Hacettepe Journal of Mathematics & Statistics
Hacet. J. Math. Stat.
Volume 48 (5) (2019), 1304 – 1311
DOI : 10.15672/HJMS.2018.567
Research Article

Resultants of quaternion polynomials

Xiangui Zhao*[1], Yang Zhang[2]

[1] Department of Mathematics, Huizhou University, Huizhou, Guangdong Province, 516007, China
[2] Department of Mathematics, University of Manitoba, Winnipeg, MB, R3T 2N2, Canada

Abstract

We generalize the concept of resultants to quaternion polynomials and investigate the relationships among resultants, greatest common right divisors and repeated right roots of quaternion polynomials.

Mathematics Subject Classification (2010). 13P15, 15B33

Keywords. quaternion, (double) resultant, greatest common right divisor

1. Introduction

Let F be a field and $F[x]$ be the algebra of univariate polynomials in x over F. Suppose $m, n \in \mathbb{N}$, $f(x) = a_m x^m + \cdots + a_1 x + a_0 \in F[x]$ and $g(x) = b_n x^n + \cdots + b_1 x + b_0 \in F[x]$ of degrees m and n respectively with all $a_i, b_j \in F$. The *Sylvester matrix* of f and g, denoted by $\mathrm{Syl}(f, g; x)$ or simply $\mathrm{Syl}(f, g)$, is defined as

$$\mathrm{Syl}(f,g) = \begin{pmatrix} a_m & & & & b_n & & & \\ a_{m-1} & a_m & & & b_{n-1} & b_n & & \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & \\ \vdots & \vdots & & a_m & \vdots & \vdots & & \ddots \\ \vdots & \vdots & & a_{m-1} & b_0 & \vdots & & b_n \\ a_0 & \vdots & & \vdots & & b_0 & & \vdots \\ & a_0 & & \vdots & & & \ddots & \vdots \\ & & \ddots & \vdots & & & \ddots & \vdots \\ & & & a_0 & & & & b_0 \end{pmatrix},$$

with n columns of a_i's and m columns of b_j's, and all entries outside the two "parallelograms" are zero. The determinant of the Sylvester matrix $\mathrm{Syl}(f, g)$ is called the *resultant* of f and g, denoted by $\mathrm{res}(f, g) = \det(\mathrm{Syl}(f, g))$. It has been well known that the resultant is a basic tool in the study of polynomials (see, for example, [16]).

Suppose $f, g \in F[x]$. Then there exist $s, t \in F[x]$ (called Bézout coefficients) such that $sf + tg = \gcd(f, g)$, where $\gcd(f, g)$ denotes the greatest common divisor of f and g. The gcd and Bézout coefficients can be computed by the (Extended) Euclidean Algorithm.

*Corresponding Author.
Email addresses: zhaoxg@hzu.edu.cn (X. Zhao), yang.zhang@umanitoba.ca (Y. Zhang)
Received: 17.03.2017; Accepted: 27.02.2018

However, the bad news is that the coefficient growth in the Euclidean Algorithm is very rapid (see, for example, Section 6.1 of [16]). To overcome this difficulty, resultants are used to control the Bézout coefficients (and it yields a modular algorithm for gcd calculations), mainly due to the following property of resultants.

Theorem 1.1 ([16, Corollary 6.17]). *Let F be a field and $f, g \in F[x]$ be nonzero. Then the following are equivalent:*

(i) $\gcd(f, g) = 1$,
(ii) $\operatorname{res}(f, g) \neq 0$,
(iii) *there do not exist $s, t \in F[x] \setminus \{0\}$ such that*

$$sf + tg = 0, \quad \deg s < \deg g, \quad \deg t < \deg f.$$

Another application of resultants in computer algebra is that, working with Gröbner bases, the resultant is one of the main tools of effective elimination theory (see, for example, [2]). For this purpose, we need the following result.

Theorem 1.2 ([2, Proposition 9 of Section 3.5]). *Given $f, g \in F[x]$ of positive degrees, there exist polynomials $s, t \in F[x]$ such that $sf + tg = \operatorname{res}(f, g)$.*

Quaternion algebra $\mathbb{H}$ was introduced by W. R. Hamilton in 1843. The quaternions $\mathbb{H}$ is an associative (but noncommutative) division algebra generated by four basic elements $1, \boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ over the reals $\mathbb{R}$ with Hamilton relations $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1$. It holds a special place in mathematics since, by Frobenius theorem, $\mathbb{H}$ is one of the only three finite dimensional division algebras over the real numbers (the other two are real numbers $\mathbb{R}$ and complex numbers $\mathbb{C}$). Quaternions $\mathbb{H}$ is also the first noncommutative division algebra to be discovered. Beyond mathematics, quaternions are also widely used, for example, in electromechanics and quantum mechanics. Since the group of unit quaternions is isomorphic to the group of the involving three dimensional (3D) rotations, the primary application of quaternions in these fields is in calculations 3D rotations such as in 3D computer animation, computer vision and orbital mechanics, see, for example, [12, 15]. Besides, many physical laws in classical, relativistic, and quantum mechanics can be written nicely using quaternions, see, for example [11].

Our goal in this paper is to extend Theorems 1.1 and 1.2 to (noncommutative) polynomials over quaternions $\mathbb{H}$. The main difficulty here is how to define a determinant for a square matrix with entries in $\mathbb{H}$ (or more generally, in a noncommutative ring). Several noncommutative determinants have been formulated, e.g., Dieudonné determinant [3], Condensed Cramer rule [13, 14], quasideterminant [6], double determinant [7]. In [4], it is shown that, for polynomials $f, g \in \mathbb{H}[x]$, Theorem 1.1 is still true if gcd is replaced by gcrd (greatest common right divisor) and *(ii)* is replaced by

(ii)' $\operatorname{res}(f, g) = \operatorname{Ddet}(S) \neq 0$ in the factor group $\mathbb{H}^*/[\mathbb{H}^*, \mathbb{H}^*]$, where S is the Sylvester matrix (Definition 4.1) of f and g, $\operatorname{Ddet}(S)$ is the Dieudonné determinant of S, and $\mathbb{H}^*$ denotes the multiplicative group of $\mathbb{H}$.

The disadvantage to use the Dieudooné determinant is that $\operatorname{Ddet}(S)$ takes a value in the factor group $\mathbb{H}^*/[\mathbb{H}^*, \mathbb{H}^*]$ instead of $\mathbb{H}$. In this paper, we will extend Theorems 1.1 and 1.2 to $\mathbb{H}[x]$ by using double determinants (in the sense of Kyrchei [7]), which take values in $\mathbb{H}$.

The paper is organized as follows. Basic notations on quaternion and quaternion polynomials are introduced in Section 2, followed by the definition and properties of double determinant in Section 3. Our main results, the generalization of the above two theorems and their application on repeated roots, are in Section 4.

2. Quaternions and quaternion polynomials

Let $\mathbb{H} = \{a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} : a, b, c, d \in \mathbb{R}\}$ be the algebra of real quaternions, i.e., the associative unital $\mathbb{R}$-algebra generated by $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ with the Hamilton relations

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1,$$

which implies that

$$\boldsymbol{ij} = \boldsymbol{k},\ \boldsymbol{jk} = \boldsymbol{i},\ \boldsymbol{ki} = \boldsymbol{j},\ \boldsymbol{ji} = -\boldsymbol{k},\ \boldsymbol{kj} = -\boldsymbol{i},\ \boldsymbol{ik} = -\boldsymbol{j}.$$

Given $h = a + b\boldsymbol{i} + c\boldsymbol{j} + d\boldsymbol{k} \in \mathbb{H}$ where $a, b, c, d \in \mathbb{R}$, the *conjugate* of h is defined as $\overline{h} = a - b\boldsymbol{i} - c\boldsymbol{j} - d\boldsymbol{k}$.

Let $\mathbb{H}[x] = \{a_m x^m + \cdots + a_1 x + a_0 : a_i \in \mathbb{H}, 0 \le i \le m, m \in \mathbb{N}\}$ be the polynomial ring in one variable x over $\mathbb{H}$, where x commutes element wise with $\mathbb{H}$. Suppose $f(x) = a_m x^m + \cdots + a_1 x + a_0 \in \mathbb{H}[x]$ and $r \in \mathbb{H}$. We define $f(r)$ (the *evaluation* of f at r) to be $f(r) = a_m r^m + \cdots + a_1 r + a_0 \in \mathbb{H}$. Note that, with the above notation, $f(x) = g(x)h(x) \in \mathbb{H}[x]$ does not imply that $f(r) = g(r)h(r)$ (see Section 16 of [8] for a counterexample), that is, evaluation at r is not a ring homomorphism from $\mathbb{H}[x]$ to $\mathbb{H}$ in general.

Suppose $f, g \in \mathbb{H}[x]$. If $f = pq$ for $p, q \in \mathbb{H}[x]$, then q is called a *right divisor* of f. A *common right divisor* of f and g is a polynomial in $\mathbb{H}[x]$ which is a right divisor of both f and g. A *greatest common right divisor* of f and g, denoted by $\mathrm{gcrd}(f, g)$, is a common right divisor h of f and g such that any common right divisor of f and g is a right divisor of h. A quaternion polynomial is *monic* if the coefficient of the highest power of x is one. By the Euclidean Algorithm for noncommutative polynomials (see, for example, [10], Section 2 of Chapter I), we can prove the following lemma.

Lemma 2.1. *Suppose $f, g \in \mathbb{H}[x]$.*

(i) *There exists a unique, monic, greatest common right divisor of f and g, denoted by* $\mathrm{gcrd}(f, g)$.

(ii) *There exist polynomials $p, q \in \mathbb{H}[x]$ such that $pf + qg = \mathrm{gcrd}(f, g)$, $\deg(p) < \deg(g)$, and $\deg(q) < \deg(f)$.*

3. Double determinants

Row and column determinants were introduced by Kyrchei [7], based on which a double determinant (cf., Chen's double determinant [1]) was defined. Let us recall from [7] some definitions and properties related to double determinants. All statements without proofs in this section are taken from [7].

Let M_n be the set of $n \times n$ square matrices with entries from $\mathbb{H}$ and let S_n be the symmetric group on the set $\{1, 2, \ldots, n\}$. Suppose $A = (a_{ij}) \in M_n$. Then, for $1 \le i \le n$, the *ith row determinant* of A is defined as

$$\mathrm{rdet}_i(A) = \sum_{\sigma \in S_n} (-1)^{n-r} a_{i i_{k_1}} a_{i_{k_1} i_{k_1+1}} \cdots a_{i_{k_1+l_1} i} \cdots a_{i_{k_r} i_{k_r+1}} a_{i_{k_r+1} i_{k_r+2}} \cdots a_{i_{k_r+l_r} i_{k_r}} \tag{3.1}$$

where $\sigma \in S_n$ is written as a product of disjoint cycles

$$\sigma = (i\ i_{k_1} i_{k_1+1} \cdots i_{k_1+l_1})(i_{k_2} i_{k_2+1} \cdots i_{k_2+l_2}) \cdots (i_{k_r} i_{k_r+1} \cdots i_{k_r+l_r})$$

such that

$$i_{k_2} < i_{k_3} < \cdots < i_{k_r},\quad i_{k_t} < i_{k_t+s},\quad 2 \le t \le r,\ 1 \le s \le l_r.$$

Column determinants $\mathrm{cdet}_i(A)$ can be defined in a similar way, see Definition 2.5 of [7]. In particular, if $A \in M_n$ is a *Hermitian matrix* (i.e., $A^* = A$, where $A^* = \bar{A}^T$ is the transpose of the conjugate of A), then

$$\mathrm{rdet}_1(A) = \cdots = \mathrm{rdet}_n(A) = \mathrm{cdet}_1(A) = \cdots = \mathrm{cdet}_n(A) \in \mathbb{R}.$$

Definition 3.1 ([7, Definition 8.2])**.** Suppose $A \in M_n$. Then the *double determinant* of A is defined as $\mathrm{ddet}(A) = \mathrm{rdet}_1(A^* A)$.

Since A^*A is Hermitian for any $A \in M_n$, we have that

$$\begin{aligned}\mathrm{ddet}(A) &= \mathrm{rdet}_1(A^*A) = \cdots = \mathrm{rdet}_n(A^*A)\\ &= \mathrm{cdet}_1(A^*A) = \cdots = \mathrm{cdet}_n(A^*A) \in \mathbb{R}.\end{aligned}$$

The double determinant enjoys some familiar properties of ordinary determinants. For example, $\mathrm{ddet}(AB) = \mathrm{ddet}(A)\,\mathrm{ddet}(B)$ for any $A, B \in M_n$.

Double determinants are also closely related to solving quaternion linear equations.

Lemma 3.2. *Suppose $A \in M_n$. Then the following statements are equivalent.*

(i) *The columns of A are not right linearly independent, i.e., there exists one column of A that is a right linear combination of the other columns of A.*
(ii) *The rows of A are not left linearly independent.*
(iii) $\mathrm{ddet}(A) = 0$.

Using the above lemma, we can prove the following lemma (where we use $\mathbf{0}$ to denote both zero row $(0, \dots, 0)$ and zero column $(0, \dots, 0)^T$).

Lemma 3.3. *Suppose $A \in M_n$, $\boldsymbol{x} = (x_1, \dots, x_n)^T$ and $\boldsymbol{y} = (y_1, \dots, y_n)$. Then the following are equivalent.*

(i) *The right system $A\boldsymbol{x} = \mathbf{0}$ of linear equations has nontrivial solutions.*
(ii) *The left system $\boldsymbol{y}A = \mathbf{0}$ of linear equations has nontrivial solutions.*
(iii) $\mathrm{ddet}(A) = 0$.

Proof. Note that the right system $A\boldsymbol{x} = \mathbf{0}$ has nontrivial solutions if and only if the columns of A are not right linearly independent and that the left system $\boldsymbol{y}A = \mathbf{0}$ has nontrivial solutions if and only if the rows of A are not left linearly independent. Thus the lemma follows from Lemma 3.2. □

Lemma 3.4 (Cramer's Rule)**.** *Let $\boldsymbol{x}A = \boldsymbol{y}$ be a left system of linear equation with coefficient matrix $A \in M_n$, constant row $\boldsymbol{y} = (y_1, \dots, y_n)$ of quaternions, and unknowns $\boldsymbol{x} = (x_1, \dots, x_n)$. If $\mathrm{ddet}(A) \neq 0$, then the system has a unique solution in $\mathbb{H}$ given by*

$$x_i = \frac{\mathrm{rdet}_i((AA^*)_{i.}(\boldsymbol{y}A^*))}{\mathrm{ddet}(A)}, \quad 1 \le i \le n,$$

where $(AA^*)_{i.}(\boldsymbol{y}A^*)$ is the matrix obtained from AA^* by replacing the ith row by the row vector $\boldsymbol{y}A^*$.

4. Resultants, gcrd and repeated roots

In this section, we define a resultant of two quaternion polynomials and investigate the relationships among resultants, gcrd and repeated roots of quaternion polynomials.

First of all, as for commutative case in the Section 1, we can define Sylvester matrices for two polynomials in $\mathbb{H}[x]$.

Definition 4.1 (Sylvester matrix and resultant)**.** Suppose $p(x) = a_m x^m + \cdots + a_1 x + a_0 \in \mathbb{H}[x]$ and $q(x) = b_n x^n + \cdots + b_1 x + b_0 \in \mathbb{H}[x]$ of degrees m and n respectively. The *Sylvester matrix* $\mathrm{Syl}(p, q) \in M_n$ of p and q is defined as

$$\mathrm{Syl}(p,q)=\begin{pmatrix} a_m & & & & b_n & & & & \\ a_{m-1} & a_m & & & b_{n-1} & b_n & & & \\ \vdots & \vdots & \ddots & & \vdots & \vdots & \ddots & & \\ \vdots & \vdots & & a_m & \vdots & \vdots & & \ddots & \\ \vdots & \vdots & & a_{m-1} & b_0 & \vdots & & & b_n \\ a_0 & \vdots & & \vdots & & b_0 & & & \vdots \\ & a_0 & & \vdots & & & \ddots & & \vdots \\ & & \ddots & \vdots & & & & \ddots & \vdots \\ & & & a_0 & & & & & b_0 \end{pmatrix},$$

with n columns of a_i's and m columns of b_j's, and all entries outside the two "parallelograms" are zero. The double determinant of the transpose of Sylvester matrix is called the *resultant* of f and g, denoted by $\mathrm{res}(f,g)=\mathrm{ddet}(\mathrm{Syl}^T(f,g))$, where $\mathrm{Syl}^T(f,g)$ is the transpose of the Sylvester matrix $\mathrm{Syl}(f,g)$. If $m=n=0$, then $\mathrm{Syl}^T(f,g)$ is the "empty" 0×0 matrix with double determinant $\mathrm{res}(f,g)=1$. It is convenient to define the resultant with the zero polynomial as

$$\mathrm{res}(f,0)=\mathrm{res}(0,f)=\begin{cases}1 & \text{if } f \text{ is a nonzero constant}\\ 0 & \text{if } f \text{ is zero or nonconstant}\end{cases}.$$

Remark 4.2. In general, $\mathrm{ddet}(A)\neq\mathrm{ddet}(A^T)$ for $A\in M_n$, e.g., if $A=\begin{pmatrix}1 & \boldsymbol{i}\\ \boldsymbol{j} & \boldsymbol{k}\end{pmatrix}$, then simple computation gives that $\mathrm{ddet}(A)=4\neq 0=\mathrm{ddet}(A^T)$. Thus, in order for desired properties, a definition of resultants of quaternion polynomials must be carefully chosen. For our purpose in this paper, we define the resultant as $\mathrm{res}(f,g)=\mathrm{ddet}(\mathrm{Syl}^T(f,g))$ instead of $\mathrm{ddet}(\mathrm{Syl}(f,g))$. That is mainly because we consider *right* divisors and *right* roots rather than the left ones.

The following theorem together with Lemma 2.1 generalizes Theorem 1.1.

Theorem 4.3. *Suppose $f,g\in\mathbb{H}[x]$ are nonzero. Then* $\mathrm{gcrd}(f,g)=1$ *if and only if* $\mathrm{res}(f,g)\neq 0$.

Proof. Suppose f and g are of degrees m and n respectively, $m,n\in\mathbb{N}$. Let

$$\begin{aligned} f&=a_mx^m+\cdots+a_1x+a_0,\\ g&=b_nx^n+\cdots+b_1x+b_0,\\ s&=u_{n-1}x^{n-1}+\cdots+u_1x+u_0,\\ t&=v_{m-1}x^{m-1}+\cdots+v_1x+v_0,\end{aligned}$$

where each $a_i,b_j,u_i,v_j\in\mathbb{H}$ and $a_m,b_n\neq 0$. Then $sf+tg=0$ if and only if $(u_{n-1},\ldots,u_0,v_{m-1},\ldots,v_0)$ is solution of the left system of linear equation $\boldsymbol{y}A=\boldsymbol{0}$ where $\boldsymbol{y}=(y_1,\ldots,y_{m+n})$ and $A=\mathrm{Syl}^T(f,g)$.

By Theorem 2.4 of [4] (cf., [16, Lemma 6.13]), $\mathrm{gcrd}(f,g)\neq 1$ if and only if there exist nonzero polynomials $s,t\in\mathbb{H}[x]$ such that $sf+tg=0$, $\deg s<\deg g$ and $\deg t<\deg f$. Thus $\mathrm{gcrd}(f,g)\neq 1$ if and only if the left system $\boldsymbol{y}A=\boldsymbol{0}$ has nontrivial solutions. Therefore, by Lemma 3.3, $\mathrm{gcrd}(f,g)\neq 1$ if and only if $\mathrm{res}(f,g)=\mathrm{ddet}(A)=0$. □

Recall that a polynomial in $\mathbb{H}[x]$ is called *integer polynomial* if its coefficients are integers. Now we can give a generalization of Theorem 1.2.

Theorem 4.4. *Suppose $f, g \in \mathbb{H}[x]$ with $\deg f > 0$ and $\deg g > 0$. Then there exist polynomials $p, q \in \mathbb{H}[x]$ such that $pf + qg = \operatorname{res}(f, g)$. Furthermore, the coefficients of p and q are integer polynomials in the coefficients of f and g.*

Proof. First note that if $\operatorname{res}(f, g) = 0$ then we can simply choose $p = q = 0$. Now assume that $\operatorname{res}(f, g) \neq 0$. Then, by Theorem 4.3, $\operatorname{gcrd}(f, g) = 1$. Hence, by Lemma 2.1, there exist $p', q' \in \mathbb{H}[x]$ such that $\deg(p') < \deg(g), \deg(q') < \deg(f)$ and $p'f + q'g = 1$. Suppose

$$f = a_m x^m + \cdots + a_0, \tag{4.1}$$

$$g = b_n x^n + \cdots + b_0, \tag{4.2}$$

$$p' = c_{n-1} x^{n-1} + \cdots + c_0, \tag{4.3}$$

$$q' = d_{m-1} x^{m-1} + \cdots + d_0. \tag{4.4}$$

where the coefficients $a_i, b_j, c_i, d_j \in \mathbb{H}$ and $a_m \neq 0, b_n \neq 0$. Substituting (4.1)–(4.4) into the equation $p'f + q'g = 1$ and equating the coefficients of powers of x, we get the following left system of linear equations

$$(c_{n-1}, \ldots, c_0, d_{m-1}, \ldots, d_0)A = (0, \ldots, 0, 1) \tag{4.5}$$

with unknowns c_i, d_j and coefficient matrix $A = \operatorname{Syl}^T(f, g)$.

Since $\operatorname{ddet}(A) = \operatorname{res}(f, g) \neq 0$, by Cramer's Rule (Lemma 3.4), System (4.5) has a unique solution in $\mathbb{H}$ given by, for example,

$$c_{n-1} = \frac{\operatorname{rdet}_1 (AA^*)_{1.}(\boldsymbol{z})}{\operatorname{ddet}(A)},$$

where $\boldsymbol{z} = (0, \ldots, 0, 1)A^*$ is the last row of A^*.

Note that, by the definition (Equation (3.1)), a row determinant of a matrix $B = (b_{ij}) \in M_n$ is a polynomial in the entries b_{ij} with integer coefficients. Thus it follows that

$$c_{n-1} = \frac{p_{n-1}}{\operatorname{ddet}(A)}, \tag{4.6}$$

where p_{n-1} is a polynomial in a_i and b_j $(1 \leq i, j \leq n)$ with integer coefficients. Similarly, all c_i and d_j $(1 \leq i \leq n-1,\ 0 \leq j \leq m-1)$ have the same form as c_{n-1} in (4.6). Hence

$$p' = c_{n-1} x^{n-1} + \cdots + c_0 = \frac{p}{\operatorname{ddet}(A)},$$

where $p \in \mathbb{H}[x]$ and the coefficients of p are polynomials in a_i and b_j, $0 \leq i \leq m, 0 \leq j \leq n$. Similarly, we can write

$$q' = \frac{q}{\operatorname{ddet}(S)},$$

where $q \in \mathbb{H}[x]$ has the same properties as p. Since p' and q' satisfy $p'f + q'g = 1$, multiplying by $\operatorname{ddet}(A)$ gives $pf + qg = \operatorname{ddet}(A)$, where p and q are integer polynomials in the coefficients of f and g as required. □

Studying the roots of quaternion polynomials is quite different from that in commutative polynomial case, see for example, [5] and [8]. As an application of our main theorem, we will consider repeated roots of a quaternion polynomial.

Let $f = a_m x^m + \cdots + a_1 x + a_0 \in \mathbb{H}[x]$, $m \in \mathbb{N}$. Sometimes we write f as $f(x)$ to emphasis the variable x. We use both f and $f(x)$ without any difference in the paper. The (formal) *derivative* of f is defined as $f' = m a_m x^{m-1} + \cdots + a_2 x + a_1$. In particular, if $m = 0$, then $f' = 0$. Then we have the following lemma, whose proof is straightforward.

Lemma 4.5. *Suppose $f(x), g(x) \in \mathbb{H}[x]$ and $c \in \mathbb{H}$. Then*

(i) $[cf(x)]' = cf'(x)$.
(ii) $[f(x) + g(x)]' = f'(x) + g'(x)$.
(iii) $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$.
(iv) $[(x + c)^n]' = n(x + c)^{n-1}$.

Remark 4.6. The Chain Rule does not hold for derivatives of quaternion polynomials in general. For example, if $f(x) = ax + b \in \mathbb{H}[x]$, then

$$[[f(x)]^2]' = [a^2x^2 + (ab + ba)x + b^2]' = 2a^2x + ab + ba$$

and

$$2f(x)f'(x) = 2(ax + b)a = 2a^2x + 2ba.$$

Hence $[[f(x)]^2]' \neq 2f(x)f'(x)$ if $ab \neq ba$.

Lemma 4.7. *Suppose $f(x)$ has a right factor $(ax+b)^n$, where $n \geq 2$, $a, b \in \mathbb{H}$, $a \neq 0$ and $ab = ba$. Then $\operatorname{res}(f, f') = 0$.*

Proof. First we suppose $f(x) = g(x)(x + b)^n$ for some $g(x) \in \mathbb{H}[x]$. Then, by Lemma 4.5,

$$\begin{aligned} f'(x) &= g'(x)(x + b)^n + ng(x)(x + b)^{n-1} \\ &= [g'(x)(x + b) + ng(x)](x + b)^{n-1}. \end{aligned}$$

Both f and f' have right factor $(x + b)^{n-1}$, which is not a unit since $n \geq 2$. Hence, by Theorem 4.3, $\operatorname{res}(f, f') = 0$.

For general case, since $ab = ba$, we have $f(x) = g(x)a^n(x + a^{-1}b)^n$. Then it follows from the last paragraph that $\operatorname{res}(f, f') = 0$. □

Remark 4.8. The condition $ab = ba$ in the above lemma is necessary. Otherwise, for example, let $f(x) = (\boldsymbol{i}x + \boldsymbol{j})^2$. Then $f' = (-x^2 - 1)' = -2x$. Thus

$$A = \operatorname{Syl}^T(f, f') = \begin{pmatrix} -1 & 0 & -1 \\ -2 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix}$$

which is a 3×3 matrix over $\mathbb{R}$. Hence, $\operatorname{res}(f, f') = \operatorname{ddet}(A) = \det(A^TA) = (\det A)^2 = 16 \neq 0$.

Definition 4.9 ([8, §16])**.** An element $r \in \mathbb{H}$ is said to be a *right root* of a nonzero polynomial $f(x) \in \mathbb{H}[x]$ if $x - r$ is a right divisor of $f(x)$. Furthermore, if $(x - r)^n$ for $n \geq 2$ is a right divisor of $f(x)$, then we call r a repeated right root of $f(x)$.

Now we are in a position to prove the following theorem.

Theorem 4.10. *Let $0 \neq f \in \mathbb{H}[x]$. Then f has a repeated right root if and only if $\operatorname{res}(f, f') = 0$.*

Proof. If r is a repeated right root of $f(x)$, i.e., $f(x) = g(x)(x - r)^n$ for some $g(x) \in \mathbb{H}[x]$ and $n \geq 2$, then, by Lemma 4.7, $\operatorname{res}(f, f') = 0$.

Now we suppose $\operatorname{res}(f, f') = 0$. Then, by Theorem 4.3, $\operatorname{gcrd}(f, f') \neq 1$. It is well known that every polynomial in $\mathbb{H}[x]$ can be factorized into a product of linear factors (see, for example, [9], Section 2). Hence we can write $\operatorname{gcrd}(f, f') = h(x)(x-r)$ for some $h(x) \in \mathbb{H}[x]$ and $r \in \mathbb{H}$. Then $x-r$ is a right divisor of both f and f'. Suppose $f(x) = f_1(x)(x-r)$ and $f'(x) = f_2(x)(x - r)$. Then, by Lemma 4.5, $f_2(x)(x - r) = f'(x) = f_1'(x)(x - r) + f_1(x)$. Hence $f_1(x) = (f_2(x) - f_1'(x))(x - r)$ and thus $f(x) = (f_2(x) - f_1'(x))(x - r)^2$. Therefore, r is a repeated root of $f(x)$. □

Acknowledgment. We would like to thank the referee for carefully reading our manuscript and valuable comments that helped improving the readability of this article. This work is supported in part by the NSERC of Canada (312386-2015), the NSF of Guangdong Province of China (2016A030310099), the NSF of China (11571220), the Science and Technology Program of Huizhou City (2016X0429044,2017C0404020), the Huizhou University (2015JB021, hzu201704, hzuxl201523).

References

[1] L. Chen, *Definition of determinant and Cramer solutions over the quaternion field*, Acta Math. Sin. **7** (2), 171-180, 1991.

[2] D.A. Cox, J. Little and D. O'Shea, *Ideals, varieties, and algorithms: an introduction to computational algebraic geometry and commutative algebra*, **10**, Springer, 2007.

[3] J. Dieudonné, *Les déterminants sur un corps non commutatif*, Bull. Soc. Math. France **71**, 27-45, 1943.

[4] A.Lj. Erić, *The resultant of non-commutative polynomials*, Mat. Vesnik **60**, 3-8, 2008.

[5] L. Feng and K. Zhao, *Classifying zeros of two-sided quaternionic polynomials and computing zeros of two-sided polynomials with complex coefficients*, Pac. J. Math. **262** (2), 317-337, 2013.

[6] I. Gelfand, S. Gelfand, V. Retakh and R. Lee Wilson, *Quasideterminants*, Adv. Math. **193** (1), 56-141, 2005.

[7] I.I. Kyrchei, *Cramer's rule for quaternionic systems of linear equations*, J. Math. Sci. **155** (6), 839-858, 2008.

[8] T.Y. Lam, *A first course in noncommutative rings*, Graduate Texts in Mathematics **131**, Springer-Verlag New York, 2001.

[9] I. Niven, *Equations in quaternions*, Amer. Math. Monthly **48** (10), 654-661, 1941.

[10] O. Ore, *Theory of non-commutative polynomials*, Ann. Math. **34** (3), 480–508, 1933.

[11] P. Rastall, *Quaternions in relativity*, Rev. Mod. Phys. **36** (3), 820-832, 1964.

[12] K. Shoemake, *Animating rotation with quaternion curves*, ACM SIGGRAPH Computer Graphics **19** (3), 245–254, 1985.

[13] G. Song and Q. Wang, *Condensed Cramer rule for some restricted quaternion linear equations*, Appl. Math. Comput. **218**, 3110-3121, 2011.

[14] G. Song, Q. Wang and H. Chang, *Cramer rule for the unique solution of restricted matrix equations over the quaternion skew fields*, Comput. Math. Appl. **61** (6), 1576-1589, 2011.

[15] R. Szeliski, *Computer vision: algorithms and applications*, Springer, 2011.

[16] J. von zur Gathen and J. Gerhard, *Modern computer algebra*, Cambridge University Press, 2013.