



Stability analysis for some numerical schemes of partial differential equation with extra measurements

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Abstract

This paper is devoted to study the stability analysis of some finite difference schemes for an inverse problem with unknowns time-dependent coefficients subject to extra measurements. We prove that the popular forward time centered space scheme is a conditional method. But the backward time centered space and Crank Nicolson methods are suitable schemes because they are unconditional methods. We justify this advantage of the stability analysis versus the some numerical methods with an example. All the results and a numerical example are in two-dimensional setting.

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1. Introduction

Various types of physical and environmental phenomena such as heat, electrostatic, electrodynamics, fluid dynamics and pollution could be modelled in the form of partial differential equations (PDEs) [1, 2, 19, 20]. Usually, in study of these phenomena all data of problem is available. However, for some practical problems, part of boundary data, or initial data, or diffusion coefficients, or source term may not be given and we have to find them by extra measurement data which will yield to some inverse problems like evolutionary inverse problems, boundary inverse problems, coefficient inverse problems and source inverse problems, respectively [4, 6, 13, 18, 22, 24, 27].

In the past several decades, various numerical techniques have been developed to solve inverse problems [3, 10, 14, 15, 21]. In these problems extra measurement data are used to obtain unique solution. Extra measurement of some problems play an important role in mathematical science, physics and engineering. These problems are widely envisaged in the modeling of many physical phenomena, namely diffusion and conduction of natural materials [29].

In this paper, we consider the following coefficient inverse problem with variable coefficients in a general bounded domain.

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Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with sufficient smooth boundary $\partial\Omega$. The coefficient inverse problem we considered is to determine unknowns coefficients $(a_{jm}(x, t))_{j,m=1}^2$ from the following equations

$$\begin{cases} \frac{\partial u}{\partial t}(x, t) = \sum_{j,m=1}^2 a_{jm}(x, t) \frac{\partial^2 u(x, t)}{\partial x_j \partial x_m} \\ \quad + \varepsilon \cdot \nabla u(x, t) + d(t)u(x, t), & x \in \Omega, \quad 0 < t \leq T, \\ u(x, 0) = f(x), & x \in \bar{\Omega}, \\ u(x, t) = g(t), & x \in \partial\Omega, \quad 0 \leq t \leq T, \end{cases} \quad (1.1)$$

subject to extra measurements

$$u(x^*, t) = E(t), \quad \nabla u(x^*, t) = F(t), \quad 0 \leq t \leq T, \quad (1.2)$$

in which $x^* \in \Omega$, d , f , g , E and F are given functions and ε is a constant function. We assume that $(a_{jm}(x, t))_{j,m=1}^2$ are sufficiently smooth coefficients.

Certain types of physical problems can be modeled by (1.1) [11, 16, 28]. In recent decades, various numerical methods have been used to approximate the solution of spacial case of (1.1). In [6] the author reduced the problem to a nonlinear integral equation. This approach depends on the explicit form of the fundamental solution of the heat operator and does not easily extend to the higher dimensional cases. In [4] backward Euler finite difference method is applied. Also, the discrete version of the maximum principle for finite difference schemes is used to show the stability of method. The finite element method is applied by the authors of [7]. Dehghan [8] used several finite difference schemes for identifying the function $u(x, t)$ and unknown coefficient $a(t)$ in a one dimensional case of (1.1). It has been shown that these methods are efficient but have not been discussed about the stability of them.

In this paper, we generalized the problem in [8] and attempt to complete the work presented in that. We use three finite difference schemes, forward time centered space (FTCS), backward time centered space (BTCS) and Crank- Nicolson (CN) method for inverse problem (1.1). Then, we discuss about the stability of them.

Various methods have been constructed for the analysis of stability among them von Neumann stability analysis is the most common method used to determine the stability. This method is based on Fourier analysis therefore it is usually limited to linear PDEs with constant coefficients [23, 26]. Although, the condition of linear equation with constant coefficients are not as restrictive as it might seem. In the case of the equation is nonlinear, we can apply linearization methods and transform it into a linear equation. When coefficients are not constant, we implement principle of frozen coefficients. According to this principle, we consider PDEs where coefficients do not change too rapidly. So, such coefficients can be considered to be almost constant at each point in space and time [9, 17].

This paper is organized as follow. In Section 2, we discuss about the von Neumann stability analysis for some PDEs with variable coefficients. Implementation of FTCS, BTCS and CN methods for inverse problem (1.1) and determine the stability region of these methods are presented in Section 3. In Section 4, we give a numerical example to illustrate our theoretical discussion. Finally, we give a conclusion in Section 5.

2. Von Neumann stability analysis for linear parabolic PDEs with variable coefficients

In this section, we explain how to apply von Neumann stability analysis as presented in [5] for linear partial differential equations (PDEs) with variable coefficients.

Consider the following problem:

$$\begin{cases} u_t(x, t) = a(x, t)u_{xx}(x, t), & x \in (0, 1), \quad 0 < t < T, \\ u(x, 0) = \varphi(x), & x \in [0, 1], \\ u(0, t) = f(t), & 0 \leq t \leq T, \\ u(1, t) = g(t), & 0 \leq t \leq T, \end{cases} \quad (2.1)$$

where T is a positive constant, $\varphi(x)$, $f(t)$, $g(t)$ and $a(x, t)$ are known functions and $a(x, t) > 0$. In this problem $a(x, t)$ is not constant, so to implement the von Neumann stability analysis, we use the principle of frozen coefficients. Suppose

$$Q_T = \{(x, t) \mid 0 \leq x \leq 1, \quad 0 \leq t \leq T\},$$

is discretized uniformly into an $M \times N$ mesh with the spatial step size $h = \frac{1}{M}$ in x direction and the time step size $k = \frac{T}{N}$, respectively. Replace the region Q_T by a set of grid points (x_p, t_q) are defined by

$$\begin{cases} x_p = ph, & p = 0, 1, \dots, M, \\ t_q = qk, & q = 0, 1, \dots, N, \end{cases}$$

in which M and N are integers. Assume that φ_p, f^q, g^q and u_p^q be the notations of finite-difference approximation of $\varphi(x_p), f(t_q), g(t_q)$ and $u(x_p, t_q)$, respectively.

We apply FTCS method to (2.1) (see [25]), so:

$$\begin{cases} \frac{u_p^{q+1} - u_p^q}{k} = a(x, t) \frac{u_{p+1}^q - 2u_p^q + u_{p-1}^q}{h^2}, & p = 1, 2, \dots, M-1, \\ u_p^0 = \varphi_p, & p = 0, 1, 2, \dots, M, \\ u_0^q = f^q, & q = 0, 1, 2, \dots, N, \\ u_M^q = g^q, & q = 0, 1, 2, \dots, N. \end{cases} \quad (2.2)$$

We freeze the coefficient $a(x, t)$ of problem (2.2), on the other hand we assume that $a(x, t)$ is fixed at each point of space and time. We apply the von Neumann stability analysis to find the stability region. Also, we assume that the solution is of the form

$$u_p^q := G^q e^{i\beta ph}.$$

Substitution of the above expression into first equation of (2.2) yields

$$G^{q+1} e^{i\beta ph} - G^q e^{i\beta ph} = ra(x, t) [G^q e^{i\beta(p+1)h} - 2G^q e^{i\beta ph} + G^q e^{i\beta(p-1)h}]. \quad (2.3)$$

Here $G = e^{i\alpha k}$ is the growth factor such that $-\pi < \alpha < \pi$ is the grid wave number and $r = \frac{k}{h^2}$. After simplifying equation (2.3), we have

$$G - 1 = ra(x, t) [e^{i\beta h} - 2 + e^{-i\beta h}],$$

therefore

$$\begin{aligned} G &= 1 + ra(x, t) [2 \cos(\beta h) - 2] \\ &= 1 - 4ra(x, t) \sin^2\left(\frac{\beta h}{2}\right). \end{aligned}$$

The stability condition for (2.2) is $|G| \leq 1$. Hence

$$-1 \leq 1 - 4ra(x, t) \sin^2\left(\frac{\beta h}{2}\right) \leq 1,$$

and

$$r \leq \frac{1}{2a(x, t) \sin^2\left(\frac{\beta h}{2}\right)}.$$

In order to place the strongest limitation on r , we have to choose βh such that the right hand side of inequality be minimum. This minimum happens when $\sin^2(\frac{\beta h}{2}) = 1$. Therefore, we have

$$r \leq \frac{1}{2a(x, t)}. \tag{2.4}$$

The above relation was achieved by the principle of frozen coefficient [9].

Relation (2.4) can be interpreted in two different ways.

(i) If constant values of k , h and hence r are employed over the entire grid, then we have to ensure that

$$r \leq \frac{1}{2 \max_{x,t} a(x, t)}.$$

(ii) If the time step k changes, then at every time level, k is to be chosen so that

$$r(t) \leq \frac{1}{2 \max_x a(x, t)}.$$

Let us consider the following problem

$$\begin{cases} u_t(x, t) = a(x, t)u_{xx}(x, t) + b(x, t)u_x(x, t) + c(x, t)u(x, t), & x \in (0, 1), \\ & 0 < t < T, \\ u(x, 0) = \varphi(x), & x \in [0, 1], \\ u(0, t) = f(t), & 0 \leq t \leq T, \\ u(1, t) = g(t), & 0 \leq t \leq T, \end{cases} \tag{2.5}$$

where T is a positive constant and a, b, c, φ, f and g are given functions, $a(x, t) > 0$ and $c(x, t) \leq 0$ for $(0, 1) \times (0, T)$. If each of $a := a(x, t)$, $b := b(x, t)$ and $c := c(x, t)$ are constant, then problem (2.5) has a solution by the form $u(x, t) = \varphi(x) \exp(ct)$.

By applying the FTCS method for (2.5), we have

$$\frac{u_p^{q+1} - u_p^q}{k} = a \frac{u_{p+1}^q - 2u_p^q + u_{p-1}^q}{h^2} + b \frac{u_{p+1}^q - u_{p-1}^q}{2h} + cu_p^q,$$

$$p = 1, 2, \dots, M - 1, \quad q = 0, 1, 2, \dots, N - 1.$$

Substitution of $u_p^q = G^q e^{i\beta p h}$ into the above difference equation leads to

$$G - 1 = ra[e^{i\beta h} - 2 + e^{-i\beta h}] + b \frac{k}{2h}[e^{i\beta h} - e^{-i\beta h}] + ck.$$

Hence

$$G \leq 1 - 4ra \sin^2(\frac{\beta h}{2}) + i(2b \frac{k}{h}) \sin(\frac{\beta h}{2}) \cos(\frac{\beta h}{2}),$$

satisfies the von Neumann condition if

$$|1 - 4ra \sin^2(\frac{\beta h}{2}) + i(2b \frac{k}{h}) \sin(\frac{\beta h}{2}) \cos(\frac{\beta h}{2})| \leq 1.$$

After simplification above relation, we have

$$r \leq \frac{2a - b^2 k \cos^2(\frac{\beta h}{2})}{4a^2 \sin^2(\frac{\beta h}{2})}.$$

Therefore, the principle of frozen coefficients yields the following stability analysis based on criterion for the FTCS method to (2.5)

$$r \leq \frac{1}{2a(x, t)}.$$

3. Von Neumann stability analysis for parabolic inverse problem

In this section, we give some theorems on the von Neumann stability analysis based on different schemes of finite difference for inverse coefficient problem (1.1) in one and two dimensional case.

Theorem 3.1. *If we consider $\Omega = (0, 1)$, $(a_{jm}(x, t))_{j,m=1}^2 := \begin{pmatrix} a(t) & 0 \\ 0 & 0 \end{pmatrix}$, $a(t) > 0$, $d(t) \leq 0$ and $\{\varepsilon_m\}_{m=1}^2 := \{\varepsilon, 0\}$ in (1.1), then the FTCS method for (1.1) is stable whenever $r \leq \frac{1}{2a(t)}$.*

Proof. By using (1.1) and (1.2), we have

$$a(t) = \frac{E'(t) - \varepsilon F(t) - d(t)E(t)}{u_{xx}(x^*, t)}. \quad (3.1)$$

Thus, problem (1.1) with some tedious manipulation yields the following problem

$$\begin{cases} u_t(x, t) = \frac{E'(t) - \varepsilon F(t) - d(t)E(t)}{u_{xx}(x^*, t)} u_{xx}(x, t) + \varepsilon u_x(x, t) + d(t)u(x, t), \\ u(x, 0) = f(x), \\ u(0, t) = g_1(t), \\ u(1, t) = g_2(t), \end{cases} \quad \begin{matrix} x \in \Omega, & 0 < t \leq T, \\ x \in \bar{\Omega}, \\ 0 \leq t \leq T, \\ 0 \leq t \leq T. \end{matrix} \quad (3.2)$$

Let $v(x, t) = u_{xx}(x, t)$, then we have

$$\begin{cases} v_t(x, t) = \frac{E'(t) - \varepsilon F(t) - d(t)E(t)}{v(x^*, t)} v_{xx}(x, t) + \varepsilon v_x(x, t) + d(t)v(x, t), \\ v(x, 0) = f''(x), \\ v(0, t) = \frac{g_1'(t) - d(t)g_1(t)}{E'(t) - \varepsilon F(t) - d(t)E(t)} v(x^*, t), \\ v(1, t) = \frac{g_2'(t) - d(t)g_2(t)}{E'(t) - \varepsilon F(t) - d(t)E(t)} v(x^*, t), \end{cases} \quad \begin{matrix} x \in \Omega, & 0 < t \leq T, \\ x \in \bar{\Omega}, \\ 0 \leq t \leq T, \\ 0 \leq t \leq T. \end{matrix} \quad (3.3)$$

This problem is nonlinear, therefore if we give the von Neumann stability analysis, then we have to reduce it to a linear form. For this purpose, we assume w is another solution of (3.3) that is nearby to v . Their difference satisfies

$$\begin{aligned} (v_t - w_t)(x, t) &= \left(\frac{E'(t) - \varepsilon F(t) - d(t)E(t)}{v(x^*, t)} v_{xx}(x, t) + \varepsilon v_x(x, t) + d(t)v(x, t) \right) \\ &\quad - \left(\frac{E'(t) - \varepsilon F(t) - d(t)E(t)}{w(x^*, t)} w_{xx}(x, t) + \varepsilon w_x(x, t) + d(t)w(x, t) \right). \end{aligned} \quad (3.4)$$

We linearized (3.4) about w . Define

$$f(v, v_x, v_{xx}) = \frac{E'(t) - \varepsilon F(t) - d(t)E(t)}{v(x^*, t)} v_{xx}(x, t) + \varepsilon v_x(x, t) + d(t)v(x, t).$$

By using an equivalent form of the chain rule for function $f(v, v_x, v_{xx})$, we have

$$df(v, v_x, v_{xx}) = f_v dv + f_{v_x} dv_x + f_{v_{xx}} dv_{xx}. \quad (3.5)$$

If we replace differentials by small but finite increase, then (3.5) will be

$$\Delta f(v, v_x, v_{xx}) \simeq f_v \Delta v + f_{v_x} \Delta v_x + f_{v_{xx}} \Delta v_{xx}, \quad (3.6)$$

where

$$\begin{aligned} \Delta v &= v - w, \quad \Delta v_x = v_x - w_x, \quad \Delta v_{xx} = v_{xx} - w_{xx}, \\ \Delta f(v, v_x, v_{xx}) &= f(v, v_x, v_{xx}) - f(w, w_x, w_{xx}). \end{aligned} \tag{3.7}$$

Based on (3.6) and (3.7), the right hand side of (3.5) is as follows:

$$\begin{aligned} & \left(d(t) - \frac{E'(t) - \varepsilon F(t) - d(t)E(t)}{v^2(x^*, t)} v_{xx} \right) \Delta v(x, t) \\ & + \varepsilon \Delta v_x(x, t) + \frac{E'(t) - \varepsilon F(t) - d(t)E(t)}{v(x^*, t)} \Delta v_{xx}(x, t). \end{aligned} \tag{3.8}$$

On the other hand, we know that

$$v_t(x, t) = f(v, v_x, v_{xx}),$$

so, we have

$$\begin{aligned} \Delta v_t(x, t) &\approx \frac{E'(t) - \varepsilon F(t) - d(t)E(t)}{v(x^*, t)} \Delta v_{xx}(x, t) \\ &+ \varepsilon \Delta v_x(x, t) + \left(d(t) - \frac{E'(t) - \varepsilon F(t) - d(t)E(t)}{v^2(x^*, t)} v_{xx}(x, t) \right) \Delta v(x, t). \end{aligned} \tag{3.9}$$

Equation (3.9) is linearized of (3.3) and dependent on Δv . If we compare (3.9) with equation $u_t = a(x, t)u_{xx}(x, t) + b(x, t)u_x(x, t) + c(x, t)u(x, t)$, we have

$$\begin{aligned} a(x, t) &:= \frac{E'(t) - \varepsilon F(t) - d(t)E(t)}{v(x^*, t)}, \quad b(x, t) := \varepsilon, \\ c(x, t) &:= d(t) - \frac{E'(t) - \varepsilon F(t) - d(t)E(t)}{v^2(x^*, t)} v_{xx}(x, t). \end{aligned}$$

According to the principle of frozen coefficients, the necessary condition for stability is

$$r \leq \frac{v(x^*, t)}{2(E'(t) - \varepsilon F(t) - d(t)E(t))}.$$

If we replace $v(x^*, t)$ by $u_{xx}(x^*, t)$ and using (3.1), then the proof is complete. □

Theorem 3.2. *If we consider $\Omega = (0, 1)$, $(a_{jm}(x, t))_{j,m=1}^2 := \begin{pmatrix} a(t) & 0 \\ 0 & 0 \end{pmatrix}$, $a(t) > 0$, $d(t) \leq 0$ and $\{\varepsilon_m\}_{m=1}^2 := \{\varepsilon, 0\}$ in (1.1), then the BTCS method for (1.1) is unconditionally stable for all $r > 0$.*

Proof. According to the proof of Theorem 3.1, we obtain an equation by the form

$$u_t(x, t) = a(x, t)u_{xx}(x, t) + b(x, t)u_x(x, t) + c(x, t)u(x, t). \tag{3.10}$$

Assume coefficients $a(x, t)$, $b(x, t)$ and $c(x, t)$ are freezed. By applying the BTCS for (3.10), we have

$$\begin{aligned} \frac{u_p^{q+1} - u_p^q}{k} &= a(x, t) \frac{u_{p+1}^{q+1} - 2u_p^{q+1} + u_{p-1}^{q+1}}{h^2} + b(x, t) \frac{u_{p+1}^{q+1} - u_{p-1}^{q+1}}{2h} + c(x, t)u_p^{q+1}, \\ p &= 1, 2, \dots, M - 1, \quad q = 0, 1, 2, \dots, N - 1. \end{aligned} \tag{3.11}$$

We apply von Neumann stability analysis for (3.11), so

$$\begin{aligned} G^{q+1}e^{i\beta ph} - G^q e^{i\beta ph} &= \\ r a(x, t) [G^{q+1}e^{i\beta(p+1)h} - 2G^{q+1}e^{i\beta ph} + G^{q+1}e^{i\beta(p-1)h}] \\ &+ b(x, t) \frac{k}{2h} [G^{q+1}e^{i\beta(p+1)h} - G^{q+1}e^{i\beta(p-1)h}] + c(x, t)kG^{q+1}e^{i\beta ph}. \end{aligned}$$

After simplification, we have

$$\begin{aligned} G - 1 &= ra(x, t)G[e^{i\beta h} - 2 + e^{-i\beta h}] + b(x, t)\frac{k}{2h}G[e^{i\beta h} - e^{-i\beta h}] + c(x, t)kG \\ &= ra(x, t)G[-4\sin^2(\frac{\beta h}{2})] + b(x, t)\frac{k}{h}G[i\sin(\beta h)] + c(x, t)kG. \end{aligned}$$

Thus

$$G[1 + 4ra(x, t)\sin^2(\frac{\beta h}{2}) - ib(x, t)\frac{k}{h}\sin(\beta h) - c(x, t)k] = 1.$$

From above equation, we have

$$G = \frac{1}{1 + 4ra(x, t)\sin^2(\frac{\beta h}{2}) - ib(x, t)\frac{k}{h}\sin(\beta h) - c(x, t)k}.$$

So for all $r > 0$, we have

$$|G| \leq 1.$$

We emphasize that if G satisfies $|G| \leq 1$, then we take $c(x, t) \leq 0$. □

Theorem 3.3. *If we consider $\Omega = (0, 1)$, $(a_{jm}(x, t))_{j,m=1}^2 := \begin{pmatrix} a(t) & 0 \\ 0 & 0 \end{pmatrix}$, $a(t) > 0$, $c(x, t) \leq 0$, $d(t) \leq 0$ and $\{\varepsilon_m\}_{m=1}^2 := \{\varepsilon, 0\}$ in (1.1), then the CN method for (1.1) is unconditionally stable for all $r > 0$.*

Proof. As seen in the proof of Theorem 3.1, we obtain an equation like (2.5). By implementation of Crank Nicolson method for (2.5), we have

$$\begin{aligned} \frac{u_p^{q+1} - u_p^q}{k} &= \frac{1}{2}a(x, t) \left[\frac{u_{p+1}^{q+1} - 2u_p^{q+1} + u_{p-1}^{q+1}}{h^2} + \frac{u_{p+1}^q - 2u_p^q + u_{p-1}^q}{h^2} \right] \\ &+ \frac{1}{2}b(x, t) \left[\frac{u_{p+1}^{q+1} - u_{p-1}^{q+1}}{2h} + \frac{u_{p+1}^q - u_{p-1}^q}{2h} \right] + c(x, t)u_p^q, \end{aligned} \tag{3.12}$$

for $p = 1, 2, \dots, M - 1$, and $q = 0, 1, 2, \dots, N - 1$.

If we put $u_p^q := G^q e^{i\beta ph}$ in (3.12), we have

$$\begin{aligned} G - 1 &= \frac{1}{2}ra(x, t)(2\cos(\beta h) - 2)(G + 1) + i(\frac{k}{2h})b(x, t)\sin(\beta h)(G + 1) \\ &+ c(x, t)k = \left(-2ra(x, t)\sin^2(\frac{\beta h}{2}) + i(\frac{k}{2h})b(x, t)\sin(\beta h) \right) (G + 1) + c(x, t)k. \end{aligned}$$

We put $A = -2ra(x, t)\sin^2(\frac{\beta h}{2}) + i(\frac{k}{2h})b(x, t)\sin(\beta h)$, so for all $r > 0$, we have

$$G = 1 + A(G + 1) + c(x, t)k \leq 1 + A(G + 1),$$

and

$$|G| \leq \left| \frac{1 + A}{1 - A} \right|$$

where $|1 - A| \geq 1$ and $|\frac{1+A}{1-A}| \leq 1$. To prove these inequalities, we have

$$|1 - A|^2 = \left(1 + 2ra(x, t)\sin^2(\frac{\beta h}{2}) \right)^2 + \frac{1}{4}krb^2(x, t)\sin^2(\beta h).$$

It is clear that $|1 - A|^2 \geq 1$ so $|1 - A| \geq 1$.

On the other hand

$$\begin{aligned} \left| \frac{1 + A}{1 - A} \right|^2 &= \left| \frac{1 - 2ra(x, t)\sin^2(\frac{\beta h}{2}) + ib(x, t)\frac{k}{2h}\sin(\beta h)}{1 + 2ra(x, t)\sin^2(\frac{\beta h}{2}) - ib(x, t)\frac{k}{2h}\sin(\beta h)} \right|^2 \\ &= \frac{(1 - 2ra(x, t)\sin^2(\frac{\beta h}{2}))^2 + rb^2(x, t)\frac{k}{4}\sin^2(\beta h)}{(1 + 2ra(x, t)\sin^2(\frac{\beta h}{2}))^2 + rb^2(x, t)\frac{k}{4}\sin^2(\beta h)}. \end{aligned} \tag{3.13}$$

Since numerator in (3.13) is less than denominator, therefore $|\frac{1+A}{1-A}| \leq 1$, hence (3.12) for all $r > 0$ is unconditionally stable. \square

Theorem 3.4. Let $(a_{jm}(x, t))_{j,m=1}^2$ be a symmetric, positive definite 2×2 matrix. For the following problem

$$\begin{cases} u_t(x, t) = \sum_{j,m=1}^2 a_{jm}(x, t) \frac{\partial^2 u(x,t)}{\partial x_j \partial x_m} \\ \quad + \sum_{m=1}^2 \varepsilon_m u_{x_m}(x, t) + d(t)u(x, t), & x \in \Omega \subseteq \mathbb{R}^2, \quad 0 < t \leq T, \\ u(x, 0) = f(x), & x \in \bar{\Omega}, \\ u(x, t) = g(t), & x \in \partial\Omega, \quad 0 \leq t \leq T. \end{cases} \quad (3.14)$$

Subject to extra measurements

$$\begin{aligned} u(x_1^*, x_2^*, t) &= E(t), & u_{x_1}(x_1^*, x_2^*, t) &= F_1(t), \\ u_{x_2}(x_1^*, x_2^*, t) &= F_2(t), & 0 \leq t \leq T, & \quad x^* \in \Omega. \end{aligned} \quad (3.15)$$

If we assume the following conditions, then (3.14) subject to (3.15) has a unique solution and

$u \in C^{4,2}(\Omega \times [0, T])$:

i- $f(x) \in C^{4+\alpha}(\Omega)$, $\Delta f(x) > 0$ and $\Delta^2 f(x^*) > 0$, and $\Delta^2 f(x) > 0$, on Ω .

ii- $g(t), E(t), F_j(t) \in C^{2+\frac{\alpha}{2}}([0, T])$, $F_j(t), E'(t) > 0$, $d(t) \leq 0$, $0 < \frac{g_i(t)}{E'(t)} < 1$,

$(\frac{g_i(t)}{E'(t)})' > 0$ on $[0, T]$ and here $0 \leq \varepsilon_m \ll 0$ and $j \in \{1, 2\}$, $i \in \{1, 2, 3, 4\}$.

Moreover, if we apply the finite difference methods then

1- The FTCS method is stable whenever $r \leq \frac{1}{\|(a_{jm}(x,t))_{j,m=1}^2\|_{L^\infty}}$.

2- The BTCS method is unconditionally stable for all $r > 0$.

3- The CN method is unconditionally stable for all $r > 0$.

Where $r = \frac{k}{\|h\|_{L^\infty}^2}$, $h = (h_1, h_2)$ the mesh-width in x_1 and x_2 , respectively. Also, k is the time step size.

Proof. The existence and uniqueness of the solutions to similar of this problem are discussed in [7]. It is shown [6] that the above conditions lead to [4] an existence and uniqueness theorem. For the numerical solution of this problem by finite differences one introduces a grid of mesh points $(x_1, x_2, t) = (x_1^i, x_2^j, t_n)$. Here $x_1^i = ih_1$, $x_2^j = jh_2$, and $t_n = nk$, where i, j and n are integers, $n \geq 0$. Define h_1 and h_2 as the mesh width in x_1, x_2 direction, respectively and k as the time step. One then seeks an approximate solution u_{ij}^n at these mesh points, determined by an equation obtained by replacing the derivatives in (3.14) by difference quotients. For functions defined on the grid we introduce thus the forward, backward and Crank Nicolson methods. By improving the above theorems, we conclude the results. \square

Table 1. L_∞ norm for error of u with different methods when $r = 0.0001$.

t	FTCS	BTCS	CN
0.2	1.7×10^{-3}	1.3×10^{-5}	2.8×10^{-5}
0.4	3.5×10^{-3}	4.6×10^{-5}	2.4×10^{-6}
0.6	4.1×10^{-3}	6.7×10^{-5}	4.8×10^{-5}
0.8	3.9×10^{-3}	7.9×10^{-6}	3.6×10^{-5}

4. Numerical results

In this section, in order to demonstrate the efficiency of our theorems, we apply the proposed methods that are FTCS, BTCS and CN to solve one example. To show the accuracy of the methods, we use maximum absolute error.

Table 2. L_∞ norm for error of u with different methods when $r = 2$.

t	FTCS	BTCS	CN
0.2	4.1×10^{-3}	7.1×10^{-5}	1.0×10^{-5}
0.4	1.1×10^{-2}	4.8×10^{-5}	1.3×10^{-6}
0.6	5.3×10^{-3}	1.6×10^{-5}	2.8×10^{-5}
0.8	2.6×10^{-2}	2.3×10^{-6}	5.1×10^{-6}

Example 4.1. We consider inverse problem (1.1) with

$$\varepsilon = (\varepsilon_1, \varepsilon_2)^T, \quad \varepsilon_1 = \varepsilon_2 = 0.1, \quad d(t) = \begin{cases} -1000t^2, & 0 < t \leq \frac{T}{2}, \\ 0, & \frac{T}{2} < t \leq T, \end{cases}$$

the initial condition

$$u(x_1, x_2, 0) = \exp(x_1 + x_2), \quad \Omega = [0, 1] \times [0, 1],$$

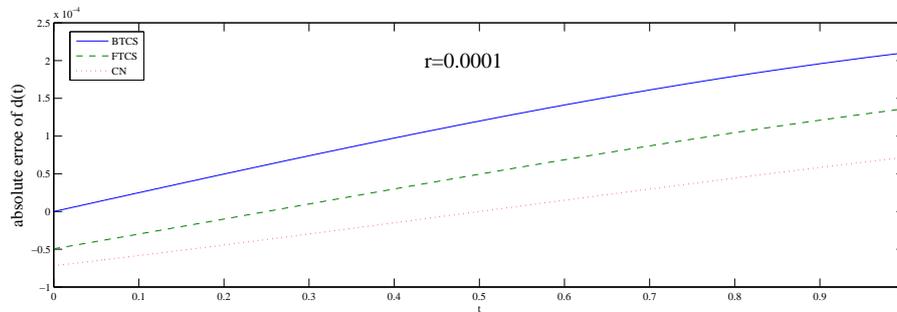


Fig. 1. The absolute error of $d(t)$ with L_∞ norm for $r = 0.0001$ related to time .

and the boundary conditions

$$\begin{aligned} u(0, 0, t) &= g_1(t) = \exp(t), \\ u(0, 1, t) &= g_2(t) = \exp(1 + t), \\ u(1, 0, t) &= g_3(t) = \exp(1 + t), \\ u(1, 1, t) &= g_4(t) = \exp(2 + t). \end{aligned}$$

We choose $x_1^* = x_2^* = 0.25$, Hence

$$E(t) = F_1(t) = F_2(t) = \exp(0.5 + t).$$

Note that the exact solution of this problem is

$$u(x_1, x_2, t) = \exp(x_1 + x_2 + t), \quad a_{j,m}(t) = \frac{1}{4}(1 - 2\varepsilon - d(t)), \quad j, m = 1, 2.$$

According to Theorem 3.4, we emphasize that $d(t)$ must be non-positive. We solve this problem at $T = 1$. Table 1 shows absolute error for u with $k = 0.01$ and $h_1 = h_2 = 0.001$ and hence $r = 0.0001$. Fig 1 shows the plot of error $d(t)$ for $r = 0.0001$. We observe that FTCS, BTCS and CN methods are stable for all positive value of $r \leq \frac{1}{\|a_{jm}(x_1, x_2, t)\|_{L_\infty}}$. The results obtained for u with $k = 0.01$, $h_1 = h_2 = 0.07$ and hence $r = 2$ are presented in Table 2. Also, Table 3 gives the absolute error of u with $k = 0.01$, $h_1 = h_2 = 0.0025$ and $r = 40$. Figs 2, 3 demonstrate the plot of error $d(t)$, for $r = 2$ and $r = 4$, respectively. In Figs 2, 3 and Tables 2, 3, we observe that for large values of r , BTCS and CN are stable. On the other hand, the absolute error obtained by applying FTCS is recorded

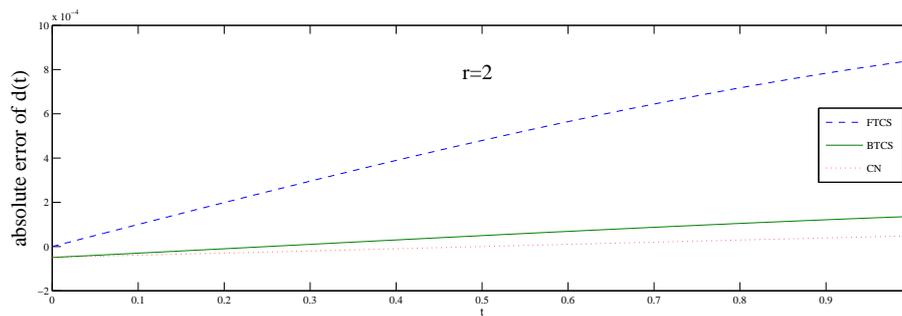


Fig. 2. The absolute error of $d(t)$ with L_∞ norm for $r = 2$ related to time .

Table 3. L_∞ norm for error of u with different methods when $r = 40$.

t	FTCS	BTCS	CN
0.2	9.1×10^{-2}	2.3×10^{-5}	4.6×10^{-5}
0.4	7.3×10^{-1}	2.8×10^{-6}	1.7×10^{-5}
0.6	8.1×10^{-2}	3.7×10^{-6}	3.7×10^{-6}
0.8	9.9×10^{-2}	5.9×10^{-5}	5.6×10^{-6}

in the tables. These results of three methods clearly indicate that the value of r is very important for FTCS and this method is valid only for $0 < r \leq \frac{1}{\|(a_{jk}(x_1, x_2, t))_{j,m=1}^2\|_{L_\infty}}$.

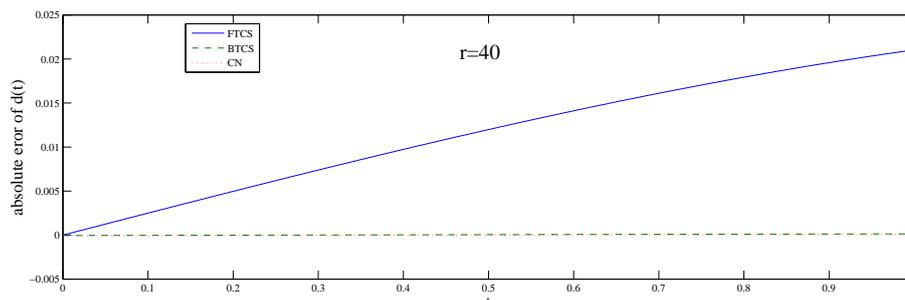


Fig. 3. The absolute error of $d(t)$ with L_∞ norm for $r = 40$ related to time .

5. Conclusion

In this paper, we proposed three numerical schemes for solving an inverse problem with extra measurements. We show that the FTCS scheme is conditional method for solving an inverse problem. But BTCS and CN methods are suitable schemes because they are unconditional method. We justified this advantage of the stability analysis versus the numerical method with an example.

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