Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 48 (5) (2019), 1312 – 1323
DOI : 10.15672/HJMS.2018.568

Research Article

Trigonometrically-fitted higher order two derivative Runge-Kutta method for solving orbital and related periodical IVPs

N. A. Ahmad*[1], N. Senu[1,2], F. Ismail[1,2]

[1] *Institute for Mathematical Research, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia*
[2] *Department of Mathematics, Universiti Putra Malaysia, 43400 Serdang, Selangor, Malaysia*

Abstract

In this paper, a trigonometrically-fitted two derivative Runge-Kutta method (TFTDRK) of high algebraic order for the numerical integration of first order Initial Value Problems (IVPs) which possesses oscillatory solutions is constructed. Using the trigonometrically-fitted property, a sixth order four stage Two Derivative Runge-Kutta (TDRK) method is designed. The numerical experiments are carried out with the comparison with other existing Runge-Kutta methods (RK) to show the accuracy and efficiency of the derived methods.

Mathematics Subject Classification (2010). 65L05, 65L06

Keywords. two derivative Runge-Kutta method, trigonometrically-fitted, ordinary differential equations, initial value problems

1. Introduction

Consider the numerical solution of the IVPs for first order Ordinary Differential Equations (ODEs) in the form of

$$q' = f(t, q), \qquad q(t_0) = q_0, \tag{1.1}$$

whose solutions show an observable oscillatory or periodically behavior. Such problems occur in several fields of applied sciences, for example, circuit simulation, molecular dynamics, orbital mechanics, mechanics, electronics, astrophysics and etc. In general, most problems with oscillatory or periodically behavior are second or higher order. Therefore, it is important to reduce the higher order problems to first order problems in order to solve the ODEs (1.1).

Several well-known authors in their papers have developed RK methods for solving oscillatory problems using several techniques, for instance, phase-fitted and amplification-fitted, trigonometrically-fitted and exponentially-fitted techniques. Simos[17] developed a fifth algebraic order trigonometrically fitted Runge-Kutta methods for the numerical integration of the radial Schrödinger equation. Anastassi and Simos[2] constructed two

*Corresponding Author.
Email addresses: nuramirah_ahmad@yahoo.com (N. A. Ahmad), norazak@upm.edu.my (N. Senu), fudziah@upm.edu.my (F. Ismail)
Received: 12.10.2016; Accepted: 01.03.2018

trigonometrically-fitted methods based on a classical fifth algebraic order England's Runge-Kutta method for the integration of the radial Schrödinger equation which have energy with lower powers in the local truncation error.

A few years later, Chen et al. [6] improved traditional RK methods by introducing frequency-depending weights in the update. With the phase-fitting and amplification-fitting conditions and algebraic order conditions, new practical RK integrators are obtained and two of the new methods have updates that are also phase-fitted and amplification-fitted. Recently Fawzi et al., in their papers [13] and [1], developed fourth algebraic order phase-fitted and amplification-fitted modified RK method and fourth order seven stage phase-fitted and amplification-fitted RK methods respectively. Monovasilis et al. in [15] constructed fourth order trigonometrically-fitted symplectic Runge-Kutta-Nyström (RKN) methods from symplectic trigonometrically fitted partitioned Runge-Kutta (PRK) methods.

Meanwhile, in [7] and [8], Demba et al. derived four-stage fourth order explicit trigonometrically -fitted Runge-Kutta-Nyström (RKN) method and fifth-order four-stage explicit trigonometrically-fitted RKN method respectively for the numerical solution of second-order initial value problems with oscillatory solutions based on Simos' RKN method. Other than that, Demba et al. in [11] and [9] constructed embedded explicit 4(3) and 5(4) pairs trigonometrically-fitted RKN methods for solving oscillatory problems respectively. In [10], again Demba et al. derived a symplectic third-order three-stage explicit trigonometrically-fitted RKN method for the numerical solution of second order initial value problems with periodic solutions.

Chan and Tsai [4] introduced special explicit TDRK methods by including the second derivative. It involves only one evaluation of f and a number of evaluations of g per step. They presented methods up to five stages and up to seventh order as well as some embedded pairs. Zhang et al. [20] proposed a new fifth order trigonometrically fitted TDRK method for the numerical solution of the radial Schrödinger equation and oscillatory problems. Meanwhile, Fang et al. [12] and Chen et al. [5] derived two fourth order and three practical exponentially fitted TDRK methods respectively. They compared the new methods with some well-known optimized codes and traditional exponentially fitted RK methods.

Hence, in this paper, a sixth order four stage trigonometrically-fitted TDRK methods are constructed. In Section 2, an overview of TDRK method is given. The new trigonometrically-fitted TDRK method is constructed in Section 3. The numerical results, discussion and conclusion are dealt in Section 4, Section 5 and Section 6 respectively.

2. Two Derivative Runge-Kutta methods

Consider the autonomous scalar ODEs (1.1) with $f : \Re^N \rightarrow \Re^N$. For this case, we assume that the second derivative is also known where

$$q'' = g(q) := f'(q)f(q), \qquad g : \Re^N \rightarrow \Re^N. \tag{2.1}$$

An explicit TDRK method for the numerical integration of IVPs (1.1) is given by

$$q_{n+1} = q_n + h\sum_{i=1}^{s} b_i f(q_i) + h^2 \sum_{i=1}^{s} \hat{b}_i g(Q_i), \tag{2.2}$$

$$Q_i = q_n + h\sum_{j=1}^{i-1} a_{ij} f(q_j) + h^2 \sum_{j=1}^{i-1} \hat{a}_{ij} g(Q_j), \tag{2.3}$$

where $i = 1, \ldots, s$. The explicit TDRK method with the coefficients in (2.2) and (2.3) is represented using the Butcher tableau as follows

c	A	$\hat{A}$
	b^T	$\hat{b}^T$

Explicit methods with minimal number of function evaluations can be developed by considering the methods in the form

$$q_{n+1} = q_n + hf(t_n, q_n) + h^2 \sum_{i=1}^{s} \hat{b}_i g(t_n + hc_i, Q_i), \tag{2.4}$$

$$Q_i = q_n + hc_i f(t_n, q_n) + h^2 \sum_{j=1}^{i-1} \hat{a}_{ij} g(Q_j), \tag{2.5}$$

where $i = 1, \ldots, s$. The above method is called special explicit TDRK method. The unique part of this method is that it involves only one evaluation of f and many evaluations of g per step compared to many evaluation of f per step in traditional explicit RK methods. Its Butcher tableau is given as follows.

c	$\hat{A}$
	$\hat{b}^T$

The TDRK parameters $\hat{a}_{ij}, \hat{b}_i$ and c_i are assumed to be real and s is the number of stages of the method. We introduce the s-dimensional vectors $\hat{b}, c$ and $s \times s$ matrix, $\hat{A}$ where $\hat{b} = [\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_s]^T, c = [c_1, c_2, \ldots, c_s]^T$ and $\hat{A} = [\hat{a}_{ij}]$ respectively.

The order conditions for special explicit TDRK methods are given in the following Table 1 as written in [4].

Table 1. Order conditions for special explicit TDRK methods.

Order	Conditions				
1	$b^T e = 1$				
2	$\hat{b}^T e = \frac{1}{2}$				
3	$\hat{b}^T c = \frac{1}{6}$				
4	$\hat{b}^T c^2 = \frac{1}{12}$				
5	$\hat{b}^T c^3 = \frac{1}{20}$	$\hat{b}^T \hat{A} c = \frac{1}{120}$			
6	$\hat{b}^T c^4 = \frac{1}{30}$	$\hat{b}^T c\hat{A} c = \frac{1}{180}$	$\hat{b}^T \hat{A} c^2 = \frac{1}{360}$		
7	$\hat{b}^T c^5 = \frac{1}{42}$	$\hat{b}^T c^2\hat{A} c = \frac{1}{252}$	$\hat{b}^T c\hat{A} c^2 = \frac{1}{504}$	$\hat{b}^T \hat{A} c^3 = \frac{1}{840}$	$\hat{b}^T \hat{A}^2 c = \frac{1}{5040}$

When equation (2.4) and (2.5) are applied to the first order ODEs (1.1), for any $(\zeta+1)$th differentiable function $g(q)$, the local truncation error (LTE), $LTE = q_{n+1} - q(t_n + h) = \mathcal{O}(h^{\zeta+1})$. Hence the method is said to have (algebraic) order ζ.

3. Derivation of the new trigonometrically-fitted method

In deriving the new method, the trigonometrically-fitted property will be applied to an existing TDRK method. Hence, the derivation of the new method will be discussed next.

For this study, a two derivative sixth algebraic order method presented by Chan et al. [4] is considered. The coefficient of the method is given by Table 2.

Table 2. Butcher Tableau for sixth order TDRK method

0				
$\frac{1}{3}$	$\frac{1}{18}$			
$\frac{1}{2}$	$\frac{1}{8}$			
$\frac{2}{3}$	$\frac{1}{9}$	$\frac{1}{9}$		
	$\frac{11}{120}$	$\frac{9}{20}$	$\frac{-4}{15}$	$\frac{9}{40}$

When the special explicit TDRK method (2.4) and (2.5) is applied to the ODEs (1.1) with $q^{'} = i\lambda q$ and $q^{''} = -\lambda^2 q$, the method becomes

$$q_{n+1} = q_n + hq_n^{'} + h^2 \sum_{i=1}^{s} \hat{b}_i g(t_n + c_i h, Q_i), \tag{3.1}$$

where

$$Q_1 = q_n + c_1 hq_n^{'}, \tag{3.2}$$

$$Q_2 = q_n + c_2 hq_n^{'} - h^2 \hat{a}_{21}\lambda^2 Q_1, \tag{3.3}$$

$$Q_3 = q_n + c_3 hq_n^{'} + h^2 \left(-\hat{a}_{31}\lambda^2 Q_1 - \hat{a}_{32}\lambda^2 Q_2\right), \tag{3.4}$$

$$Q_4 = q_n + c_4 hq_n^{'} + h^2 \left(-\hat{a}_{41}\lambda^2 Q_1 - \hat{a}_{42}\lambda^2 Q_2 - \hat{a}_{43}\lambda^2 Q_3\right), \tag{3.5}$$

which results in

$$q_{n+1} = q_n + hq_n^{'} + h^2 \sum_{i=1}^{s} \hat{b}_i(-\lambda^2 Q_j). \tag{3.6}$$

Let $q_n = e^{i\lambda t}$ and $f(t_n, q_n) = i\lambda q_n$, compute the values for q_{n+1} and substitute those values in equation (3.1)–(3.5). By using $e^{iv} = \cos(v) + i\sin(v)$, the following equation is obtained

$$\cos(v) + i\sin(v) = 1 + z + z^2 \sum_{i=1}^{s} \hat{b}_i \left(1 + zc_i + z^2 \sum_{j=1}^{i-1} \hat{a}_{ij} Q_j e^{-I\lambda t_n}\right), \tag{3.7}$$

where $z = iv$. For optimized value of maximum global error, the combination of $\hat{a}_{31}$ and $\hat{a}_{32}$ are chosen as free parameters. By using the coefficients of the method in Table 2, separate the real part and the imaginary part of (3.7) and this leads to

$$\cos(v) = 1 - \frac{1}{2}v^2 + \frac{3}{40}v^4 - \frac{4}{15}v^4\hat{a}_{3,1} - \frac{4}{15}\hat{a}_{3,2}v^4 + \frac{2}{135}v^6\hat{a}_{3,2} - \frac{1}{720}v^6, \tag{3.8}$$

$$\sin(v) = v - \frac{1}{6}v^3 - \frac{4}{45}v^5\hat{a}_{3,2} + \frac{1}{120}v^5. \tag{3.9}$$

Solving equation (3.8) and (3.9) we will obtain

$$\hat{a}_{31} = -\frac{1}{24}\,\frac{90\cos(v)\,v + 180\,v - 15\,v^3 - 2\,v^5 - 270\sin(v) + 15\,v^2\sin(v)}{v^5}, \tag{3.10}$$

$$\hat{a}_{32} = -\frac{3}{32}\,\frac{120\sin(v) - 120\,v + 20\,v^3 - v^5}{v^5}. \tag{3.11}$$

As $v \to 0$, the following Taylor expansions are obtained

$$\hat{a}_{31} = \frac{1}{8} - \frac{1}{448}v^2 + \frac{1}{16128}v^4 - \frac{31}{31933440}v^6 + \frac{1}{103783680}v^8 + \dots,$$

$$\hat{a}_{32} = \frac{1}{448}v^2 - \frac{1}{32256}v^4 + \frac{1}{3548160}v^6 - \frac{1}{553512960}v^8 + \dots.$$

The following expansions are obtained by direct calculation:

$$\begin{aligned} &\hat{b}^T e = \frac{1}{2}, \qquad \hat{b}^T c = \frac{1}{6}, \qquad \hat{b}^T c^2 = \frac{1}{12}, \qquad \hat{b}^T c^3 = \frac{1}{20}, \qquad \hat{b}^T c^4 = \frac{1}{30}, \\ &\hat{b}^T \hat{A} c = \frac{1}{120} + \frac{1}{120}\,\frac{120\,\sin(v) - 120\,v + 20\,v^3 - v^5}{v^5} = \frac{1}{120} + \mathcal{O}(v^2), \\ &\hat{b}^T c\hat{A} c = \frac{1}{180} + \frac{1}{240}\,\frac{120\,\sin(v) - 120\,v + 20\,v^3 - v^5}{v^5} = \frac{1}{180} + \mathcal{O}(v^2), \\ &\hat{b}^T \hat{A} c^2 = \frac{1}{360} + \frac{1}{360}\,\frac{120\,\sin(v) - 120\,v + 20\,v^3 - v^5}{v^5} = \frac{1}{360} + \mathcal{O}(v^2). \end{aligned} \tag{3.12}$$

Hence, the coefficients given in Table 2 satisfy all the order conditions of order two to order six. But it failed to satisfy the order condition for order seven. For example,

$$\hat{b}^T c^2 \hat{A} c = \frac{1}{270} - \frac{1}{20160}\,v^2 + \frac{1}{1451520}\,v^4 - \frac{1}{159667200}\,v^6 \neq \frac{1}{252} + \mathcal{O}(v^2). \tag{3.13}$$

Hence, it is a sixth order method. The original method is obtained by Chan and Tsai [4] and it is denoted as TDRK4(6). Therefore the LTE for TFTDRK4(6) is given below.

$$\begin{aligned} LTE =& \frac{1}{181440}\Big[(23\,f_{yyyyy}f_y + 54\,f_{yyyy}f_{yy} + 6\,f_{xyyyyy})\,f^5 + \Big(195\,f_{yyyy}{f_y}^2 \\ &+15\,f_{yyyyy}f_x + 88\,f_{xy}f_{yyyy} + 100\,f_y f_{xyyyy} + 15\,f_{xxyyyy})\,f^4 + \\ &\Big(633\,f_{yyy}{f_y}^3 + 1188\,{f_{yy}}^2{f_y}^2 + 366\,f_{yyy}f_{yy}f_x + 1386\,f_{xyy}f_y f_{yy} + \\ &576\,f_{xyyy}{f_y}^2 + 60\,f_{xyyyy}f_x + 34\,f_{yyyy}f_{xx} + 170\,f_y f_{xxyyy} + \\ &284\,f_{xyyy}f_{xy} + 20\,f_{xxxyyy})\,f^3 + \Big(708\,f_{yy}{f_y}^4 + 1068\,{f_{yy}}^2 f_y f_x + \\ &2496\,f_{xy}f_{yy}{f_y}^2 + 45\,f_{yyyy}{f_x}^2 + 666\,f_{xyy}f_x f_{yy} + 522\,f_{xyyy}f_y f_x + \\ &156\,{f_{yy}}^2 f_{xx} + 912\,f_{xxy}f_{yy}f_y + 603\,f_{xxyy}{f_y}^2 + 636\,{f_{xy}}^2 f_{yy} + \\ &90\,f_{xxyyy}f_x + 102\,f_{xyyy}f_{xx} + 36\,f_{yyy}f_{xxx} + 114\,f_{xxxy}f_{yy} + \\ &140\,f_{xxxyy}f_y + 324\,f_{xxyy}f_{xy} + 15\,f_{xxxxyy})\,f^2 + \Big(36\,{f_y}^6 + 948\,f_{yy}{f_y}^3 f_x \\ &+168\,{f_x}^2{f_{yy}}^2 + 480\,f_{yy}{f_y}^2 f_{xx} + 1116\,{f_{xy}}^2{f_y}^2 + 90\,f_{xyyy}{f_x}^2 + \\ &336\,f_{xxy}f_{yy}f_x + 432\,f_{xxyy}f_y f_x + 312\,f_{xy}f_{yy}f_{xx} + 180\,f_{yy}f_y f_{xxx} + \\ &258\,f_{xxxy}{f_y}^2 + 948\,f_{xxy}f_{xy}f_y + 60\,f_{xxxyy}f_x + 102\,f_{xxyy}f_{xx} + \\ &20\,f_{yy}f_{xxxx} + 55\,f_y f_{xxxxy} + 148\,f_{xxxy}f_{xy})\,f + 36\,{f_y}^5 f_x + \\ &276\,f_{yy}{f_y}^2{f_x}^2 + 396\,f_{xy}{f_y}^3 f_x + 36\,{f_y}^4 f_{xx} + 15\,f_{yyy}{f_x}^3 + \\ &222\,f_{xyy}f_y{f_x}^2 + 192\,f_{yy}f_y f_{xx}f_x + 540\,{f_{xy}}^2 f_y f_x + 288\,f_{xy}{f_y}^2 f_{xx} \\ &+36\,{f_y}^3 f_{xxx} + 45\,f_{xxyy}{f_x}^2 + 102\,f_{xyy}f_{xx}f_x + 36\,f_{yy}f_{xxx}f_x + \\ &114\,f_{xxxy}f_y f_x + 144\,f_{xy}f_y f_{xxx} + 36\,{f_y}^2 f_{xxxx} + 15\,f_{xxxxy}f_x + \\ &34\,f_{xxxy}f_{xx} + f_{xxxxxx}\Big]\,v^7 + \mathcal{O}(v^8). \end{aligned} \tag{3.14}$$

From (3.14), we can see that the order of TFTDRK4(6) is 6 because all of the coefficients up to v^6 vanished.

4. Problems tested and numerical results

In this section, the performance of the proposed method TFTDRK4(6) is compared with existing RK methods by considering the following problems. All problems below are tested using C code for solving differential equations where the solutions are periodic.

Problem 1 (Harmonic Oscillator).

$$q_1{}'(t) = q_2(t), \qquad q_1(0) = q_{0_1}, \qquad t \in [0, 1000],$$
$$q_2{}'(t) = -\omega^2 q_1(t), \qquad q_2(0) = q_{0_2}.$$

Exact solution is

$$q_1(t) = \bar{c}_1 \sin(\omega t) + \bar{c}_2 \cos(\omega t), \qquad q_2(t) = \bar{c}_3 \omega \cos(\omega t) - \bar{c}_4 \omega \sin(\omega t).$$

Total energy as given in [16]

$$E(q_1, q_2) = \frac{q_1{}^2}{2} + \frac{q_2{}^2}{2} = \frac{\Psi^2}{2},$$

where Ψ depends on the initial conditions.

Problem 2 (Inhomogeneous problem [19]).

$$q_1{}' = q_2, \qquad q_1(0) = 1, \qquad t \in [0, 1000\pi],$$
$$q_2{}' = -100 q_1 + 99 \sin(t), \qquad q_2(0) = 11.$$

Exact solution is

$$q_1(t) = \cos(10t) + \sin(10t) + \sin(t), \qquad q_2(t) = -10 \sin(10t) + 10 \cos(10t) + \cos(t).$$

Problem 3 (An "almost" Periodic Orbit problem [18]).

$$q_1{}' = q_2, \qquad q_1(0) = 1, \qquad t \in [0, 1000\pi],$$
$$q_2{}' = -q_1 + 0.001 \cos(t), \qquad q_2(0) = 1,$$
$$q_3{}' = q_4, \qquad q_3(0) = 0,$$
$$q_4{}' = -q_3 + 0.001 \sin(t), \qquad q_4(0) = 0.995.$$

Exact solution is

$$q_1(t) = \cos(t) + 0.0005t \sin(t), \qquad q_2(t) = -\sin(t) + 0.0005t \cos(t) + 0.0005t \sin(t),$$
$$q_3(t) = \sin(t) - 0.0005t \cos(t), \qquad q_4(t) = \cos(t) + 0.0005t \sin(t) - 0.0005 \cos(t).$$

Problem 4 (Two-body problem[17]).

$$q_1{}' = q_2, \qquad q_1(0) = 1, \qquad t \in [0, 1000\pi],$$
$$q_2{}' = -\frac{q_1}{(\sqrt{q_1{}^2 + q_3{}^2})^3}, \qquad q_2(0) = 0,$$
$$q_3{}' = q_4, \qquad q_3(0) = 0,$$
$$q_4{}' = -\frac{q_3}{(\sqrt{q_1{}^2 + q_3{}^2})^3}, \qquad q_4(0) = 1.$$

Exact solution is

$$q_1(t) = \cos(t), \qquad q_2(t) = -\sin(t), \qquad q_3(t) = \sin(t), \qquad q_4(t) = \cos(t).$$

Problem 5 (Duffing problem[14]).

$$q_1{}' = q_2, \qquad q_1(0) = 0.200426728067,$$
$$q_2{}' = -q_1 - q_1{}^3 + 0.002 \cos(1.01t), \qquad q_2(0) = 0, \qquad t \in [0, 1000\pi].$$

Exact solution is

$$q_1(t) = 0.200179477536\cos(1.01t) + 2.46946143\times 10^{-4}\cos(3.03t) + 3.04014\times 10^{-7}\cos(5.05t) + 3.74\times 10^{-10}\cos(7.07t),$$

$$q_2(t) = -0.2021812723\sin(1.01t) - 7.482468133\times 10^{-4}\sin(3.03t) - 1.53527070\times 10^{-6}\sin(5.05t) - 2.64418\times 10^{-9}\sin(7.07t).$$

The following notations are used in Figures 1–11 :

- **TFTDRK4(6):** New trigonometrically-fitted TDRK method of sixth order four stage derived in this paper.
- **TFTDRK3(5):** Existing fifth order three stage trigononometrically-fitted TDRK method developed by Zhang et al. [20].
- **TFRKS6(5)**: Existing fifth order six stage trigononometrically-fitted RK method derived by Simos [2].
- **TFRKAS6(5)**: Existing fifth order six stage trigononometrically-fitted RK method given in Anastassi and Simos [2].
- **PFAFRKC7(5)**: Existing fifth order seven stage phase-fitted and amplification-fitted RK method developed by Chen et al. [6].
- **RKB7(6)**: Existing sixth order seven stage RK method developed by Butcher [3].
- **TDRK4(6)**: Existing sixth order four stage TDRK method given by Chan and Tsai [4].

The performance of these numerical results are represented graphically in the following Figures 1–11:

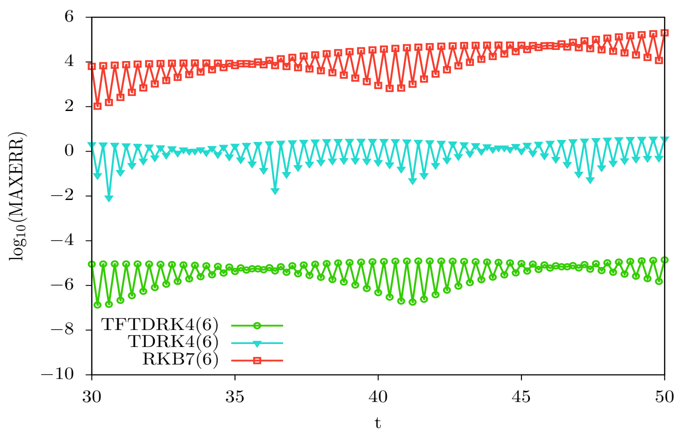

Figure 1. (Energy Conservation). The logarithm error of energy (MAXERR) at each integration point when solving the harmonic oscillator (Problem 1) for $\omega = 8$ with initial condition $q_{0_1} = 1, q_{0_2} = -2$ and $h = 1/5$ with $t_{end} = 1000$.

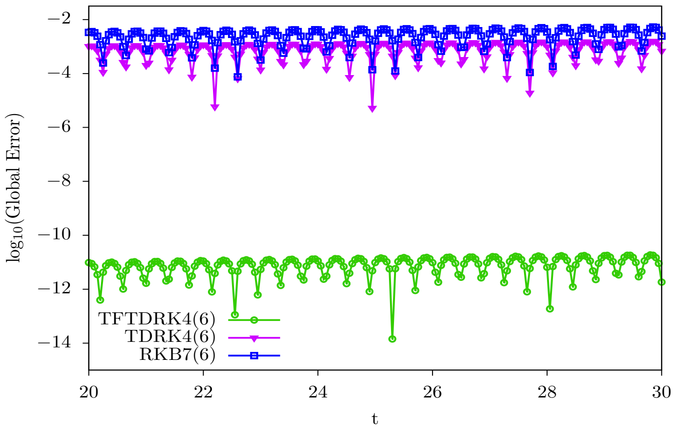

Figure 2. The error at each integration point when solving the harmonic oscillator (Problem 1) with $\omega = 8$ with initial condition $q_{0_1} = 1, q_{0_2} = -2$ and $h = 1/20$ with $t_{end} = 1000$.

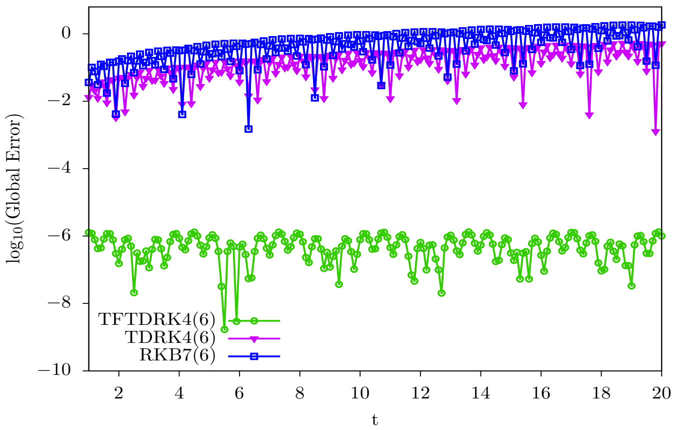

Figure 3. The global error at each integration point when solving the inhomogeneous problem (Problem 2) with $h = 1/10$.

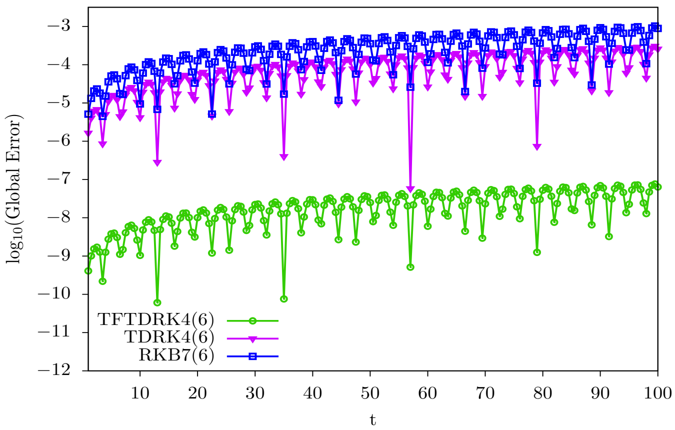

Figure 4. The global error at each integration point when solving the "almost" periodic problem (Problem 3) with $h = 1/2$.

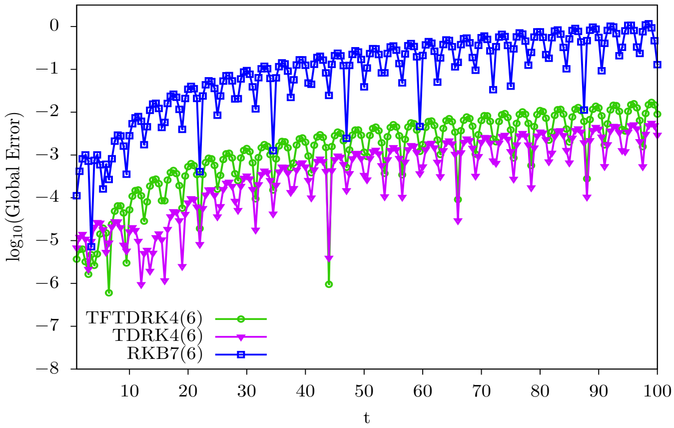

Figure 5. The global error at each integration point when solving the two-body problem (Problem 4) with $h = 1/2$.

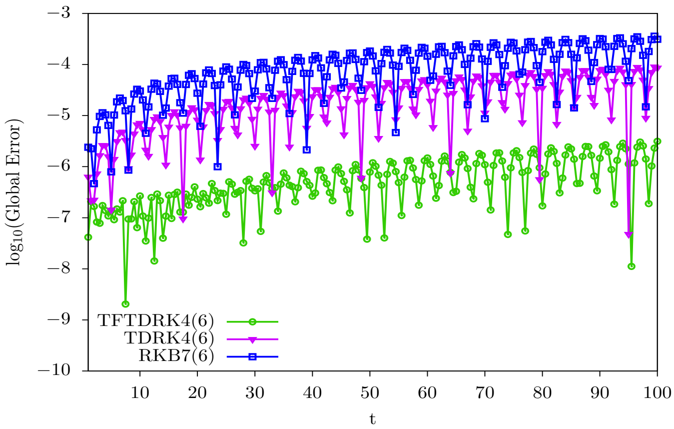

Figure 6. The global error at each integration point when solving the Duffing problem (Problem 5) with $h = 1/2$.

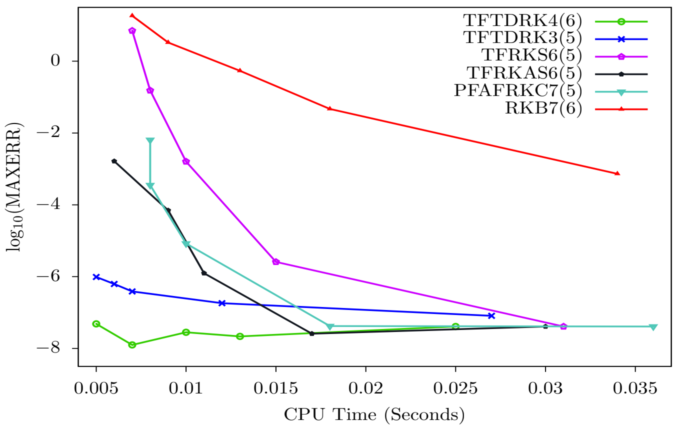

Figure 7. The efficiency curve for the harmonic oscillator (Problem 1) with $\lambda = 8$ and time step $h = 1/10 - 1/50(i), i = 0, \ldots, 4$ with $t_{end} = 1000$.

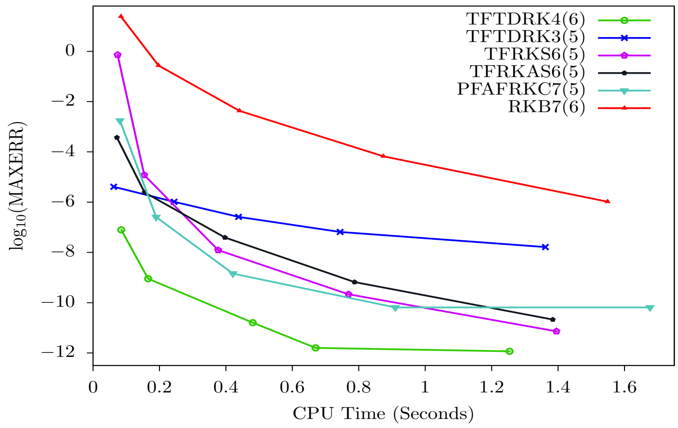

Figure 8. The efficiency curve for the inhomogeneous problem (Problem 2) with time step $h = 1/2^i, i = 4, \ldots, 8$.

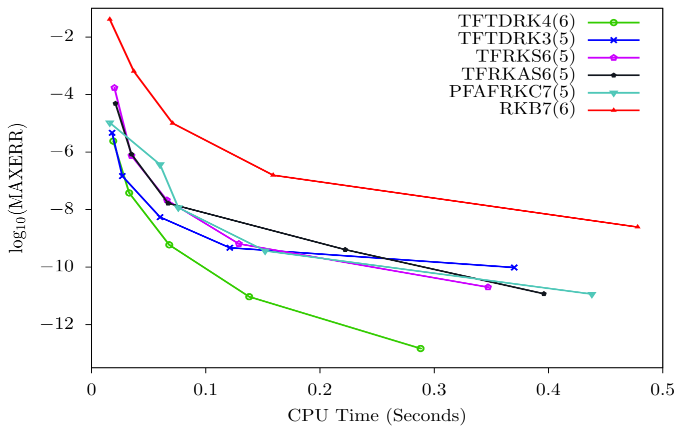

Figure 9. The efficiency curve for the "almost" periodic problem (Problem 3) with time step $h = 1/2^i, i = 1, \ldots, 5$.

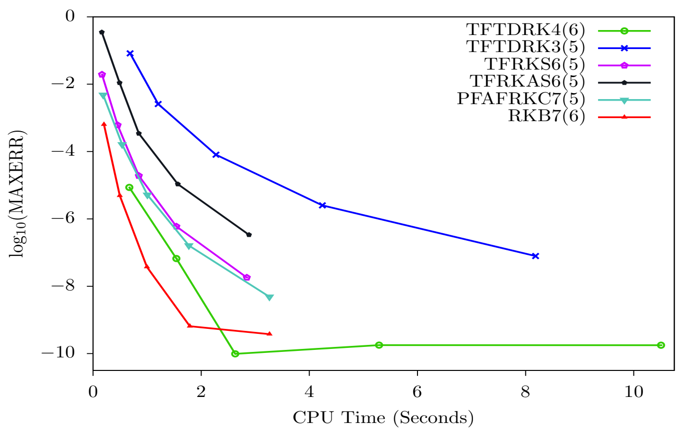

Figure 10. The efficiency curve for the two-body problem (Problem 4) with time step $h = 1/2^i, i = 4, \ldots, 8$.

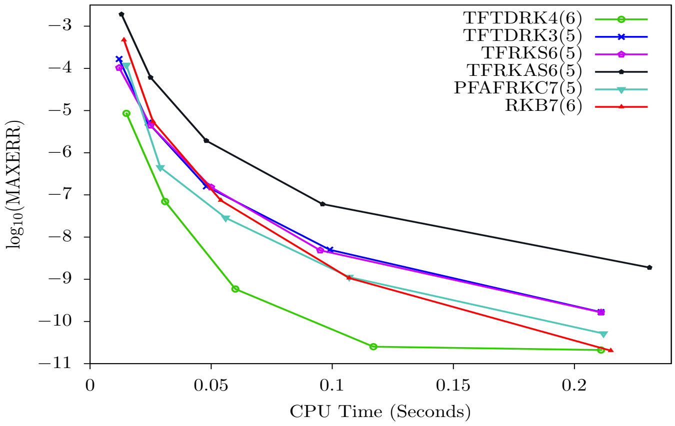

Figure 11. The efficiency curve for the Duffing problem (Problem 5) with time step $h = 1/2^i, i = 1, \ldots, 5$.

5. Discussion

The results show the typical properties of the new trigonometrically-fitted TDRK method, TFTDRK4(6) which have been derived earlier. The derived method are compared with some well-known existing RK methods. Figure 1 shows the error of the energy at each integration point. Based on Figure 1, the trigonometrically-fitted TDRK method conserved the energy by having smaller magnitude of energy error compared to TDRK4(6) and RKB7(6). In Figures 2-6, the logarithm number of global error versus the time of integration for different time step, h for various physical problems are plotted. From Figures 2–4 and Figure 6, the global error produced by the TFTDRK4(6) method have smaller global error compared to TDRK4(6) and RKB7(6). Meanwhile in Figures 5, TFTDRK4(6) have slightly bigger global error compared to TDRK4(6).

Next, the global error and the efficiency of the method over a long period of integration are plotted. Figures 7–11 represent the efficiency and accuracy of the method developed by plotting the graph of the logarithm of the maximum global error against the CPU time (seconds) for a longer periods of computations. From the graphs plotted, TFTDRK4(6) method has the smallest maximum global error compared to other existing RK methods which have trigonometrically-fitted and phase-fitted and amplification-fitted properties. In Figure 7, as the value of h becomes smaller, the maximum global error of the TFTDRK4(6) method seems to flatten at the end of the curve. The accuracy of the method depends on the step-size, h and the frequency, λ. The derived method will converge to its original method as the value of h becomes smaller. In Figure 10, TFTDRK4(6) method has a longer CPU time but its maximum global error still outperform other existing methods.

6. Conclusion

In this research, a trigonometically-fitted higher order TDRK method are developed. Based on the numerical results obtained, it can be concluded that the new TFTDRK4(6) method is more promising compared to other well-known existing explicit RK methods in terms of accuracy and the CPU seconds.

References

[1] F. Adel, N. Senu, F. Ismail, and Z.A. Majid, *A New Efficient Phase-Fitted and Amplification-Fitted Runge-Kutta Method for Oscillatory Problems*, Int. J. Pure Appl. Math. **107**, 69-86, 2016.

[2] Z.A. Anastassi and T.E. Simos, *Trigonometrically fitted Runge-Kutta methods for the numerical solution of the Schrödinger equatio*n, J. Math. Chem. **37** (3), 281-293, 2005.

[3] J.C. Butcher, *On Runge-Kutta processes of high order*, J. Aust. Math. Soc. **4** (2), 179-194, 1964.

[4] R.P. Chan and A.Y. Tsai, *On explicit two-derivative Runge-Kutta methods*, Numer. Algorithms **53**, 171–194, 2010.

[5] Z. Chen, J. Li, R. Zhang and X. You, *Exponentially Fitted Two-Derivative Runge-Kutta Methods for Simulation of Oscillatory Genetic Regulatory Systems*, Comput. Math. Methods Med. **2015**, 689137, 2015.

[6] Z. Chen, X. You, X. Shu and M. Zhang, *A new family of phase-fitted and amplification-fitted Runge-Kutta type methods for oscillators*, J. Appl. Math. **2012**, 1-27, 2012.

[7] M.A. Demba, N. Senu and F. Ismail, *Trigonometrically-fitted explicit four-stage fourth-order Runge-Kutta-Nyström method for the solution of initial value problems with oscillatory behavior*, Global Journal of Pure and Applied Mathematics, **12** (1), 67-80, 2016.

[8] M.A. Demba, N. Senu and F. Ismail, *Fifth-Order Four-Stage Explicit Trigonometrically-Fitted Runge-Kutta-Nyström Methods*, Recent Advances in Mathematical Sciences, 27-36, 2016.

[9] M.A. Demba, N. Senu and F. Ismail, *A 5(4) Embedded Pair of Explicit Trigonometrically-Fitted Runge-Kutta-Nyström Methods for the Numerical Solution of Oscillatory Initial Value Problems*, Math. Comput. Appl. **21** (4), 46, 2016.

[10] M.A. Demba, N. Senu and F. Ismail, *A symplectic explicit trigonometrically-fitted Runge-Kutta-Nyströ method for the numerical solution of periodic problems*, Int. J. Appl. Eng. Res. **11** (11), 7495-7500, 2016.

[11] M.A. Demba, N. Senu and F. Ismail, *An Embedded 4(3) Pair of Explicit Trigonometrically-Fitted Runge-Kutta-Nyström Method for Solving Periodic Initial Value Problems*, Appl. Math. Sci. **17**, 819-838, 2017.

[12] Y. Fang, X. You and Q. Ming, *Exponentially Fitted Two-Derivative Runge-Kutta Methods For The Schrödinger Equation*, Int. J. Mod. Phys. C **24** (10), 1350073, 2013.

[13] F.A. Fawzi, N. Senu, F. Ismail, and Z.A. Majid, *A Phase-Fitted and Amplification-Fitted Modified Runge-Kutta Method of Fourth Order for Periodic Initial Value Problems*, Research and Education in Mathematics (ICREM7), 2015 International Conference on. IEEE, 25-28, 2015.

[14] A.A. Kosti, Z.A. Anastassi and T.E. Simos, *An optimized explicit Runge-Kutta-Nyström method for the numerical solution of orbital and related periodical initial value problems*, Comput. Phys. Commun. **183**, 470-479, 2012.

[15] T. Monovasilis, Z. Kalogiratou and T.E. Simos, *Construction of Exponentially Fitted Symplectic Runge-Kutta-Nyström Methods from Partitioned RungeKutta Methods*, Mediterr. J. Math. **13** (4), 2271-2285, 2015.

[16] P. Pokorny, *Continuation of periodic solutions of dissipative and conservative systems: application to elastic pendulum*, Math. Probl. Eng. **2009**, Article ID 104547, 2009.

[17] T.E. Simos, *Family of fifth algebraic order trigonometrically fitted Runge-Kutta methods for the numerical solution of the Schrödinger equation*, Comp. Mater. Sci. **34** (4), 342-354, 2005.

[18] E. Stiefel and D.G. Bettis, *Stabilization of Cowell's method*, Numer. Math. **13**, 154-175, 1969.

[19] H. Van de Vyver, *An Explicit Numerov-Type Method for Second-Order Differential Equations with Oscillating Solutions*, Comput. Math. Appl. **53**, 1339-1348, 2007.

[20] Y. Zhang, H. Che, Y. Fang and X. You, *A new trigonometrically fitted two-derivative Runge-Kutta method for the numerical solution of the Schrödinger equation and related problems*, J. Appl. Math. **2013**, Article ID 937858, 2013.