)$ is isomorphic to the Heisenberg group H_{2p+1} if and only if $\dim \mathcal{Z} = r = 1$. Easily for the derived Lie group $H'(p,r)$ we have $H'(p,r) = \mathcal{Z}$. If we use the coordinates (x_α), (y_i^α) and (z_i) for the Lie groups $\mathcal{M}_{1\times p}(\mathbb{R})$, $\mathcal{M}_{p\times r}(\mathbb{R})$ and $\mathcal{M}_{1\times r}(\mathbb{R})$ respectively, where $i = 1, \cdots, r$ and $\alpha = 1, \cdots, p$, then the following left invariant vector fields constitute a basis for the Lie algebra $\mathfrak{H}(\mathfrak{p}, \mathfrak{r})$ of $H(p,r)$,

$$E_\alpha = \frac{\partial}{\partial x_\alpha} - \sum_i \frac{1}{2} y_i^\alpha \frac{\partial}{\partial z_i}, \quad E_{(\alpha,i)} = \frac{\partial}{\partial y_i^\alpha} + \frac{1}{2} x_\alpha \frac{\partial}{\partial z_i}, \quad E_i = \frac{\partial}{\partial z_i}. \tag{2.3}$$

The only non-zero commutator between two elements of the above basis is of the form

$$[E_{(\alpha,i)}, E_\alpha] = -E_i, \quad i = 1, \cdots, r \text{ and } \alpha = 1, \cdots, p. \tag{2.4}$$

So for the derived Lie algebra $\mathfrak{H}(\mathfrak{p}, \mathfrak{r})'$ we have $\mathfrak{H}(\mathfrak{p}, \mathfrak{r})' = \mathtt{span}\{E_i | i = 1, \cdots, r\}$.
Now let **a** be the left invariant Riemannian metric on $H(p,r)$ considered in [10], which is the left invariant Riemannian metric such that the above basis is an orthonormal basis. Suppose that $\langle \cdot, \cdot \rangle$ is the inner product induced by **a** on the Lie algebra $\mathfrak{H}(\mathfrak{p}, \mathfrak{r})$. From now on, for simplicity, we use the same notation $\langle \cdot, \cdot \rangle$ for the left invariant Riemannian metric **a** and its induced inner product on $\mathfrak{H}(\mathfrak{p}, \mathfrak{r})$.

Lemma 2.1. *For $i = 1, \cdots, r$ and $\alpha = 1, \cdots, p$, the Levi-Civita connection ∇ of the Riemannian manifold $(H(p,r), \langle \cdot, \cdot \rangle)$ is given as follows,*

$$\begin{aligned}
\nabla_{E_\alpha} E_i &= \nabla_{E_i} E_\alpha = -\frac{1}{2} E_{(\alpha,i)}, \\
\nabla_{E_i} E_{(\alpha,i)} &= \nabla_{E_{(\alpha,i)}} E_i = \frac{1}{2} E_\alpha, \\
\nabla_{E_{(\alpha,i)}} E_\alpha &= \frac{1}{2} E_i, \\
\nabla_{E_\alpha} E_{(\alpha,i)} &= -\frac{1}{2} E_i,
\end{aligned}$$

and for other $X, Y \in \{E_\alpha, E_{(\alpha,i)}, E_i | i = 1, \cdots, r$ and $\alpha = 1, \cdots, p\}$, we have $\nabla_X Y = 0$.

Proof. Using the Koszul formula

$$2\langle \nabla_X Y, Z \rangle = \langle [Z,X], Y \rangle + \langle [Z,Y], X \rangle + \langle Z, [X,Y] \rangle, \tag{2.5}$$

and the fact that the basis $\{E_\alpha, E_{(\alpha,i)}, E_i | i = 1, \cdots, r$ and $\alpha = 1, \cdots, p\}$ is an orthonormal basis with respect to the inner product $\langle \cdot, \cdot \rangle$, complete the proof. □

Lemma 2.2. *For the curvature tensor R of the Riemannian manifold $(H(p,r), \langle \cdot, \cdot \rangle)$ we have*

$$\begin{aligned}
R(E_i, E_\alpha)E_\alpha &= -R(E_\alpha, E_i)E_\alpha = \frac{1}{4} E_i, \\
R(E_\alpha, E_{(\alpha,i)})E_\alpha &= -R(E_{(\alpha,i)}, E_\alpha)E_\alpha = \frac{3}{4} E_{(\alpha,i)}, \\
R(E_{(\alpha,i)}, E_\alpha)E_{(\alpha,i)} &= -R(E_\alpha, E_{(\alpha,i)})E_{(\alpha,i)} = \frac{3}{4} E_\alpha, \\
R(E_\alpha, E_\beta)E_{(\alpha,i)} &= -R(E_\beta, E_\alpha)E_{(\alpha,i)} = \frac{1}{4} E_{(\beta,i)}, \quad \alpha \neq \beta, \\
R(E_\alpha, E_{(\alpha,i)})E_\beta &= -R(E_{(\alpha,i)}, E_\alpha)E_\beta = \frac{1}{2} E_{(\beta,i)}, \quad \alpha \neq \beta, \\
R(E_{(\beta,i)}, E_\alpha)E_{(\alpha,i)} &= -R(E_\alpha, E_{(\beta,i)})E_{(\alpha,i)} = \frac{1}{4} E_\beta, \quad \alpha \neq \beta,
\end{aligned}$$

and for other cases $R = 0$.

Proof. It is sufficient to use the previous lemma and the formula of curvature tensor which is defined by $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$. □

Now we compute the sectional curvature of the Riemannian manifold $(H(p,r), \langle\cdot,\cdot\rangle)$.

Theorem 2.1. *Suppose that $P = \mathtt{span}\{u,v\}$ is a two-dimensional subspace of the Lie algebra $\mathfrak{H}(\mathfrak{p},\mathfrak{r})$ such that*

$$u = \sum_{\alpha} \lambda_\alpha E_\alpha + \sum_{\alpha,i} \lambda_{(\alpha,i)} E_{(\alpha,i)} + \sum_i \lambda_i E_i,$$

and

$$v = \sum_{\beta} \eta_\beta E_\beta + \sum_{\beta,j} \eta_{(\beta,j)} E_{(\beta,j)} + \sum_j \eta_j E_j,$$

and the set $\{u,v\}$ is an ortonormal basis for P. Then for the sectional curvature $K(P)$ we have,

$$\begin{aligned} K(P) \quad = \quad & \frac{1}{4}\sum_{\alpha,j} \lambda_\alpha^2 \eta_j^2 - \frac{3}{4}\sum_{\alpha,j} \lambda_\alpha^2 \eta_{(\alpha,j)}^2 + \frac{1}{4}\sum_{\alpha,i} \lambda_{(\alpha,i)}^2 \eta_i^2 \\ & - \frac{3}{4}\sum_{\alpha,i} \lambda_{(\alpha,i)}^2 \eta_\alpha^2 + \frac{1}{4}\sum_{\beta,i} \lambda_i^2 \eta_\beta^2 + \frac{1}{4}\sum_{\beta,i} \lambda_i^2 \eta_{(\beta,i)}^2 \\ & + \frac{3}{2}\sum_{\alpha,j} \lambda_\alpha \lambda_{(\alpha,j)} \eta_\alpha \eta_{(\alpha,j)} + \frac{1}{4}\sum_{\alpha,j} \lambda_\alpha \lambda_{(\alpha,j)} \eta_{(\alpha,j)} \eta_j \\ & - \frac{1}{4}\sum_{\alpha,i} \lambda_{(\alpha,i)} \lambda_i \eta_{(\alpha,i)} \eta_i - \frac{1}{2}\sum_{\beta,i,j} \lambda_i \lambda_{(\beta,i)} \eta_{(\beta,j)} \eta_j. \end{aligned}$$

Proof. It is a direct consequence of lemma 2.2 and the sectional curvature formula for Riemannian manifolds. □

3. Some (α,β)-metrics on the Lie groups $H(p,r)$

In this section we study left invariant Randers metrics of Douglas type and (α,β)-metrics of Berwald type on the Lie groups $H(p,r)$ induced by the left invariant Riemannian metric $\langle\cdot,\cdot\rangle$ discussed in the previous section.

Theorem 3.1. *There is not any non-Riemannian $(\alpha,\beta)-$metric of Berwald type on the Lie group $H(p,r)$ induced by the Riemannian metric $\langle\cdot,\cdot\rangle$ and a left invariant vector field X.*

Proof. Let F be an arbitrary $(\alpha,\beta)-$metric defined by the left invariant Riemannian metric $\langle\cdot,\cdot\rangle$ and a left invariant vector field X. It is well known that the $(\alpha,\beta)-$metric F is of Berwald type if and only if the vector field X is parallel with respect to the Levi-Civita connection ∇ of $\langle\cdot,\cdot\rangle$. But lemma 2.1 shows that there is not any non-zero left invariant vector field X such that $\nabla_Y X = 0$, for all $Y \in \mathfrak{H}(\mathfrak{p},\mathfrak{r})$. □

Remark 3.1. This is a generalization of corollary 5.2 of [7] to the Riemannian Lie groups $(H(p,r), \langle\cdot,\cdot\rangle)$.

By attention to the previous theorem we have,

Theorem 3.2. *Let $F(x,y) = \sqrt{\langle y,y\rangle} + \langle X(x),y\rangle$ be a left invariant Randers metric defined by the Riemannian metric $\langle\cdot,\cdot\rangle$ and a left invariant vector field X on $H(p,r)$. F is a non-Berwaldian Douglas metric if and only if $X \in \mathtt{span}\{E_\alpha, E_{(\alpha,i)} | \alpha = 1\cdots p \text{ and } i = 1\cdots r\}$ and $\langle X,X\rangle < 1$.*

Proof. By considering theorem 3.2 of [1], F is of Douglas type if and only if X is orthogonal to $\mathfrak{H}(\mathfrak{p},\mathfrak{r})$ so the proof is completed. □

In the following theorem we give the flag curvature formula of left invariant Randers metrics of Berwald type on $H(p,r)$.

Theorem 3.3. *Let $F(x,y) = \sqrt{\langle y,y\rangle} + \langle X(x),y\rangle$ be a left invariant Randers metric of Douglas type defined by the Riemannian metric $\langle\cdot,\cdot\rangle$ and a left invariant vector field $X = \sum_\beta \mu_\beta E_\beta + \sum_{(\beta,j)} \mu_{(\beta,j)} E_{(\beta,j)}$. Suppose that (P,y) is a flag such that $\{y,v\}$ is an orthonormal basis for P with respect to $\langle\cdot,\cdot\rangle$ and $y = \sum_\alpha \lambda_\alpha E_\alpha + \sum_{(\alpha,i)} \lambda_{(\alpha,i)} E_{(\alpha,i)} + \sum_i \lambda_i E_i$. Then the flag curvature is given by*

$$\tilde{K}(P,y) = \frac{K(P)}{1 + (\langle X,y\rangle)^2} + \frac{3}{4(\langle X,y\rangle)^4}\Big(\sum_{\beta,i} \mu_\beta \lambda_{(\beta,i)} \lambda_i - \sum_{\beta,j} \mu_{(\beta,j)} \lambda_\beta \lambda_j\Big)^2,$$

where $\tilde{K}$ and K denote the flag curvature of F and the sectional curvature of $\langle\cdot,\cdot\rangle$, respectively.

Proof. By attention to the formula 2.3 of [8] for the flag curvature we have

$$\tilde{K}(P,y) = \frac{\langle y,y\rangle}{F(y)^2}K(P) + \frac{1}{4F(y)^4}\Big(3\langle U(y,y),X\rangle^2 - 4F(y)\langle U(y,U(y,y)),X\rangle\Big), \tag{3.1}$$

where $U : \mathfrak{H}(\mathfrak{p},\mathfrak{r}) \times \mathfrak{H}(\mathfrak{p},\mathfrak{r}) \longrightarrow \mathfrak{H}(\mathfrak{p},\mathfrak{r})$ is a symmetric function defined by the following equation,

$$2\langle U(v_1,v_2),v_3\rangle = \langle [v_3,v_1],v_2\rangle + \langle [v_3,v_2],v_1\rangle.$$

Now the equations

$$\langle U(y,y),X\rangle = \sum_{\beta,i}\mu_\beta\lambda_{(\beta,i)}\lambda_i - \sum\mu_{(\beta,j)}\lambda_\beta\lambda_j,$$

and

$$\langle U(y,U(y,y)),X\rangle = 0.$$

together with the formula 3.1 complete the proof. □

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