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Research Article

Approximation by $p-$Faber-Laurent rational functions in doubly-connected domain

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Abstract

Let G be a doubly-connected domain bounded by regular curves. In this work, the approximation properties of the $p-$Faber-Laurent rational series expansions in the $\omega-$weighted Smirnov classes $E^p(G,\omega)$ are studied.

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1. Introduction

Let $\Gamma \subset \mathbb{C}$ be a Jordan rectifiable curve. For $p > 1$ we define a class $L^p(\Gamma)$ of Lebesgue measurable functions $f : \Gamma \to \mathbb{R}$ satisfying the condition

$$\left(\int_{\Gamma} |f(z)|^p \, |dz| < \infty \right)^{\frac{1}{p}} < \infty.$$

This class $L^p(\Gamma)$ is a Banach space with respect to the norm

$$\|f\|_{L^p(\Gamma)} := \left(\int_{\Gamma} |f(z)|^p \, |dz| < \infty \right)^{\frac{1}{p}}.$$

A Jordan curve Γ is called regular, if there exists a number $c > 0$ such that for every $r > 0$, $\sup \{|\Gamma \cap D(z,r)| : z \in \Gamma\} \leq cr$, where $D(z,r)$ is an open disk with radius r and centered at z and $|\Gamma \cap D(z,r)|$ is the length of the set $\Gamma \cap D(z,r)$.

Let ω be a weight function on Γ. ω is said to satisfy Muckenhoupt's A_p-condition on Γ if

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$$\sup_{z\in\Gamma}\sup_{r>0}\left(\frac{1}{r}\int\limits_{\Gamma\cap D(z,r)}\omega(\zeta)\,|d\zeta|\right)\left(\frac{1}{r}\int\limits_{\Gamma\cap D(z,r)}[\omega(\zeta)]^{-\frac{1}{p-1}}\,|d\zeta|\right)^{p-1}<\infty$$

Let us further assume that B is a simply-connected domain with a rectifiable Jordan boundary Γ and $B^- := ext\Gamma$, further let

$$\mathbb{T}=\{\omega\in\mathbb{C}:|\omega|=1\}\,,\quad D:=int\mathbb{T},\ \ D^-:=ext\mathbb{T}.$$

Also, ϕ^* stand for the conformal mapping of B^- onto D^- normalized by

$$\phi^*(\infty)=\infty$$

and

$$\lim_{z\to\infty}\frac{\phi^*(z)}{z}>0,$$

and let ψ^* be the inverse of ϕ^*. Let ϕ_1^* be the conformal mapping of B onto D^-, normalized by

$$\phi_1^*(0)=0$$

and

$$\lim_{z\to 0} z\phi^*(z>0.$$

The inverse mapping of ϕ_1^* will be denoted by ψ_1^*.

Note that the mappings ψ^* and ψ_1^* have in some deleted neighborhood of ∞ representations

$$\psi^*(w)=\alpha w+\alpha_0+\frac{\alpha_1}{w}+\frac{\alpha_2}{w^2}+...+\frac{\alpha_k}{w^k}+...,\quad \alpha>0$$

and

$$\psi_1^*(w)=\frac{\beta_1}{w}+\frac{\beta_2}{w^2}+...+\frac{\beta_k}{w^k}+...,\quad \beta_1>0.$$

The functions

$$\frac{\left(\frac{d\psi^*(w)}{dw}\right)^{1-\frac{1}{p}}}{\psi^*(w)-z},\ \ z\in B$$

and

$$\frac{w^{-\frac{2}{p}}\left(\frac{d\psi_1^*(w)}{dw}\right)^{1-\frac{1}{p}}}{\psi_1^*(w)-z},\ \ z\in B^-.$$

are analytic in the domain D^-. The following expansions hold:

$$\frac{\left(\frac{d\psi^*(w)}{dw}\right)^{1-\frac{1}{p}}}{\psi^*(w)-z}=\sum_{k=0}^{\infty}\frac{\Phi_{k,p}(z)}{w^{k+1}},\ \ z\in B,\ \ w\in D^-$$

and

$$\frac{w^{-\frac{2}{p}}\left(\frac{d\psi_1^*(w)}{dw}\right)^{1-\frac{1}{p}}}{\psi_1^*(w)-z}=\sum_{k=1}^{\infty}-\frac{F_{k,p}(\frac{1}{z})}{w^{k+1}},z\in B^-,w\in D^-,$$

where $\Phi_{k,p}(z)$ and $F_{k,p}(\frac{1}{z})$ are the $p-$Faber polynomials of degree k with respect to z and $\frac{1}{z}$ for the continuums $\overline{B}$ and $\overline{B}\backslash B$, respectively [3], [14], [24, pp.255-257].

Let $E^1(B)$ be a classical Smirnov class of analytic functions in B.The set $L^p(\Gamma,\omega) := \{f \in L^1(\Gamma) : |f|^p \omega \in L^1(\Gamma)\}$ is called the $\omega-$weighted L^p-space. The set $E^p(B,\omega) := \{f \in E^1(B) : f \in L^p(\Gamma,\omega)\}$ is called the $\omega-$weighted Smirnov class of order $p-$analytic functions in B.

Note that detailed information about properties of the non-weighted Smirnov class $E^p(B)$, $p > 1$, can be found in [4, pp. 168-185] and [7, pp. 438-453].

Let $\omega \in A_p(\mathbb{T})$. For $f \in L^p(\mathbb{T},\omega)$ we define the operator

$$(\nu_h f)(\omega) := \frac{1}{2h}\int_{-h}^{h} f\left(\omega e^{it}\right) dt,\ \omega \in \mathbb{T},\ \ 0 < h < \pi.$$

If $\omega \in A_p(\mathbb{T})$ and $f \in L^p(\mathbb{T},\omega)$, then the operator ν_h is a bounded linear operator on $L^p(\mathbb{T},\omega)$:

$$\|\nu_h(f)\|_{L^p(\mathbb{T},\omega)} \le c_2 \|f\|_{L^p(\mathbb{T},\omega)}.$$

Let $1 < p < \infty$, $\omega \in A_p(\mathbb{T})$ and $f \in L^p(\mathbb{T},\omega)$. The function

$$\Omega_{p,\omega,k}(f,\delta) := \sup_{\substack{0<h_i \le\delta \\ i=1,2,\ldots,k}} \left\| \prod_{i=1}^{k} (I-\nu_{h_i}) f \right\|_{L^p(\mathbb{T},\omega)},\ \ \delta > 0$$

is called *the $k-th$ modulus of continuity* of $f \in L^p(\mathbb{T},\omega)$.

It can easily be shown that $\Omega_{p,\omega,k}(f,\cdot)$ is a continuous, nonnegative and nondecreasing function satisfying the conditions

$$\lim_{\delta\to 0}\Omega_{p,\omega,k}(f,\delta) = 0,\ \Omega_{p,\omega,k}(f+g,\delta) \le \Omega_{p,\omega,k}(f,\delta) + \Omega_{p,\omega,k}(g,\delta),\ \ \delta > 0$$

for $f, g \in L^p(\mathbb{T},\omega)$.

Let G be a doubly-connected domain in the complex plane $\mathbb{C}$, bounded by the rectifiable Jordan curves Γ_1 and Γ_2 (the closed curve Γ_2 is in the closed curve Γ_1). Without loss of generality we assume $0 \in \text{int}\Gamma_2$. Let $G_1^0 := \text{int}\Gamma_1$, $G_1^\infty := \text{ext}\Gamma_1$, $G_2^0 := \text{int}\Gamma_2$, $G_2^\infty := \text{ext}\Gamma_2$.

We denote by $\omega = \phi(z)$ the conformal mapping of G_1^∞ onto domain D^- normalized by the conditions

$$\phi(\infty) = \infty,\ \lim_{z\to\infty}\frac{\phi(z)}{z} = 1$$

and let ψ be the inverse mapping of ϕ.

We denote by $\omega = \phi_1(z)$ the conformal mapping of G_2^0 onto domain D^- normalized by the conditions

$$\phi_1(0) = \infty\ ,\ \lim_{z\to\infty}(z.\phi_1(z)) = 1,$$

and let ψ_1 be the inverse mapping of ϕ_1.

Let us take

$$C_{\rho_0} := \{z : |\phi(z)| = \rho_0 > 1\},\ \Gamma_{r_0} := \{z : |\phi_1(z)| = r_0 > 1\}.$$

For $\Phi_{k,p}(z)$ and $F_{k,p}\left(\frac{1}{z}\right)$ the following integral representations hold [3], [14] and [24, pp.255-257]:

(1) If $z \in intC_{\rho_0}$, then

$$\Phi_{k,p}(z) = \frac{1}{2\pi i}\int_{C_{\rho_0}} \frac{[\phi(\zeta)]^k (\phi'(\zeta))^{\frac{1}{p}}}{\zeta - z} d\zeta. \tag{1.1}$$

(2) If $z \in extC_{\rho_0}$, then

$$\Phi_{k,p}(z) = [\phi(z)]^k (\phi'(z))^{\frac{1}{p}} + \frac{1}{2\pi i} \int\limits_{C_{\rho_0}} \frac{[\phi(\zeta)]^k (\phi'(\zeta))^{\frac{1}{p}}}{\zeta - z} d\zeta. \tag{1.2}$$

(3) If $z \in intC_{r_0}$, then

$$F_{k,p}(\frac{1}{z}) = [\phi_1(z)]^{k-\frac{2}{p}} (\phi'(z))^{\frac{1}{p}} - \frac{1}{2\pi i} \int\limits_{C_{r_0}} \frac{[\phi_1(\zeta)]^{k-\frac{2}{p}} (\phi_1'(\zeta))^{\frac{1}{p}}}{\zeta - z} d\zeta. \tag{1.3}$$

(4) If $z \in extC_{r_0}$, then

$$F_{k,p}\left(\frac{1}{z}\right) = -\frac{1}{2\pi i} \int\limits_{C_{r_0}} \frac{[\phi_1(\zeta)]^{k-\frac{2}{p}} (\phi_1'(\zeta))^{\frac{1}{p}}}{\zeta - z} d\zeta. \tag{1.4}$$

Note that in the classical case $p = \infty$ these integral representations are proved in [23]. If a function $f(z)$ is analytic in the doubly-connected domain bounded by the curves C_{ρ_0} and Γ_{r_0}, then the following series expansion holds:

$$f(z) = \sum_{k=0}^{\infty} a_k \Phi_{k,p}(z) + \sum_{k=1}^{\infty} b_k F_{k,p}\left(\frac{1}{z}\right), \tag{1.5}$$

where

$$a_k = \frac{1}{2\pi i} \int\limits_{|\omega|=\rho_1} \frac{f[\psi(\omega)]\,\psi'(\omega)^{\frac{1}{p}}}{\omega^{k+1}} d\omega, \quad (1 < \rho_1 < \rho_0),\ k = 0, 1, 2, ...$$

and

$$b_k = \frac{1}{2\pi i} \int\limits_{|\omega|=r_1} \frac{f[\psi_1(\omega)]\,(\psi_1'(\omega))^{\frac{1}{p}} \omega^{\frac{2}{p}}}{\omega^{k+1}} d\omega, \quad (1 < r_1 < r_0),\ k = 1, 2, ...$$

The series (1.5) is called the $p-$Faber -Laurent series of f, and the coefficients a_k and b_k are said to be the $p-$Faber -Laurent coefficients of f . For $z \in G$ by Cauchy's theorem we have

$$f(z) = \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{f(\xi)}{\xi - z} d\xi.$$

If $z \in int\Gamma_2$ and $z \in ext\Gamma_1$, then

$$\frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{f(\xi)}{\xi - z} d\xi = 0. \tag{1.6}$$

Let us consider

$$I_1(z) = \frac{1}{2\pi i} \int\limits_{\Gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad I_2(z) = \frac{1}{2\pi i} \int\limits_{\Gamma_2} \frac{f(\xi)}{\xi - z} d\xi.$$

The function $I_1(z)$ determines the functions $I_1^+(z)$ and $I_1^-(z)$ while the function $I_2(z)$ determines the functions $I_2^+(z)$ and $I_2^-(z)$. The functions $I_1^+(z)$ and $I_1^-(z)$ are analytic in $int\Gamma_1$ and $ext\Gamma_1$, respectively. The functions $I_2^+(z)$ and $I_2^-(z)$ are analytic in $int\Gamma_2$ and $ext\Gamma_2$, respectively.

Let B be a finite domain in the complex plane bounded by a rectifiable Jordan curve Γ and $f \in L_1(\Gamma)$. Then the functions f^+ and f^- defined by

$$f^+(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in B$$

and

$$f^-(z) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta, \quad z \in B^-$$

are analytic in B and B^-respectively, and $f^-(\infty) = 0$. Thus the limit

$$S_\Gamma(f)(z) := \lim_{\varepsilon\to\infty} \frac{1}{2\pi i} \int\limits_{\Gamma\cap\{\zeta:|\zeta-z|>\varepsilon\}} \frac{f(\zeta)}{\zeta - z} d\zeta$$

exists and is finite for almost all $z \in \Gamma$.

The quantity $S_\Gamma(f)(z)$ is called the Cauchy singular integral of f at $z \in \Gamma$.

According to the Privalov theorem [7, pp. 431], if one of the functions f^+ or f^- has the non-tangential limits a.e. on Γ, then $S_\Gamma(f)(z)$ exists a.e. on Γ and also the other one has the non-tangential limits a.e. on Γ. Conversely, if $S_\Gamma(f)(z)$ exists a.e. on Γ, then the functions $f^+(z)$ and $f^-(z)$ have non-tangential limits a.e. on Γ. In both cases, the formulae

$$f^+(z) = S_\Gamma(f)(z) + \frac{1}{2}f(z), \quad f^-(z) = S_\Gamma(f)(z) - \frac{1}{2}f(z)$$

and hence

$$f = f^+ - f^- \tag{1.7}$$

holds a.e. on Γ. From the results given in [6], it follows that if Γ is a regular curve, then S_Γ is bounded on $L^p(\Gamma, \omega)$.

We will say that the doubly connected domain G is bounded by the regular curve if the domains G_1^0 and G_2^0 are bounded by the closed regular curves.

Let Γ_i $(i = 1, 2)$ be a regular curve and let $f_0 := f[\psi(\omega)]\,\psi'(\omega)^{\frac{1}{p}}$ for $f \in L^p(\Gamma_1, \omega)$ and let $f_1(\omega) := f[\psi_1(\omega)]\,(\psi_1'(\omega))^{\frac{1}{p}}\omega^{\frac{2}{p}}$ for $f \in L^p(\Gamma_2, \omega)$.We also set $\omega_0(\omega) := \omega[\psi(\omega)]$, $\omega_1(\omega) := \omega[\psi_1(\omega)]$,Then , if $f \in L^p(\Gamma_1, \omega)$ and $f \in L^p(\Gamma_2, \omega)$ we obtain $f_0 \in L^p(\mathbb{T}, \omega_0)$ and $f_1 \in L^p(\mathbb{T}, \omega_1)$.

Moreover, $f_0^-(\infty) = f_1^-(\infty) = 0$ and by (1.7)

$$\left.\begin{array}{l} f_0(\omega) = f_0^+(\omega) - f_0^-(\omega) \\ f_1(\omega) = f_1^+(\omega) - f_1^-(\omega) \end{array}\right\} \tag{1.8}$$

a.e. on T.

Now, in the doubly-connected domain we define the $\omega-$weighted Smirnov class . The set $E^p(G) := \{f \in E^1(G) : f \in L^p(\Gamma, \omega)\}$ is called the $\omega-$weighted Smirnov space of order p of analytic functions in G.

Lemma 1.1 ([14]). *Let $g \in E^p(D, \omega)$ and $\omega \in A_p(\mathbb{T})$. If $\sum\limits_{k=0}^{n} d_k(g)\omega^k$ is the nth partial sum of the Taylor series of g at the origin, then there exists a constant $c_9 > 0$ such that*

$$\left\| g(\omega) - \sum_{k=0}^{n} d_k(\omega)\omega^k \right\|_{L^p(\mathbb{T},\omega)} \le c_9 \Omega_{p,\omega,k}\left(g, \frac{1}{n}\right)$$

for every natural number n.

We set

$$R_n(f,z) := \sum_{k=0}^{n} a_k \Phi_{k,p}(z) + \sum_{k=1}^{n} b_k F_{k,p}(\frac{1}{z}).$$

The rational function $R_n(f,z)$ is called the $p-$Faber-Laurent rational function of degree n of f.

Since series of Faber polynomials are a generalization of Taylor series to the case of a simply connected domain, it is natural to consider the construction of a similar generalization of Laurent series to the case of a doubly-connected domain.

The problems of approximation of the functions in the non-weighted and weighted Smirnov classes were investigated in [1, 2, 5, 9–13, 16, 17, 22]. Similar problems in the different spaces defined on the continuums of the complex plane were investigated by several authors (see for example, [3, 8, 14, 15, 18–21, 25–27]).We remark, that when Γ is a closed regular Jordan curve, the approximation properties of the $p-$Faber-Laurent rational series expansions in the $\omega-$ Lebesgue spaces $L^p(\Gamma,\omega)$ have been investigated by Israfilov [16]. In this work the approximation problems of the functions by $p-$Faber-Laurent rational functions in the weighted Smirnov classes $E_p(G,\omega)$, defined in the doubly-connected domains with the regular boundaries are investigated.

Our main result can be formulated as follows.

Theorem 1.2. *Let G be a finite doubly-connected domain with the regular boundary $\Gamma = \Gamma_1 \cup \Gamma_2$. If $\omega \in A_p(\Gamma), \omega_0, \omega_1 \in A_p(\mathbb{T})$ and $f \in E^p(G,\omega)$ then there is a constant $c_{10} > 0$ such that for any $n = 1, 2, 3, ...$*

$$\|f - R_n(\cdot, f)\|_{L^p(\Gamma,\omega)} \leq c_{10} \left\{ \Omega_{p,\omega_0,k}(f_0, 1/n) + \Omega_{p,\omega_1,k}(f_1, 1/n) \right\},$$

where $R_n(.,f)$ is the $p-$Faber-Laurent rational function of degree n of f.

2. Proof of the main result

Proof of Theorem 1.2. We take the curves Γ_1, Γ_2 and $\mathbb{T} := \{\omega \in \mathbb{C} : |\omega| = 1\}$ as the curves of integration in the formulas (1.2)-(1.5) and (1.6), respectively. (This is possible due to the conditions of theorem 1.2). Let $f \in E^p(G,\omega)$. Then $f_0 \in L^p(\mathbb{T},\omega_0)$, $f_1 \in L^p(\mathbb{T},\omega_1)$. According to (1.8)

$$\left.\begin{aligned} f(\zeta) &= [f_0^+(\phi(\zeta)) - f_0^-(\phi(\zeta))](\phi(\zeta))^{\frac{1}{p}} \\ f(\xi) &= [f_1^+(\phi_1(\xi)) - f_1^-(\phi_1(\xi))](\phi_1(\xi))^{-\frac{2}{p}}(\phi_1'(\xi))^{\frac{1}{p}}. \end{aligned}\right\} \tag{2.1}$$

Let $z \in ext\Gamma_1$. Then from (1.2) and (2.1) we have

$$\begin{aligned}\sum_{k=0}^{n} a_k \Phi_{k,p}(z) &= \sum_{k=0}^{n} a_k \left[\phi\left(z\right)\right]^k \left(\phi'(z)\right)^{\frac{1}{p}} + \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{\left(\phi'(\zeta)\right)^{\frac{1}{p}} \sum\limits_{k=0}^{n} a_k \left[\phi\left(\zeta\right)\right]^k}{\zeta - z} d\zeta \\ &= \sum_{k=0}^{n} a_k \left[\phi\left(z\right)\right]^k \left(\phi'(z)\right)^{\frac{1}{p}} \\ &\quad + \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{\left(\phi'(\zeta)\right)^{\frac{1}{p}} \sum\limits_{k=0}^{n} a_k \left[\phi\left(\zeta\right)\right]^k - f_0^{+}\left[\phi\left(\zeta\right)\right]}{\zeta - z} d\zeta \\ &\quad + \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{f\left(\zeta\right)}{\zeta - z} d\zeta - f_0^{-}\left[\phi\left(z\right)\right]\left(\phi'(z)\right)^{\frac{1}{p}} \ . \end{aligned} \tag{2.2}$$

For $z \in ext\Gamma_2$, consideration of (1.4) and (2.1) gives

$$\begin{aligned}\sum_{k=1}^{n} b_k F_k\left(\frac{1}{z}\right) &= -\frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{\left(\phi_1'(\xi)\right)^{\frac{1}{p}} \phi_1(\xi)^{-\frac{2}{p}} \sum\limits_{k=1}^{n} b_k \left[\phi_1\left(\xi\right)\right]^k}{\xi - z} d\xi \\ &\quad -\frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{\sum\limits_{k=0}^{n} b_k \left[\phi_1\left(\xi\right)\right]^k}{\xi - z} d\xi \\ &= \frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{\left(\left(\phi_1(\xi)\right)^{-\frac{2}{p}} \left(\phi_1'(\xi)\right)^{\frac{1}{p}} \left[f_1^{+}(\phi_1\left(\xi\right)) - \sum\limits_{k=0}^{n} b_k \left[\phi_1\left(\xi\right)\right]^k\right]}{\xi - z} d\xi \\ &\quad -\frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{f\left(\xi\right)}{\xi - z} d\xi. \end{aligned} \tag{2.3}$$

For $z \in ext\Gamma_1$, using (2.2), (2.3) we have

$$\begin{aligned}&\sum_{k=0}^{n} a_k \left[\Phi_k\left(z\right)\right]^k + \sum_{k=1}^{n} a_k F_k\left(\frac{1}{z}\right) \\ &= \sum_{k=0}^{n} a_k \left[\phi\left(z\right)\right]^k \left(\phi'(z)\right)^{\frac{1}{p}} + \frac{1}{2\pi i}\int\limits_{\Gamma_1} \frac{\left(\phi'(\zeta)\right)^{\frac{1}{p}} \sum\limits_{k=0}^{n} a_k \left[\phi\left(\zeta\right)\right]^k - f_0^{+}\left[\phi\left(\zeta\right)\right]}{\zeta - z} d\zeta \\ &\quad -f_0^{-}\left[\phi\left(z\right)\right] + \frac{1}{2\pi i}\int\limits_{\Gamma_2} \frac{\left(\left(\phi_1(\xi)\right)^{-\frac{2}{p}} \left(\phi_1'(\xi)\right)^{\frac{1}{p}} \left[f_1^{+}(\phi_1\left(\xi\right)) - \sum\limits_{k=0}^{n} b_k \left[\phi_1\left(\xi\right)\right]^k\right]}{\xi - z} d\xi. \end{aligned}$$

Taking the limit as $z \to z^* \in \Gamma_1$ along all non-tangential paths outside Γ_1, we obtain

$$\begin{aligned} f(z^*) - \sum_{k=0}^{n} a_k \Phi_k(z^*) - \sum_{k=1}^{n} b_k F_k\left(\frac{1}{z^*}\right) &= f_0^+[\phi(z^*)] - \sum_{k=0}^{n} a_k [\phi(z^*)]^k (\phi'(z^*))^{\frac{1}{p}} \\ &+\frac{1}{2}(\phi'(z^*))^{\frac{1}{p}}\left(f_0^+[\phi(z^*)] - \sum_{k=0}^{n} a_k[\phi(z^*)]^k\right) \\ &+S_{\Gamma_1}\left[(\phi')^{\frac{1}{p}}(f_0^+\circ\phi - \sum_{k=0}^{n} a_k\phi^k)\right](z^*) \\ &-\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{f_1^+[\phi_1(\xi)] - \sum\limits_{k=1}^{n} b_k[\phi_1(\xi)]^k}{\xi - z^*}d\xi \quad (2.4) \end{aligned}$$

a.e. on Γ_1.

Now using (2.4), Minkowski's inequality and the boundedness of S_{Γ_1} in $L^p(\Gamma_1,\omega)$ [14, Theorem 5] we have

$$\|f - R_n(f,z)\|_{L^p(\Gamma_1,\,\omega)} \le c_{11}\left\|f_0^+(\omega) - \sum_{k=0}^{n} a_k\omega^k\right\|_{L^p(\mathbb{T},\omega_0)} + c_{12}\left\|f_1^+(w) - \sum_{k=0}^{n} b_k\omega^k\right\|_{L^p(\mathbb{T},\omega_1)} \quad (2.5)$$

That is, the Faber-Laurent coefficients a_k and b_k of the function f are the Taylor coefficients of the functions f_0^+ and f_1^+, respectively. Then by (2.5), Lemma 1.1 and [14, Lemma 4] we have

$$\begin{aligned} \|f - R_n(.,f)\|_{L^p(\Gamma_1,\,\omega)} &\le c_{10}(p)\left\{\Omega_{p,\omega_0,k}\left(f_0^+,1/n\right) + \Omega_{p,\omega_1,k}\left(f_1^+,1/n\right)\right\} \\ &\le c_{11}(p)\left\{\Omega_{p,\omega_0,k}(f_0,1/n) + \Omega_{p,\omega_1,k}(f_1,1/n)\right\}. \end{aligned}$$

Let $z \in int\Gamma_2$. Then by (1.3) and (2.1) we have

$$\begin{aligned} \sum_{k=1}^{n} b_k F_{k,p}\left(\frac{1}{z}\right) &= (\phi_1'(z))^{\frac{1}{p}}(\phi_1(z))^{-\frac{2}{p}}\sum_{k=1}^{n} b_k[\phi_1(z)^k \\ &-\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{(\phi_1'(\zeta))^{\frac{1}{p}}(\phi_1(\zeta))^{-\frac{2}{p}}\sum\limits_{k=1}^{n} b_k[\phi_1(\xi)]^k}{\xi - z}d\xi \\ &= (\phi_1'(z))^{\frac{1}{p}}(\phi_1(z))^{-\frac{2}{p}}\sum_{k=1}^{n} b_k[\phi_1(z)]^k \\ &-\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{(\phi_1'(\zeta))^{\frac{1}{p}}(\phi_1(\zeta))^{-\frac{2}{p}}\left(\sum\limits_{k=1}^{n} b_k[\phi_1(\xi)]^k - f_1^+[\phi_1(\xi)]\right)}{\xi - z}d\xi \\ &-\frac{1}{2\pi i}\int\limits_{\Gamma_2}\frac{f(\xi)}{\xi - z}d\xi - f_1^-[\phi_1(z)](\phi_1'(z))^{\frac{1}{p}}(\phi_1(z))^{-\frac{2}{p}}. \quad (2.6) \end{aligned}$$

For $z \in int\Gamma_1$, from (1.1) and (2.1) we obtain

$$\sum_{k=1}^{n} a_k \Phi_k(z) = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{(\phi'(\zeta))^{\frac{1}{p}} \sum_{k=1}^{n} a_k [\phi(\zeta)]^k}{\zeta - z} d\zeta.$$

$$= \frac{1}{2\pi i}\int_{\Gamma_1} \frac{(\phi'(\zeta))^{\frac{1}{p}} \left(\sum_{k=1}^{n} a_k [\phi(\zeta)]^k - f_0^+[\phi(\zeta)]\right)}{\zeta - z} d\zeta + \frac{1}{2\pi i}\int_{\Gamma_1} \frac{f(\zeta)}{\zeta - z} d\zeta. \quad (2.7)$$

Use of (2.6) and (2.7) for $z \in int\Gamma_2$ gives

$$\sum_{k=0}^{n} a_k \Phi_k(z) + \sum_{k=1}^{n} b_k F_k\left(\frac{1}{z}\right) = \frac{1}{2\pi i}\int_{\Gamma_1} \frac{(\phi'(\zeta))^{\frac{1}{p}} \left(\sum_{k=0}^{n} a_k [\phi(\zeta)]^k - f_0^+[\phi(\zeta)]\right)}{\zeta - z} d\zeta$$

$$+ (\phi_1'(z))^{\frac{1}{p}} (\phi_1(z))^{-\frac{2}{p}} \sum_{k=1}^{n} b_k [\phi_1(z)]^k$$

$$- \frac{1}{2\pi i}\int_{\Gamma_2} \frac{(\phi_1'(\zeta))^{\frac{1}{p}} (\phi_1(\zeta))^{-\frac{2}{p}} \left(\sum_{k=1}^{n} b_k [\phi_1(\xi)]^k - f_1^+[\phi_1(\xi)]\right)}{\xi - z} d\xi$$

$$- f_1^-[\phi_1(z)] (\phi_1'(z))^{\frac{1}{p}} (\phi_1(z))^{-\frac{2}{p}}.$$

Taking the limit as $z \to z^* \in \Gamma_2$ along all non-tangential paths inside Γ_2, we reach

$$f(z^*) - \sum_{k=0}^{n} a_k \Phi_{k,p}(z^*) - \sum_{k=1}^{n} b_k F_{k,p}\left(\frac{1}{z^*}\right)$$

$$= f_1^+[\phi_1(z^*)] - \frac{1}{2} (\phi_1'(z^*))^{\frac{1}{p}} (\phi_1(z^*))^{-\frac{2}{p}} \left[\sum_{k=1}^{n} b_k [\phi_1(z^*)]^k - f_1^+[\phi_1(z^*)]\right]$$

$$- S_{\Gamma_2}\left[(\phi_1')^{\frac{1}{p}} (\phi_1)^{-\frac{2}{p}} \left(\sum_{k=1}^{n} b_k \phi_1^k - \left(f_1^+ \circ \phi_1\right)\right)\right](z^*)$$

$$- \frac{1}{2\pi i}\int_{\Gamma_1} \frac{(\phi'(\zeta))^{\frac{1}{p}} \left(\sum_{k=0}^{n} a_k [\phi(\zeta)]^k - f_0^+[\phi(\zeta)]\right)}{\zeta - z^*} d\zeta \quad (2.8)$$

a.e. on Γ_2.

Consideration of (2.8), the Minkowski's inequality and the boundedness of S_{Γ_2} in $L^p(\Gamma_2, \omega)$ [14, Theorem 5] gives the inequality

$$\|f - R_n(f,z)\|_{L^p(\Gamma_2,\omega)} \le c_{14} \left\| f_1^+(\omega) - \sum_{k=1}^{n} b_k \omega^k \right\|_{L^p(\mathbb{T},\omega_1)} + c_{15} \left\| f_0^+(w) - \sum_{k=0}^{n} a_k \omega^k \right\|_{L^p(\mathbb{T},\omega_0)}. \quad (2.9)$$

Use of (2.9), Lemma 1.1 and [14, Lemma 4] leads to

$$\begin{aligned} \|f - R_n(.,f)\|_{L^p(\Gamma,\omega)} &\le c_{10}\left\{\Omega_{p,\omega_1,k}\left(f_1^+, 1/n\right) + \Omega_{p,\omega_0,k}\left(f_0^+, 1/n\right)\right\} \\ &\le c_{11}\left\{\Omega_{p,\omega_1,k}(f_1, 1/n) + \Omega_{p,\omega_0,k}(f_0, 1/n)\right\}. \end{aligned}$$

Then, the proof of theorem 1.2 is completed. □

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