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Research Article

On Fixed Point Results for Multivalued Operators in b_2 Metric Spaces

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Abstract

The aim of this paper is to establish fixed point theorem for multivalued operators in b_2-metric space and generalize the results of Chifu and Petruşel (2017) and Yihao, Ren, and Zhong (2018). Moreover, an example is given to illustrate the main theorem.

Keywords: Fixed point, b_2 metric space, multi valued operators, Hardy-Rogers contractions.

b_2 Metrik Uzayında Çok Değişkenli Operatörler için Sabit Nokta Sonuçları Üzerine

Öz

Bu çalışmanın amacı, b_2-metrik uzayında çok değişkenli operatörler için sabit nokta teoremi oluşturmak ve Chifu ve Petruşel (2017) ve Yihao, Ren ve Zhong (2018) sonuçlarını genelleştirmektir. Ayrıca, ana teoremi göstermek için bir örnek verilmiştir.

Anahtar Kelimeler: Sabit nokta, b_2 metrik uzayı, çok değişkenli operatörler, Hardy-Rogers daralmaları

1. Introduction

Banach (1922) introduced the Banach contraction principle, a well-known fixed point theorem. 2-metric spaces were first introduced by Gähler (1963), and then many mathematicians proved important theorems in 2-metric spaces. Some of these are: Dung (2013), Alioucheand Simpson (2012). Bakhtin (1989) and Czerwik (1993) presented the representation of b-metric spaces, respectively. Demma (2016) introduced the simulation function representation in b-metric spaces, and Piao (2013)] introduced the b_2 -metric spaces, the generalization of 2-metric spaces and b metric spaces. , Zead Mustafa (2014), Fadail, Ahmad and Ozturk (2015), Cui, Zhao and Zhong (2017), Yihao, Ren, Zhong (2018), Suzuki (2018) are some of the authors who brought new results in the $b_2$metric space.

On the other hand, Hardy and Rogers (1973) gave the generalization of the Reich fixed point theorem. Chifu and Petruşel (2017) obtained results for the valuable Hardy-Rogers contractions in b-metric space. Fixed point theory is a subject studied not only in the

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specified metric spaces, but also in many different metric spaces. (Girgin and Öztürk, 2019).

2. Preliminaries

Definition 2.1 Let X be a nonempty set and let s ≥1 be a given real number. A function $d: X \times X \to \mathbb{R}_+$ is said to be a b-metric space if and only if for all $x, y, z \in X$ the following conditions are satisfied:

1. $d(x,y) = 0 \Leftrightarrow x = y$;
2. $d(x,y) = d(y,x)$;
3. $d(x,y) \leq s[d(x,z) + d(z,y)]$.

In this case, the pair (X,d) is called b-metric space with constant s.

Gähler (1963) proved that in a nonempty set X, the map $d: X \to R$ is called 2-metric, if it satisfies the following condition;

1.If there exists a point $z \in X$, such that $d(x,y,z) \neq 0$,
for every pair of distinct points $x, y \in X$.
2. $d(x,y,z) = 0$, when at least two of $x, y, z \in X$ are the same,
3. For all $x, y, z, t \in X$,
i) $d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x)$,
ii) $d(x,y,z) \leq d(x,y,t) + d(y,z,t) + d(z,x,t)$,
(X,d) is called 2-metric space.

Let
$D(A,B) = \inf\{d(a,b) | a \in A, b \in B\}$
and
$H(a,b) = \max\{\rho(A,B), \rho(B,A)\}$.
$D: P(X) \times P(X) \to \mathbb{R}_+$ is called the gap functional and $H: P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$ is called Pompeiu-Hausdorff generalized functional (Rockafellar, Tyrrell, Roger, 2005). Here

$$\rho: P(X) \times P(X) \to \mathbb{R}_+ \cup \{+\infty\}$$

is the excess generalized functional and defined as

$$\rho(A,B) = supf\{D(a,B) | a \in A\}.$$

Throughout the paper we will use the generalized versions of these functionalities in b_2 metric spaces.

Definition 2.2 [Piao (2013)] Let X be a non-empty set and $s \geq 1$ be a real number and let $d: X \times X \times X \to \mathbb{R}$ be a map satisfying the following conditions;

1) For every pair of distinct points $x, y \in X$, there exists a point $z \in X$ such that $d(x,y,z) \neq 0$.
2) If at least two of x, y, z are the same, then $d(x,y,z) = 0$.
3) The symmetry:

$$d(x,y,z) = d(x,z,y) = d(y,x,z) = d(y,z,x) = d(z,x,y) = d(z,y,x)$$

for all $x, y, z \in X$.
4) The rectangle inequality: $d(x,y,z) \leq s[d(x,y,a) + d(y,z,a) + d(z,x,a)]$
for all $x, y, z, a \in X$.

Then d is called a b_2 metric in X and (X,d) is called a b_2 metric space with parameter s. Obviously, for $s = 1$, b_2 metric reduces to 2-metric.

Definition 2.3 [Suzuki, 2018)] Let x_n be a sequence in b_2 metric space (X,d).

1. A sequence x_n is said to be convergent to $x \in X$, written as $\lim_{n\to\infty} x_n = x$, if for all $a \in X$, $\lim_{n\to\infty} d(x_n, x, a) = 0$.
2. x_n is a Cauchy sequence if and only if $d(x_n, x_m, a) \to 0$, when $n, m \to \infty$.
3. (X,d) is said to be b_2 −complete if every b_2 −Cauchy sequence is a b_2 − convergent sequence.

Lemma 2.4 Let (X,d) be a b_2 − metric space with constant $s \geq 1$ and A, B closed subsets of X, then for all $x, y \in X$ and $a \in A, b \in B$,

1)$D(x,B,y) \leq d(x,b,y)$,

2) $D(a,B,y) \leq H(A,B,y)$,
3) $D(x,A,y) \leq s[d(x,y,z) + D(z,A,y) + D(x,A,z)]$,
4) $D(x,A,y) = 0$ if and only if $x \in \bar{A}$ or $y \in \bar{A}$,
5) For $q > 1$, there exists $b \in B$ such that $d(a,b,x) \leq qH(A,B,x)$.

3. Fixed Point Results

Lemma 3.1 $\forall x \in X$, let us construct a sequence as follows;

$$T(x_n) = x_{n+1}.$$

Then $H(T(x_n),T(x_{n+1}),x_n) = 0$.

Proof: Since

$$\begin{aligned} H(H_n, x_{n+1}, x_{n+2} &= H(T(x_n),T(x_{n+1}),x_n) \\ &\leq \{a_1 d(x_n,x_{n+1},x_n) \\ &+ a_2[D(x_n,T(x_n),x_n) \\ &+ D(x_{n+1},T(x_{n+1}),x_n)]\} \\ &+ a_3[D(x_n,T(x_{n+1}),x_n) \\ &+ D(x_{n+1},T(x_n),x_n)] \end{aligned}$$

using Lemma 2.4,

$$\begin{aligned} &H(T(x_n),T(x_{n+1}),x_n) \\ &\leq (a_2 + a_3 s)H(T(x_n),T(x_{n+1}),x_n) \end{aligned}$$

is obtained. Since $(a_2 + a_3 s) < 1/s$ this is a contradiction, therefore

$$H(T(x_n),T(x_{n+1}),x_n) = 0.$$

Theorem 3.2 Let $T: X \to P(x)$ be a multivalued operator in complete b_2 metric space (X,d), and let

i) there exists $a_1, a_2, a_3 \in \mathbb{R}$, such that $a_1 + a_2 + 2a_3 s < \frac{s-1}{s^2}$ and $a_2 + 2a_3 s < \frac{1}{s}$,
ii) $H(T(x),T(y),a) \leq a_1 d(x,y,a)$

$$\begin{aligned} &+a_2[D(x,T(x),a) \\ &+ D(y,T(y),a)] \\ &+ a_3[D(x,T(y),a) \\ &+ D(y,T(x),a)] \end{aligned}$$

$for\ all\ x,y,a \in X$,
iii) T is closed,
then T has a fixed point.

Proof: First, we will show that the considered sequence is a Cauchy sequence.

$$\begin{aligned} H(T(x_n),T(x_{n+1}),a) &\leq a_1 d(x_n,x_{n+1},a) \\ &+a_2[d(x_n,x_{n+1},a) \\ &+ d(T(x_n),T(x_{n+1}),a)] \\ &+a_3 s[d(x_n,x_{n+1},a) \\ &+ H(x_n,T(x_{n+1}),T(x_n)) \\ &+ H(T(x_{n+1}),a,T(x_n))]. \end{aligned}$$

Thus;

$$(1 - a_2 - a_3 s)H(T(x_n),T(x_{n+1}),a)$$

$$\leq (a_1 + a_2 + a_3 s)d(x_n,x_{n+1},a).$$

Using the conditions,

$$a_1 + a_2 + a_3 s < a_1 + a_2 + 2a_3 s < \frac{s-1}{s^2},$$

and $a_2 + a_3 s < \frac{1}{s}$,

$$\frac{a_1 + a_2 + a_3 s}{a_1 - a_2 - a_3 s} < \frac{1}{s} < 1$$

is obtained. Let $1 < q < \frac{a_1 - a_2 - a_3 s}{s(a_1 + a_2 + a_3)}$, then

$$d(x_{n+1},x_{n+2},a) \leq qH(T(x_n),T(x_{n+1}),a)$$

$$\leq q\frac{a_1 + a_2 + a_3 s}{a_1 - a_2 - a_3 s}d(x_n,x_{n+1},a).$$

Let

$\frac{a_1 + a_2 + a_3 s}{a_1 - a_2 - a_3 s} = \alpha$ and $\alpha < \frac{1}{s} < 1$, for every $a \in X$ and $n = 1, 2, \ldots$.

Continuing the above process for the sequence $(x_n)_{n\in N}$ with $x_n \in T(x_{n-1})$,

$$d(x_{n+1}, x_{n+2}, a) \le \alpha^n d(x_0, x_1, a)$$

is obtained. Hence,

$$\lim_{n\to\infty} d(x_n, x_{n+1}, a) = 0.$$

For arbitrary, $\epsilon > 0, \exists n_0 \in N_+$, when $n \ge n_0$, the following holds.

$$d(x_n, x_{n+1}, a) < \frac{1}{3s}\epsilon < \epsilon, \forall a \in X.$$

Using mathematical induction $\forall m > n > n_0$, $\forall a \in X$ we will prove that $d(x_m, x_n, a) < \epsilon$.

For $m = n + 1$, $d(x_{n+1}, x_n, a) < \epsilon$ holds. Suppose that for $m = n + k$ and $k \ge 1$, then

$d(x_m, x_n, a) < \frac{1}{3s}\epsilon$ also holds.

Now we will prove that it is also true for $m = n + k + 1$.

Since $d(x_m, x_{m+1}, a) < \frac{1}{3s}\epsilon$ and $d(x_n, x_{m+1}, a) < \epsilon$ using rectangle inequality,

$d(x_n, x_{m+1}, a) \le s\big[d(x_m, x_{m+1}, a)\big) + d(x_m, x_n, a) + d(x_m, x_{m+1}, x_n)] < s\left(\frac{1}{3s}\epsilon + \frac{1}{3s}\epsilon + \frac{1}{3s}\epsilon\right) \le \epsilon$

This implies that, the considered sequence $\{x_n\}$, is a Cauchy sequence.

Let

$$\lim_{n\to\infty} x_{n+1} = x.$$

By rectangular inequality, we obtain

$$d(x_{n+2}, x, a) \le s\{d(x_{n+2}, x_{n+1}, a) + d(x_{n+1}, x, a) + d(x_{n+2}, x, x_{n+1})\}.$$

Since $\{x_n\}$ is a Cauchy sequence, for $n \to \infty$ $\lim_{n\to\infty} d(x_{n+2}, x, a) \le$ $\lim_{n\to\infty} s\, d(x_{n+2}, x_{n+1}, a) \le s\frac{1}{3s}\epsilon < \epsilon$,

is provided and $\lim_{n\to\infty} x_{n+2} = x$ is obtained. This indicates that $\{x_n\}$ is convergent and converges to x.

Now we will prove that x is the unique common fixed point for the mapping T.

Let

$H(T(x), T(y), a) \le a_1 d(x, y, a) + a_2[D(x, T(x), a) + D(y, T(y), a)] + a_3[D(x, T(y), a) + D(y, T(x), a)]$.

$$H(T(x), x_{n+2}, a) \le a_1 d(x, x_{n+1}, a) + a_2[D(x, T(x), a) + D(x_{n+1}, x_{n+2}, a)] + a_3[D(x, x_{n+2}, a) + D(x_{n+1}, T(x), a)]$$

Suppose that $n \to \infty$, then

$H(T(x), x, a) \le a_2 D(x, T(x), a) + a_3 s D(x_{n+1}, T(x), a)$
$\le a_2 H(x, T(x), a) + a_3 s H(T(x_n), T(x), a)$
$\le \frac{1}{s} H(x, T(x), a)$.

Thus, $H(x, T(x), a) = 0, \forall a \in X$ and this indicates that $T(x) = x$ (x is the fixed point of T).

We will show the uniqueness by contradiction.

Let x, y be two different fix points of $T, i.e. Tx = x, Ty = y$ and $x \ne y$.Then

$$d(x, y, a) \ge 0, \qquad \forall a \in X.$$

$$d(x,y,a) = D(T(x),y,a) \leq H(T(x),T(y),a) \leq a_1 d(x,y,a) + a_2[D(x,T(x),a) + D(y,T(y),a)] + a_3[D(x,T(y),a) + D(y,T\,x,a)] \leq \{a_1 d(x,y,a) + 2a_3 d(x,y,a)\}.$$

Using the inequality,

$a_1 + 2a_3 < a_1 + a_2 + 2a_3 s < \frac{s-1}{s^2}$,

we get $d(x,y,a) = 0, \forall a \in X$, which is a contradiction. As a result, x is the unique fixed point of T, thus $Fix(T) = \{x\}$.

Given example illustrates the main theorem.

Example:

Let $X = (x,0): x \in [0,1] \cup \{(0,2)\} \subset R^2$, and let we denote the absolute value of the cube of area of a triangle by $d(x,y,z)$ where $x,y,a \in X$, are vertices. Let

$$d((x,0),(y,0),(0,2)) = |x - y|^3.$$

given in complete b_2 metric space X with $s = 3$, and the mapping $f : X \rightarrow X$ is given by:

$$f(x,0) = \left(\frac{x}{3},0\right), x \in [0,1], f(0,2) = (0,0),$$

Then f has a unique fixed point.

Case 1: For $x = (x,0),\ y = (y,0), a = (0,2)$

$$H(T(x),T(y),a) = \left|\frac{x}{3} - \frac{y}{3}\right|^3 = \frac{1}{81} d(x,y,a).$$

Case 2: For $x = (x,0),\ y = (0,2),\ a = (a,0)$

$$H(T(x),T(y),a) = \left|\frac{x}{3} - \frac{a}{3}\right|^3 = \frac{1}{81} d(x,T(y),a).$$

Case3: For $x = (0,2),\ y = (y,0),\ a = (a,0)$,

$$H(T(x),T(y),a) = \left|\frac{y}{3} - \frac{a}{3}\right|^3 = \frac{1}{81} d(T(x),y,a).$$

Hence it can be concluded that

$H(T(x),T(y),a) = \frac{1}{81}\{d(x,y,a) + d(x,T(y),a) + d(T(x),y,a)\}$.

Here $a_1 = a_3 = \frac{1}{81}, a_2 = 0$, and all the assumptions on a_1, a_2, a_3 in Theorem 2.1 are fulfilled for $s = 3$ and the operator defined above satisfies the conditions of the theorem, therefore it is proved that the f has a unique fixed point $(0,0)$.

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