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Research Article

Simple continuous modules

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Abstract

A module M is called a simple continuous module if it satisfies the conditions $(min-C_1)$ and $(min-C_2)$. A module M is called singular simple-direct-injective if for any singular simple submodules A, B of M with $A \cong B \mid M$, then $A \mid M$. Various basic properties of these modules are proved, and some well-studied rings are characterized using simple continuous modules and singular simple-direct-injective modules. For instance, it is shown that a ring R is a right V-ring if and only if every right R-module is a simple continuous modules, and that a regular ring R is a right GV-ring if and only if every cyclic right R-module is a singular simple-direct-injective module.

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1. Introduction and Preliminaries

Throughout this paper, R is an associative ring with identity and all modules are unital right R-modules. For a module M, we denote by $Soc(M)$ and $E(M)$ the socle and the injective hull of M, respectively. We write $N \leq M$ if N is a submodule of M, $N \leq_e M$ if N is an essential submodule of M, $N \mid M$ if N is a direct summand of M, and $N \leq_c M$ if N is a closed submodule of M.

Recall the following conditions for a module M:

(C_1) If each submodule A of M is essential in a direct summand of M;

(C_2) If a submodule A of M is isomorphic to a direct summand of M, then A is a direct summand of M;

(C_3) $K \oplus L$ is a direct summand of M whenever K and L are direct summands of M with $K \cap L = 0$;

$(min-C_1)$ If each simple submodule A of M is essential in a direct summand of M;

$(min-C_2)$ If a simple submodule A of M is isomorphic to a direct summand of M, then A is a direct summand of M.

Let M be a module. M is called a CS module if it satisfies the condition (C_1); M is called a direct-injective module if it satisfies the condition (C_2); M is called a continuous module if it satisfies the conditions (C_1) and (C_2); M is called a simple-direct-injective module [5] if it satisfies the condition $(min-C_2)$.

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Extending modules (CS-modules) play important roles in rings and categories of modules, their generalizations and related modules have been studied extensively by many authors. The concept of simple-direct-injective modules was introduced by V. Camillo, Y. Ibrahim, M. Yousif and Y. Q. Zhou [5], and some well-studied rings are characterized using simple-direct-injective modules. Motivated by this, simple continuous modules are given in Section 2 and V-rings are characterized in terms of simple continuous modules. It is shown that a ring R is a right V-ring (i.e., every simple right R-module is injective) if and only if every right R-module is a simple continuous module. In [5], the authors proved that a ring R is a right V-ring if and only if every right R-module is a simple-direct-injective module. As a proper generalization of V-rings, the notion of GV-rings was posed by V. S. Ramamurthi, K. M. Rangaswamy [14]. A ring R is called a right GV-ring if every singular simple right R-module is injective. Inspired by those, singular simple-direct-injective modules are introduced in Section 5. It is shown that a ring R is a right GV-ring if and only if every right R-module is a singular simple-direct-injective module and a regular ring R is a right GV-ring if and only if every cyclic right R-module is a singular simple-direct-injective module. For standard definitions we refer to [3, 4, 6–12, 15–17].

2. Simple continuous modules

In this section, the notion of simple continuous modules are introduced and some basic properties of simple continuous modules are proved.

Definition 2.1. A module M is called a simple continuous module if it satisfies the conditions $(min - C_1)$ and $(min - C_2)$.

Example 2.2.

(1) $\mathbb{Z}_{\mathbb{Z}}$ is a simple continuous module, but not a continuous module.

(2) Let $M = \mathbb{Z}_p \oplus \mathbb{Q}$, where p is a prime. Then M is a simple continuous $\mathbb{Z}$-module, but not continuous.

We do not know whether a direct summand of a simple continuous module is a simple continuous module. We have the following.

Recall that a submodule X of M is called fully invariant if for every $h \in S$, $h(X) \subseteq X$, where $S = End(M)$, [13].

Proposition 2.3. *Any fully invariant direct summand of a simple continuous module is a simple continuous module.*

Proof. Let M be a simple continuous module and K a fully invariant direct summand of M. It is easy to see that K satisfies the condition $(min - C_2)$. Next we shall show that K satisfies the condition $(min - C_1)$. Let S be a simple submodule of K. Since M satisfies the condition $(min - C_1)$, there is a direct summand H of M such that $S \leq_e H$. Write $M = H \oplus H'$, then $S \oplus H' \leq_e M$, and hence $S \oplus (H' \cap K) \leq_e K$. So $S \leq_e H \cap K$. Since K is a fully invariant direct summand of M and $M = H \oplus H'$, $K = (H \cap K) \oplus (H' \cap K)$ by [13, Lemma 2.1], as required. □

Recall that a module M is called a (weakly) duo module if any (direct summand) submodule is a fully invariant submodule of M, [13].

Corollary 2.4. *Any direct summand of a simple continuous (weakly) duo is a simple continuous module.*

A module M is said to be a UC-module if every submodule of M has a unique closure in M, [16].

Proposition 2.5. *Let M be a simple continuous UC module. Then any summand of M is a simple continuous module.*

Proof. Let M be a simple continuous UC module and K a direct summand of M. It is easy to see that K satisfies the condition $(min - C_2)$. Next we shall show that K satisfies the condition $(min - C_1)$. Let S be a simple submodule of K. Since M satisfies the condition $(min - C_1)$, there exists a direct summand H of M such that $S \leq_e H$. Let L denote the closure of S in K. So that $S \leq_e L \leq_c K$, and hence $L \leq_c M$. Thus $S \leq_e L \leq_c M$ and $S \leq_e H \leq_c M$. Since M is a UC module, $L = H$. Since H is a direct summand of M, L is a direct summand of K. Therefore S is essential in a direct summand L of K, as desired. □

Example 2.6. $\mathbb{Z}_2$ and $\mathbb{Z}_8$ are simple continuous $\mathbb{Z}$-modules, but $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ is not a simple continuous $\mathbb{Z}$-module because the non-summand $0 \oplus \mathbb{Z}(4+8\mathbb{Z})$ is isomorphic to the simple summand $\mathbb{Z}_2 \oplus 0$.

Example 2.7. ([11, Example 2.9]) Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is any field. Let $A = \begin{pmatrix} F & F \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & F \end{pmatrix}$. It is clear that A and B are simple continuous as R-modules. However $R = A \oplus B$ is not simple continuous.

The above two examples show that a direct sum of simple continuous modules need not be a simple continuous module, so we have the following.

Proposition 2.8. *Let $M = M_1 \oplus M_2$, where M_1 and M_2 satisfy the condition $(min - C_1)$ and M_1 is M_2-injective, then M satisfies the condition $(min - C_1)$.*

Proof. Let S be a simple submodule of M. We shall prove that S is essential in a direct summand of M by considering two cases.

Case 1: $S \cap M_1 = 0$. In this case, since M_1 is M_2-injective, there exists a direct summand N of M such that $N \cong M_2$, $S \leq N$ and $M = M_1 \oplus N$. Then N satisfies the condition $(min - C_1)$, and so there is a direct summand K of N such that $S \leq_e K$, as required.

Case 2: $S \cap M_1 \neq 0$. Since S is simple, $S \leq M_1$. The rest is obvious.

□

Lemma 2.9 ([5, Lemma 3.3])**.** *If M is an indecomposable module that is not simple, then $M \oplus E(M)$ is simple-direct-injective.*

Corollary 2.10. *If M is a uniform module that is not simple, then $M \oplus E(M)$ is a simple continuous module.*

Proof. It follows by Proposition 2.8 and Lemma 2.9. □

The following examples reveal the relationships among simple-direct-injective modules, modules satisfying the condition $(min - C_1)$ and modules satisfying the condition (C_1).

Example 2.11.

(1) Let p be any rational prime and $M_1 = \mathbb{Z}_p$, $M_2 = \mathbb{Z}_\infty$. Then $M = M_1 \oplus M_2$ satisfies the condition $(min - C_1)$, but not the condition (C_1).

(2) Let $R = \begin{pmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z} \end{pmatrix}$ be the upper triangular generalized triangular matrix ring. Then R_R satisfies the condition $(min - C_1)$, but not the condition (C_1).

(3) $\mathbb{Z}_2 \oplus \mathbb{Z}_8$ satisfies the condition $(min - C_1)$, but it is not a simple-direct-injective module because the non-summand $0 \oplus \mathbb{Z}(4 + 8\mathbb{Z})$ is isomorphic to the simple summand $\mathbb{Z}_2 \oplus 0$.

(4) Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is any field. Then R_R satisfies the condition $(min - C_1)$, but it is not a simple-direct-injective module. As $Soc(R_R)$ is projective, if R_R is a simple-direct-injective module, then R is a mininjective ring by [5, P44]. It is impossible.

(5) (Björk example) Let F be a field and assume that $a \mapsto \bar{a}$ is an isomorphism $F \to \bar{F} \subseteq F$, where the subfield $\bar{F} \neq F$. Let R denote the left vector space on basis $\{1, t\}$, and make R into an F-algebra by defining $t^2 = 0$ and $ta = \bar{a}t$ for all $a \in F$. Then R is a right mininjective ring, and hence R_R is a simple-direct-injective module. However, R_R does not satisfy the condition $(min - C_1)$.

3. Simple continuous modules and V-rings

In this section, some connections between right V-rings and simple continuous modules are presented.

Theorem 3.1. *The following conditions are equivalent for a ring R:*

(1) *R is a right V-ring.*
(2) *Every right R-module is a simple continuous module.*
(3) *Every finitely cogenerated right R-module is a simple continuous module.*
(4) *Direct sums of simple continuous modules are simple continuous modules.*
(5) *Every 2-generated right R-module is a simple continuous module.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and (1) $\Rightarrow$ (5). They are clear.

(3) $\Rightarrow$ (1) Let S be a simple right R-module. Since $S \oplus E(S)$ is finitely cogenerated, it is a simple continuous module by hypothesis. Thus $S \oplus E(S)$ is simple-direct-injective, and hence $S = E(S)$ by [5, Proposition 2.1]. Therefore S is injective and R is a right V-ring.

(4) $\Rightarrow$ (1) Let S be a simple right R-module. Since S and $E(S)$ are simple continuous modules, $S \oplus E(S)$ is a simple continuous module by hypothesis. Thus $S \oplus E(S)$ is simple-direct-injective, and hence $S = E(S)$ by [5, Proposition 2.1]. Therefore S is injective and R is a right V-ring.

(5) $\Rightarrow$ (1) Let $S = xR$ be a simple right R-module and $0 \neq y \in E(S)$. Then $xR \leq_e yR$. By hypothesis, $xR \oplus yR$ is a simple continuous module, and so it is simple-direct-injective. Thus, $xR = yR$ by [5, Proposition 2.1] and hence $S = E(S)$. Therefore S is injective and R is a right V-ring. □

It is well known that a ring R is semisimple if and only if every right R-module is a continuous module. From Theorem 3.1, if a ring R is a right V-ring which is not semisimple, then there is a simple continuous module which is not a continuous module. See the following example.

Example 3.2. Let $\mathbb{F}$ be a field and $\mathfrak{J}$ be an infinite index set. Let $R = \Pi_{i \in \mathfrak{J}} F_i$, where $F_i = \mathbb{F}$ for each $i \in \mathfrak{J}$. Then R is a right V-ring which is not semisimple, and hence there is a simple continuous module which is not a continuous module.

Proposition 3.3. *A regular ring R is a right V-ring if and only if every cyclic right R-module is a simple continuous module.*

Proof. " $\Rightarrow$ ". This is clear by Theorem 3.1.

" $\Leftarrow$ ". Since every cyclic right R-module is a simple continuous module, it is simple-direct-injective. The rest is obvious by [5, Theorem 4.4]. □

Lemma 3.4. *Any direct sum of injective modules is a simple continuous module.*

Proof. It is clear by [5, Lemma 3.1]. □

A module M is called strongly soc-injective if for any module N and any semisimple submodule K of N, every homomorphism $f : K \to M$ extends to N, [2].

Lemma 3.5 ([2, Proposition 16])**.** *A module M is strongly soc-injective if and only if $M = E \oplus T$, where E is injective and $Soc(T) = 0$.*

Proposition 3.6. *The following are equivalent for a ring R:*

(1) *R is a right noetherian right V-ring;*
(2) *Every simple continuous module is strongly Soc-injective.*

Proof. Similar to [5, Proposition 4.3]. □

4. When are simple continuous modules continuous?

We characterize the rings whose simple continuous modules are continuous.

Lemma 4.1 ([1, Corollary 2.4 and 2.6])**.** (1) *If $M = A_1 \oplus A_2$ is a C_3-module and $f : A_1 \to A_2$ is an R-monomorphism, then Imf is a direct summand of A_2.*
(2) *If $M \oplus M$ is a C_3-module, then M is a C_2-module.*

A module is uniserial if the lattice of its submodules is totally ordered under inclusion. A ring R is called left uniserial if ${}_RR$ is a uniserial module. A ring R is called serial if both modules ${}_RR$ and R_R are direct sums of uniserial modules.

A ring R is said to satisfy the condition $(*)$ if every finitely generated right R-module satisfies the condition $(min-C_1)$. For instance, a dedekind domain satisfies the condition $(*)$.

Theorem 4.2. *The following are equivalent for a ring R with the condition $(*)$:*

(1) *Every simple continuous right R-module is a C_3-module.*
(2) *Every simple continuous right R-module is continuous.*
(3) *Every simple continuous right R-module is quasi-injective.*
(4) *Every right R-module is a direct sum of a semisimple module and a family of injective uniserial modules of length 2.*
(5) *R is an artinian serial ring with $J(R)^2 = 0$.*

Proof. (3) $\Rightarrow$ (2) $\Rightarrow$ (1) They are clear.

(1) $\Rightarrow$ (4) We claim that R is right artinian. First we show that R is right semiartinian. Assume on the contrary that M is a right R-module with $Soc(M) = 0$. If $0 \neq N \leq M$, then $Soc(N \oplus M) = 0$ and $N \oplus M$ is a simple continuous module. Thus $N \oplus M$ is a C_3-module by hypothesis, and the inclusion map $i : N \hookrightarrow M$ splits by Lemma 4.1. This shows that M is semisimple, a contradiction. Thus $Soc(M) \neq 0$ for every right R-module M, and hence R is right semiartinian. Next we show that R is right noetherian. It suffices to show that, for any family $K_i (i \in I)$ of simple right R-modules, $M = \oplus_{i\in I} E(K_i)$ is injective. By Lemma 3.4, $M \oplus E(M)$ is a simple continuous module, so $M \oplus E(M)$ is a C_3-module by hypothesis. By Lemma 4.1, the inclusion map $i : M \hookrightarrow E(M)$ splits, and hence $M = E(M)$ is injective, as required. So R is right noetherian, and hence R is right artinian.

We next show that every indecomposable injective right R-module E has a unique composition series of length at most 2. Note that E has a simple socle X and $E = E(X)$. If $E = X$, we are done. Suppose that $X \subset Y \subseteq E$. It suffices to show that $Y = E$. Let $M = Y \oplus E$. Then M is a simple continuous module by Corollary 2.10, and hence M is a C_3-module. So $Y = E$ by Lemma 4.1, as desired.

We now show that every finitely generated indecomposable right R-module has a unique composition series of length at most 2. To see this, let M be a finitely generated indecomposable right R-module. If M is simple, we are done. If M is not simple, since R satisfies the condition $(*)$, M satisfies the condition $(min - C_1)$, and hence $M \oplus E(M)$ satisfies the condition $(min - C_1)$ by Proposition 2.8. Therefore $M \oplus E(M)$ is a simple continuous module by Lemma 2.9. Thus $M \oplus E(M)$ is a C_3-module by hypothesis, and so $M = E(M)$ is injective by Lemma 4.1. Thus M is an indecomposable injective right R-module, and, as above, it has a unique composition series of length at most 2.

Finally, consider an arbitrary right R-module M. Since R is right noetherian, M contains a maximal injective submodule N. Write $M = N \oplus K$, where K contains no nonzero injective submodules. The injective module N is a direct sum of indecomposable injective modules each of which has a unique composition series of length at most 2. Thus there is a decomposition $N = E_1 \oplus E_2$, where E_1 is semisimple and E_2 is a direct sum of injective uniserial modules of length 2. So, to finish the proof, it suffices to show that K is semisimple. Without loss of generality, we may assume that K is a cyclic module. Since R is right artinian, K is artinian, so it is a direct sum of indecomposable modules. Therefore we can further assume that K is a cyclic indecomposable module. As above, K is a uniserial module of length at most 2. If K is of length 2, then $K = E(K)$ because $E(K)$ is a uniserial module of length at most 2. This contradicts the fact that K contains no nonzero injective submodules. Hence K is simple, as desired.

The rest follow by [5, Theorem 3.4]. □

Corollary 4.3. *A dedekind domain R is semisimple artinian if and only if every simple continuous module is injective.*

Proof. " $\Rightarrow$ " is clear.

" $\Leftarrow$ " if every simple continuous module is injective, then R is a V-ring. But R is artinian by Theorem 4.2, so R is semisimple artinian. □

5. Singular simple-direct-injective modules and GV-rings

In [5], the authors proved that a ring R is a right V-ring if and only if every right R-module is a simple-direct-injective module. As a generalization of V-rings, the notion of GV-rings was posed by V. S. Ramamurthi, K. M. Rangaswamy [14]. A ring R is called a right GV-ring if every singular simple right R-module is injective. Inspired by those, singular simple-direct-injective modules are introduced in this Section. It is shown that a ring R is a right GV-ring if and only if every right R-module is a singular simple-direct-injective module and a regular ring R is a right GV-ring if and only if every cyclic right R-module is a singular simple-direct-injective module.

Definition 5.1 ([14])**.** A ring R is a right GV-ring if each simple right R-module is either projective or injective if and only if every singular simple right R-module is injective.

Proposition 5.2. *The following are equivalent for a module M:*

(1) *For any singular simple submodules A, B of M with $A \cong B \mid M$, $A \mid M$.*
(2) *For any singular simple summands A, B of M with $A \cap B = 0$, $A \oplus B \mid M$.*
(3) *If $M = A_1 \oplus A_2$ with A_1 singular simple and $f : A_1 \to A_2$ an R-homomorphism, then $Imf \mid A_2$.*

Proof. (1) $\Rightarrow$ (2) Let A, B be singular simple summands of M with $A \cap B = 0$. Write $M = A \oplus T$ for a submodule $T \leq M$, and let $\pi : A \oplus T \to T$ be the canonical projection. Clearly $A \oplus B = A \oplus \pi(B)$. Since $\pi(B) \cong B$ and B is a singular simple summand of M, $\pi(B) \mid M$ by hypothesis, and so $\pi(B) \mid T$. Thus $M = A \oplus T = A \oplus \pi(B) \oplus K = A \oplus B \oplus K$ for a submodule $K \leq T \leq M$. Therefore $A \oplus B \mid M$.

(2) $\Rightarrow$ (3) Without loss of generality we may assume that $f \neq 0$. This means that f is an R-monomorphism. Let $T = \{a + f(a) : a \in A_1\}$ be the graph submodule of M. We claim that $M = T \oplus A_2$. For, if $x \in M$, then $x = a + b$, where $a \in A_1$, $b \in A_2$. Now $x = a + f(a) - f(a) + b \in T + A_2$, and so $M = T + A_2$. If $x \in T \cap A_2$, then $x = a + f(a)$ for some $a \in A_1$, and consequently $a = x - f(a) \in A_1 \cap A_2 = 0$. This shows that $x = 0$, so $M = T \oplus A_2$, and $T \mid M$. Next we show that $A_1 \cap T = 0$. For, if $x \in A_1 \cap T$, then $x = a + f(a)$ for some $a \in A_1$, and consequently $x - a = f(a) \in A_1 \cap A_2 = 0$. Now, since f is monic, $a = 0$, and hence $x = 0$. Since $T \cong M/A_2 \cong A_1$ is singular simple, $A_1 \oplus T \mid M$ by hypothesis. Finally we show that $A_1 \oplus T = A_1 \oplus Imf$. For, if $x \in Imf$, then $x = f(a)$

for some $a \in A_1$, and so $x = -a + a + f(a) \in A_1 + T$, and hence $A_1 \oplus T = A_1 \oplus Imf$. Since $A_1 \oplus T \mid M$, $A_1 \oplus Imf \mid M$, and so $Imf \mid A_2$, as required.

(3) $\Rightarrow$ (1) Let A, B be singular simple submodules of M with $B \overset{\sigma}{\cong} A \mid M$. We need to show that $B \mid M$. If $A \cap B \neq 0$, there is nothing to prove. Otherwise, assume that $A \cap B = 0$, and write $M = A \oplus T$ for some submodule T of M. If $\pi : A \oplus T \to T$ be the canonical projection, then clearly $A \oplus B = A \oplus \pi(B)$ and $\pi(B) \cong B$ is singular simple. Now, since A is singular simple, $M = A \oplus T$, and $\pi \mid_B \sigma^{-1} : A \to T$ is monic with $Im(\pi \mid_B \sigma^{-1}) = \pi(B)$. By hypothesis, $\pi(B) \mid T$. If $T = \pi(B) \oplus K$ for some submodule K of T, then $M = A \oplus T = A \oplus \pi(B) \oplus K = A \oplus B \oplus K$ and $B \mid M$, as desired.

□

Definition 5.3. A module M is called singular simple-direct-injective if M satisfies the equivalent conditions of Proposition 5.2.

Theorem 5.4. *The following conditions are equivalent for a ring R:*

(1) *R is a right GV-ring.*
(2) *Every right R-module is a singular simple-direct-injective module.*
(3) *Every finitely cogenerated right R-module is a singular simple-direct-injective module.*
(4) *Direct sums of singular simple-direct-injective modules are singular simple-direct-injective modules.*
(5) *Every 2-generated right R-module is a singular simple-direct-injective module.*

Proof. Similar to Theorem 3.1. □

Example 5.5. Let $R = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$, where F is any field. Then R is a right GV-ring and not a right V-ring. Since a ring R is a right V-ring if and only if every right R-module is simple-direct-injective, there is a singular simple-direct-injective module is not simple-direct-injective by Theorem 5.4.

Theorem 5.6. *A regular ring R is a right GV-ring if and only if every cyclic right R-module is singular simple-direct-injective.*

Proof. " $\Rightarrow$ " is clear by Theorem 5.4.

" $\Leftarrow$ " Let S be a singular simple right R-module and $E = E(S)$ the injective hull of S. Assume to the contrary, there is a nonzero element $x \in E$ such that $x \bar{\in} S$. Clearly, $S \leq_e xR$. Define the epimorphism $f : R \to xR$ by $f(r) = xr, r \in R$, and set $X = Kerf$. Now the map f induces an isomorphism $\sigma : xR \to R/X$. If $T/X = \sigma(S)$ is singular simple, then $T/X = (tR+X)/X$ for some nonzero element $t \in R$. Since R is regular, there is $s \in R$ such that $tst = t$. If we set $e = ts$, then $e^2 = e$ and $T/X = (tR + X)/X = (eR + X)/X$. Inasmuch as $S \leq_e xR$, we infer that T/X is a minimal essential right ideal of R/X. If $M = \{r \in R : er \in X\}$, then $R/M \cong T/X$ and M is a maximal right ideal of R.

Now we claim that, for $N = M \cap X$, $X/N \cong R/M$. To see this, observe first since $(eR + X)/X$ is a singular simple essential submodule of R/X and $((1 - e)R + X)/X$ is a nonzero submodule of R/X, it follows that $(eR + X)/X \subseteq ((1 - e)R + X)/X$, and hence $e + X = (1 - e)(-r) + X$ for some $r \in R$. Therefore $y = e + (1 - e)r \in X$, and if we multiply on the left by e, we get $ey = e$. Now $N = M \cap X \subseteq X \subset T$, and if $y \in N$, then $y \in M$, which implies that $ey \in X$, and so $e \in X$, a contradiction. Thus $y \bar{\in} N$, and it follows that X is not contained in M. Now $X/N = X/(M \cap X) \cong (X + M)/M = R/M$.

Next we show that $(eR + N)/N \cong R/M$. If $g : R \to (eR + N)/N$ is given by $g(r) = er + N$, where $r \in R$, then g is a well-defined R-epimorphism. Since M is a maximal right ideal of R and $M \subseteq Kerg$, we infer that $M = Kerg$ and $(eR+N)/N \cong R/M$, as required.

Next we show that $((1 - e)yR + N)/N \cong R/M$. Observe first that if $m \in M$, then it follows, from the definition of M and the fact $ey = e$, that $em = eym \in X$, and hence

$ym \in M$. Therefore $ym \in M \cap X = N$, and so $yM \subseteq N$. Since $eM \subseteq N$ and $ey = e$, it follows that $eyM \subseteq N$, and consequently $(1-e)yM \subseteq yM + eyM \subseteq N$. Now if we define $h : R \to ((1-e)yR+N)/N$ by $h(r) = (1-e)yr+N$, where $r \in R$, then h is a well-defined R-epimorphism. Since $(1-e)yM \subseteq N$, it follows that $R/M \cong ((1-e)yR+N)/N$, as desired. Therefore $((1-e)yR+N)/N \cong (eR+N)/N \cong R/M \cong X/N \cong (eR+X)/X \cong T/X \cong S$ are singular simple.

As $eM \subseteq N$, $eN \subseteq eM \subseteq N$ and N is invariant under left multiplication by e. Therefore $R/N = (eR+N)/N \oplus ((1-e)R+N)/N$. Since $((1-e)yR+N)/N \cong (eR+N)/N$ and $(eR+N)/N$ is a singular simple summand of R/N, $((1-e)yR+N)/N$ is a singular simple summand of R/N by hypothesis, and hence $((1-e)yR+N)/N$ is a singular simple summand of $((1-e)R+N)/N$. Thus $R/N = (eR+N)/N \oplus ((1-e)yR+N)/N \oplus A/N$, where $A/N \leq R/N$.

Finally, we only need to show that $R/N = (eR+N)/N \oplus X/N$. Since if this happens, then $(R/N)_R$ has uniform dimension 2. So A/N must be zero and $R/N = (eR+N)/N \oplus ((1-e)yR+N)/N$, and consequently $R/X \cong (eR+N)/N$ is singular simple, a contradiction. First, we have $(eR+N)/N \cap X/N = 0$. To see this, let $er + N = x + N \in (eR+N)/N \cap X/N, r \in R, x \in X$, then $er - x \in N$, and since $N \subseteq X$, it follows that $er \in X$. This means $r \in M$, and hence $er \in N$. Therefore $(eR+N)/N \cap X/N = 0$. Since $(eR+N)/N$ and X/N are singular simple submodules of R/N and $X/N \cong (eR+N)/N \mid R/N$, $(eR+N)/N \oplus X/N$ is a direct summand of R/N by hypothesis. Hence it suffices to show that $(eR+N)/N \oplus X/N \leq_e R/N$. Now, let $(aR+N)/N$ be a nonzero submodule of R/N. If $a \in X$, then $0 + N \neq a + N \in X/N \subseteq (eR+N)/N \oplus X/N$. Otherwise, assume that $a \bar{\in} X$. In this case $(aR+X)/X$ is a nonzero submodule of R/X. Consequently, since $(eR+X)/X$ is a singular simple essential submodule of R/X, it follows that $(eR+X)/X \subseteq (aR+X)/X$. Therefore, $e + X = ar + X, r \in R$. Thus $ar = e + l$ for some $l \in X$ and $0+N \neq ar+N = (e+N)+(l+N) \in (eR+N)/N \oplus X/N$, as desired. □

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