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Research Article

On generalized Mathieu series and its companions

P. Cerone*, C. Lenard

Department of Mathematics and Statistics, La Trobe University, Melbourne (Bundoora) VIC 3086, Australia

Abstract

Integral representations for a generalized Mathieu series and its companions are used to obtain bounds for their corresponding series. The bounds are procured mainly using results pertaining to the Čebyšev functional. The relationship to Zeta type functions are also examined. It is demonstrated that the Zeta companion relations are a particular case of the generalised Mathieu companions.

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1. Introduction

The series, known in the literature as the Mathieu series,

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^2}, \quad r > 0, \tag{1.1}$$

has been extensively studied in the past since its introduction by Mathieu [18] in 1890, where it arose in connection with work on elasticity of solid bodies. The reader is directed to the references and the books [4] and [23] for further illustration of various representations and bounds. The various applications areas involve the solution of the biharmonic equation in a rectangular two dimensional domain using the so called *superposition method* and the interested reader is referred to the work of Meleshko ([19–21]) for excellent coverage and further references. A Literature search in MathScinet with 'Mathieu series' results in over 700 hits demnostrates that the area continues to attract many avenues of research and application. See also some of the recent activity such as in [3, 9, 11, 22, 26, 30].

One of the main questions addressed in relation to the series is obtaining sharp bounds.

Building on some results from [33], Alzer, Brenner and Ruehr [1] showed that the best constants a and b in

$$\frac{1}{x^2+a} < S(x) < \frac{1}{x^2+b}, \qquad x \neq 0 \tag{1.2}$$

*Corresponding Author.
Email addresses: p.cerone@latrobe.edu.au (P. Cerone), c.lenard@latrobe.edu.au (C. Lenard)
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are $a=\frac{1}{2\zeta(3)}$ and $b=\frac{1}{6}$ where $\zeta(\cdot)$ denotes the Riemann zeta function defined by

$$\zeta(p)=\sum_{n=1}^{\infty}\frac{1}{n^p}. \tag{1.3}$$

An integral representation for $S(r)$ as given in (1.1) was presented in [10] and [12] as

$$S(r)=\frac{1}{r}\int_0^{\infty}\frac{x}{e^x-1}\sin(rx)\,dx. \tag{1.4}$$

Guo [15] utilised (1.4) to obtain bounds on $S(r)$. Alternate bounds to (1.1) were obtained by Qi and coworkers in [27–29].

Guo in [15] posed the interesting problem as to whether there is an integral representation of the generalized Mathieu series

$$S_\mu(r)=\sum_{n=1}^{\infty}\frac{2n}{(n^2+r^2)^{1+\mu}},\qquad r>0,\ \mu>0. \tag{1.5}$$

In [31] an integral representation was obtained for $S_m(r)$, where $m\in\mathbb{N}$, namely

$$\begin{aligned}S_m(r)=&\frac{2}{(2r)^m\,m!}\int_0^{\infty}\frac{t^m}{e^t-1}\cos\left(\frac{m\pi}{2}-rt\right)dt\\ &-2\sum_{k=1}^{m}\left[\frac{(k-1)(2r)^{k-2m-1}}{k!\,(m-k+1)}(-(m+1)m-k)\right.\\ &\left.\times\int_0^{\infty}\frac{t^k\cos\left[\frac{\pi}{2}(2m-k+1)-rt\right]}{e^t-1}dt\right].\end{aligned} \tag{1.6}$$

The challenge of Guo [15] to obtain an integral representation for $S_\mu(r)$ as defined in (1.5) was successfully answered by Cerone and Lenard [8] in which the following two theorems were proved.

Theorem 1.1. *The generalized Mathieu series $S_\mu(r)$ defined by (1.5) may be represented in the integral form*

$$S_\mu(r)=C_\mu(r)\int_0^{\infty}\frac{x^{\mu+\frac{1}{2}}}{e^x-1}J_{\mu-\frac{1}{2}}(rx)\,dx,\qquad \mu>0, \tag{1.7}$$

where

$$C_\mu(r)=\frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\,\Gamma(\mu+1)} \tag{1.8}$$

and $J_\nu(z)$ is the ν^{th} order Bessel function of the first kind.

Theorem 1.2. *For m a positive integer we have*

$$S_m(r)=\frac{1}{2^{m-1}}\cdot\frac{1}{r^{2m-1}}\cdot\frac{1}{m}\sum_{k=0}^{m-1}\frac{(-1)^{\left\lfloor\frac{3k}{2}\right\rfloor}}{k!}r^k\left[\delta_{k\ even}A_k(r)+\delta_{k\ odd}B_k(r)\right], \tag{1.9}$$

where

$$A_k(r)=\int_0^{\infty}\frac{x^{k+1}}{e^x-1}\sin(rx)\,dx,\quad B_k(r)=\int_0^{\infty}\frac{x^{k+1}}{e^x-1}\cos(rx)\,dx, \tag{1.10}$$

with $\delta_{condition}=1$ if condition holds and zero otherwise and $\lfloor x\rfloor$ is the largest integer not greater than x.

The emphasis as in [8], became the derivation of bounds for the generalized Mathieu series $S_\mu(r)$. The first approach utilized sharp bounds for the Bessel function $|J_\nu(z)|$. To this end, in an article by Landau [16], the best possible uniform bounds were obtained for Bessel functions using monotonicity arguments. Landau showed

$$|J_\nu(z)| < \frac{b_L}{\nu^{\frac{1}{3}}} \tag{1.11}$$

uniformly in the argument $z > 0$ and is best possible in the exponent $\frac{1}{3}$ and constant

$$b_L = 2^{\frac{1}{3}} \sup_z Ai(z) = 0.674885\cdots, \tag{1.12}$$

where $Ai(z)$ is the Airy function satisfying

$$w'' - zw = 0.$$

Landau also showed that for $z > 0$

$$|J_\nu(z)| \leq \frac{c_L}{z^{\frac{1}{3}}} \tag{1.13}$$

uniformly in the order $\nu > 0$ and the exponent $\frac{1}{3}$ is best possible with

$$c_L = \sup_z z^{\frac{1}{3}} J_0(z) = 0.78574687\ldots. \tag{1.14}$$

The following theorem, based on the Landau bounds (1.11) – (1.14), was obtained in [8].

Theorem 1.3. *The generalized Mathieu series $S_\mu(r)$ satisfies are bounded above for $\mu > \frac{1}{2}$ and $r > 0$*

$$S_\mu(r) \leq b_L \frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}} \cdot \frac{1}{\left(\mu - \frac{1}{2}\right)^{\frac{1}{3}}} \cdot \frac{\Gamma\left(\mu + \frac{3}{2}\right)}{\Gamma(\mu+1)} \zeta\left(\mu + \frac{3}{2}\right), \tag{1.15}$$

and

$$S_\mu(r) \leq c_L \cdot \frac{\sqrt{\pi}}{2^{\mu-\frac{1}{2}} r^{\mu-\frac{1}{6}}} \cdot \Gamma\left(\mu + \frac{7}{6}\right) \zeta\left(\mu + \frac{7}{6}\right), \tag{1.16}$$

where b_L and c_L are given by (1.12) and (1.14) respectively.

The following corollary was also obtained in [8] for $S(r) = S_1(r)$. The first of of these results is corrected below.

Corollary 1.4. *The Mathieu series $S(r)$ satisfies the following bounds*

$$S(r) \leq \frac{3\pi}{2^{\frac{13}{6}}} b_L \cdot \zeta\left(\frac{5}{2}\right) \cdot \frac{1}{\sqrt{r}} \tag{1.17}$$

and

$$S(r) \leq \frac{7c_L}{36} \cdot \sqrt{\frac{\pi}{2}} \cdot \Gamma\left(\frac{1}{6}\right) \zeta\left(\frac{13}{6}\right) \cdot r^{-\frac{5}{6}}, \tag{1.18}$$

where b_L and c_L are given by (1.12) and (1.14) respectively.

The following results were obtained in [8] using a weighted Čebyšev functional approach. See also [6] where the approach was utilized for a greater variety of special functions.

Theorem 1.5. *For $\mu > 0$ and $r > 0$ the generalized Mathieu series $S_\mu(r)$ satisfies*

$$\left| S_\mu(r) - \frac{\pi^2}{12\mu\left(r^2 + \frac{1}{4}\right)^\mu} \right| \tag{1.19}$$

$$\leq \kappa \left[\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(2\mu - \frac{1}{2}\right)}{2^{2\mu-1}\Gamma^2(\mu+1)} \int_0^{\frac{\pi}{2}} \frac{\cos^{2\mu-1}\phi}{\left[\left(\frac{1}{4}\right)^2 + r^2\cos^2\phi\right]^{2\mu-\frac{1}{2}}} d\phi - \frac{1}{2\mu^2\left(r^2+\frac{1}{4}\right)^{2\mu}} \right]^{\frac{1}{2}}$$

$$\leq \kappa \left[\frac{\Gamma\left(2\mu - \frac{1}{2}\right)\Gamma\left(\mu + \frac{1}{2}\right)}{2^{2\mu}\Gamma^3(\mu+1)} \cdot \frac{1}{r^{4\mu-1}} - \frac{1}{2\mu^2\left(r^2+\frac{1}{4}\right)^{2\mu}} \right]^{\frac{1}{2}},$$

where

$$\kappa = \left[\pi^2\left(1 - \frac{\pi^2}{72}\right) - 7\zeta(3)\right]^{\frac{1}{2}} = 0.3198468959\ldots. \tag{1.20}$$

Corollary 1.6. *The Mathieu series $S(r)$, satisfies the following bounds*

$$\left| \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^2} - \frac{\pi^2}{12\left(r^2+\frac{1}{4}\right)} \right| \leq 2\sqrt{2} \cdot \kappa \left\{ \frac{2}{1+(4r)^2} - \frac{1}{\left[1+(2r)^2\right]^2} \right\}^{\frac{1}{2}} \tag{1.21}$$

where κ is as given by (1.20).

As explained in Pogany *et al.*[22], motivated by [8], a family of *Mathieu* **a-***series* were introduced by Pogany *et al.*[25] together with their integral representations with various approaches and results to procure bounds.

The *alternating generalized Mathieu series*, companion to $S_\mu(r)$,was introduced by Pogany *et al.* [25] and is represented by

$$\tilde{S}_\mu(r) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \cdot 2n}{(n^2+r^2)^{1+\mu}}, \qquad r>0,\ \mu>0. \tag{1.22}$$

which can be also expressed in the following integral form

$$\tilde{S}_\mu(r) = C_\mu(r) \int_0^\infty \frac{x^{\mu+\frac{1}{2}}}{e^x+1} J_{\mu-\frac{1}{2}}(rx)\, dx, \qquad \mu>0, \tag{1.23}$$

where $C_\mu(r)$ is as given in (1.8).

In the current paper, further bounds are obtained for the alternating generalized Mathieu series $\tilde{S}_\mu(r)$, and the odd $\phi_\mu(r)$ and even $\psi_\mu(r)$ generalized Mathieu series, as defined in Section 4. This is accomplished by using their integral representations via Čebyšev Functional bounds. The methodology produces both the approximation and bounds for the companion series of the *generalized Mathieu series.* In Section 5 some properties of the generalized Mathieu series and its companions are given with an emphasis on the moments in terms of Beta and Zeta functions. The paper's emphasis is to analyze the odd and even counterparts for the generalized Mathieu series as has been accomplished for the odd and even Zeta functions. It is further demonstrated that the relationship between the Zeta function, the alternating Zeta function and the odd Zeta function is recaptured by allowing $r- > 0$ in the relationship between the generalised ; Mathieu series , the alternating and odd Mathieu series in Theorem 4.7 and Remark 4.8.

2. Some results on bounding the Čebyšev functional

The weighted Čebyšev functional defined by

$$T(f,g;p) := \mathcal{M}(fg;p) - \mathcal{M}(f;p)\,\mathcal{M}(g;p), \tag{2.1}$$

where $\mathcal{M}$ is the weighted integral mean

$$\mathcal{M}(h;p) := \frac{\int_a^b p(x)\,h(x)\,dx}{\int_a^b p(x)\,dx}, \tag{2.2}$$

has been extensively investigated in the literature with the view of determining its bounds. The unweighted Čebyšev functional $T(f,g;1)$,was bounded by Grüss in [14] by the product of the difference of the functions and their function bounds.

There has been much activity in procuring bounds for $T(f,g;p)$ and the interested reader is referred to [5, 7]. The functional $T(f,g;p)$ is known to satisfy a number of identities. Included amongst these are identities of Sönin type, namely

$$P \cdot T(f,g;p) = \int_a^b p(t)\,[f(t) - K]\,[g(t) - \mathcal{M}(g;p)]\,dt, \tag{2.3}$$

where K is a constant and,

$$P = \int_a^b p(x)\,dx\ . \tag{2.4}$$

The constant $K \in \mathbb{R}$, but in the literature some of the more popular values have been taken as

$$0,\ \frac{\Delta+\delta}{2},\ f\left(\frac{a+b}{2}\right) \text{ and } \mathcal{M}(f;p),$$

where $-\infty < \delta \le f(t) \le \Delta < \infty$ and $t \in [a,b]$.

An identity attributed to Körkine viz

$$P^2 \cdot T(f,g;p) = \frac{1}{2}\int_a^b \int_a^b p(x)\,p(y)\,(f(x) - f(y))\,(g(x) - g(y))\,dxdy \tag{2.5}$$

may also easily be shown to hold.

Remark 2.1. For $-\infty < \delta \le f(t) \le \Delta < \infty$ for $t \in [a,b]$ Cerone and Dragomir [7] showed that

$$\begin{aligned} P \cdot |T(f,g;p)| &\le \frac{1}{2}(\Delta - \delta)\int_a^b p(t)\,|g(t) - \mathcal{M}(g;p)|\,dt \\ &\le \frac{1}{2}(\Delta - \delta)\left(\int_a^b p(t)\,|g(t) - \mathcal{M}(g;p)|^{\alpha}\,dt\right)^{\frac{1}{\alpha}},\ 1 \le \alpha < \infty \\ &\le \frac{1}{2}(\Delta - \delta)\, ess \sup_{t\in[a,b]} |g(t) - \mathcal{M}(g;p)|\,. \end{aligned} \tag{2.6}$$

Specifically, if $-\infty < \phi \le g(t) \le \Phi < \infty$ for $t \in [a,b]$, then

$$\begin{aligned} |T(f,g;p)| &\le \frac{1}{2}(\Delta - \delta)\int_a^b p(t)\,|g(t) - \mathcal{M}(g;p)|\,dt \\ &\le \frac{1}{2}(\Delta - \delta)\left[\frac{1}{P}\int_a^b p(t)\,g^2(t)\,dt - \mathcal{M}^2(g;p)\right]^{\frac{1}{2}} \\ &\le \frac{1}{4}(\Delta - \delta)(\Phi - \phi)\,. \end{aligned} \tag{2.7}$$

The results in (2.6) were obtained from the Sönin type identity (2.3) on taking $K = \frac{\Delta+\delta}{2}$.

It is instructive to show from (2.3) that the best K, in the sense of providing the sharpest bound for the Euclidean or $2-$norm, results when $K = \mathcal{M}(f;p)$.

Lemma 2.2. *The sharpest bound for the Čebyšev functional involving the Euclidean norm is given by*

$$P \cdot |T(f,g;p)| \le \inf_K \left[\int_a^b p(t)(f(t)-K)^2\,dt\right]^{\frac{1}{2}} \left[\int_a^b p(t)(g(t)-\mathcal{M}(g;p))^2\,dt\right]^{\frac{1}{2}}$$
$$= \left[\int_a^b p(t) f^2(t)\,dt - \mathcal{M}^2(f;p)\right]^{\frac{1}{2}} \left[\int_a^b p(t) g^2(t)\,dt - \mathcal{M}^2(g;p)\right]^{\frac{1}{2}}.$$

Proof. From (2.3) we have, on using the Cauchy-Buniakowsky-Schwartz inequality

$$P \cdot |T(f,g;p)| \le \left(\int_a^b p(t)(f(t)-K)^2\,dt\right)^{\frac{1}{2}} \left(\int_a^b p(t)(g(t)-\mathcal{M}(g;p))^2\,dt\right)^{\frac{1}{2}}.$$

Now, the sharpest bound is obtained by taking the infimum over $K \in \mathbb{R}$. That is

$$\inf_{K\in\mathbb{R}} \left(\int_a^b p(t)(f(t)-K)^2\,dt\right)^{\frac{1}{2}} = \inf_{K\in\mathbb{R}} \left(\int_a^b p(t)\left(f^2(t)-2Kf(t)+K^2\right)dt\right)^{\frac{1}{2}}$$
$$= \inf_{K\in\mathbb{R}} \left[\int_a^b p(t) f^2(t)\,dt + P\cdot K(K-2\mathcal{M}(f;p))\right]^{\frac{1}{2}}$$
$$= \left(\int_a^b p(t) f^2(t)\,dt - P\cdot\mathcal{M}^2(f;p)\right)^{\frac{1}{2}},$$

and the infimum occurs when $K = \mathcal{M}(f;p)$. □

In the next section Lemma 2.2 is used to obtain bounds for the alternating generalized Mathieu series $\tilde{S}_\mu(r)$.

We note that the first inequality in (2.6) results from

$$|P\cdot T(f,g;p)| \le \inf_K \|f(\cdot)-K\|_\infty \int_a^b p(t)\,|g(t)-\mathcal{M}(g;p)|\,dt \tag{2.8}$$
$$\le \|f(\cdot)-K\|_\infty \int_a^b p(t)\,|g(t)-\mathcal{M}(g;p)|\,dt,$$

which are tighter than those in Lemma 2.2.

However, (2.8) relies on knowing where the shifted functions are positive and where they are negative. This is not always an easy task.

The first result in (2.6) arises from (2.8) with $K = \frac{\Delta+\delta}{2}$ so that

$$\left\|f(\cdot)-\frac{\Delta+\delta}{2}\right\|_\infty = \sup_{t\in[a,b]}\left|f(t)-\frac{\Delta+\delta}{2}\right| = \frac{\Delta-\delta}{2},$$

where $-\infty < \delta \le f(t) \le \Delta < \infty$ for $t \in [a,b]$.

3. Bounds for $\tilde{S}_\mu(r)$ via the Čebyšev functional

Bounds on the Čebyšev functional (2.1) may be looked upon as estimating the distance of the weighted mean of the product of two functions from the product of the weighted means of the two functions. This proves to be quite useful since the individual means are invariably easier to evaluate.

Here we investigate the bounding of the alternating generalized Mathieu series, $\tilde{S}_\mu(r)$ through the identities (1.22) – (1.23). We notice that bounding $\tilde{S}_\mu(r)$ is accomplished via $\tilde{\chi}_\mu(r)$ where

$$\tilde{\chi}_\mu(r) := \int_0^\infty \frac{x^{\mu+\frac{1}{2}}}{e^x+1} J_{\mu-\frac{1}{2}}(rx)\,dx \qquad \mu, r > 0, \tag{3.1}$$

since from (1.23)

$$\tilde{S}_{\mu}(r)=C_{\mu}(r)\,\tilde{\chi}_{\mu}(r)\,, \tag{3.2}$$

where $C_{\mu}(r)$ is positive as defined in (1.8).

The following technical lemma involving the Euler beta function will be required, $B(x,y)$ and which is represented in terms of the gamma function by

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{3.3}$$

Lemma 3.1. *The following result holds*

$$\frac{1}{2}\cdot\frac{B(\frac{1}{2},\mu)}{[\alpha^2+r^2]^{2\mu-\frac{1}{2}}}\le\int_0^{\frac{\pi}{2}}\frac{\cos^{2\mu-1}\phi}{[\alpha^2+r^2\cos^2\phi]^{2\mu-\frac{1}{2}}}d\phi\le\frac{1}{2}\cdot\frac{B(\frac{1}{2},\mu)}{\alpha^{4\mu-1}}, \tag{3.4}$$

It is noted that equality follows in (3.4) when $r=0$.

Proof. Making the substitution $p=r\cos\phi$ gives

$$\int_0^{\frac{\pi}{2}}\frac{\cos^{2\mu-1}\phi}{[\alpha^2+r^2\cos^2\phi]^{2\mu-\frac{1}{2}}}d\phi=\frac{1}{r^{2\mu-1}}\int_0^r\frac{p^{2\mu-1}}{[\alpha^2+p^2]^{2\mu-\frac{1}{2}}\cdot\sqrt{r^2-p^2}}dp. \tag{3.5}$$

Now, since $0\le p\le r$ then

$$\frac{1}{\alpha^2+r^2}\le\frac{1}{\alpha^2+p^2}\le\frac{1}{\alpha^2}$$

and so from (3.5)

$$\begin{aligned}\frac{1}{r^{2\mu-1}}\cdot\frac{1}{[\alpha^2+r^2]^{2\mu-\frac{1}{2}}}\cdot\int_0^r\frac{p^{2\mu-1}}{\sqrt{r^2-p^2}}dp\ &\le\ \frac{1}{r^{2\mu-1}}\cdot\int_0^r\frac{p^{2\mu-1}}{[\alpha^2+p^2]^{2\mu-\frac{1}{2}}\cdot\sqrt{r^2-p^2}}dp\\ &\le\ \frac{1}{\alpha^{4\mu-1}}\cdot\frac{1}{r^{2\mu-1}}\cdot\int_0^r\frac{p^{2\mu-1}}{\sqrt{r^2-p^2}}dp.\end{aligned}$$

Further, since $\int_0^r\frac{p^{2\mu-1}}{\sqrt{r^2-p^2}}dp=r^{2\mu-1}\int_0^1\frac{\rho^{2\mu-1}}{\sqrt{1-\rho^2}}d\rho=\frac{r^{2\mu-1}}{2}B(\frac{1}{2},\mu)=\frac{r^{2\mu-1}}{2}\frac{\Gamma(\frac{1}{2})\Gamma(\mu)}{\Gamma(\mu+\frac{1}{2})}$ where $\Gamma(\frac{1}{2})=\sqrt{\pi}$ then the result (3.4) follows. □

The following lemma examines the behavior of $\tilde{\chi}_{\mu}(r)$ as defined by (3.1)

Lemma 3.2.

$$\begin{aligned}&\left|\tilde{\chi}_{\mu}(r)-\frac{1}{2}\cdot\frac{(2r)^{\mu-\frac{1}{2}}}{\sqrt{\pi}}\cdot\frac{\Gamma(\mu)}{\left(r^2+\frac{1}{4}\right)^{\mu}}\cdot\frac{\pi^2}{12}\right|\\ &\qquad\le\tilde{\kappa}\left[\frac{\Gamma\left(2\mu-\frac{1}{2}\right)r^{2\mu-1}}{\pi^{\frac{3}{2}}}\int_0^{\frac{\pi}{2}}\frac{\cos^{2\mu-1}\phi}{\left[\left(\frac{1}{4}\right)^2+r^2\cos^2\phi\right]^{2\mu-\frac{1}{2}}}d\phi-2K_*^2\right]^{\frac{1}{2}},\end{aligned} \tag{3.6}$$

where

$$K_*=\frac{(2r)^{\mu-\frac{1}{2}}\,\Gamma(\mu)}{2\sqrt{\pi}\left(r^2+\frac{1}{4}\right)^{\mu}}\quad\textit{is defined in (3.16),}$$

and

$$\tilde{\kappa}=\left[\frac{\pi^3}{4}-8\cdot G-2\cdot\left(\frac{\pi^2}{24}\right)^2\right]^{\frac{1}{2}}=0.29260623049\ldots. \tag{3.7}$$

and G is Catalan's constant, ([24, p. 610]).

Proof. Firstly, we notice that $\tilde{\chi}_\mu(r)$ from (3.1) may be written in the form

$$\tilde{\chi}_\mu(r)=\int_0^\infty e^{-\frac{x}{2}}\cdot\frac{x}{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}\cdot x^{\mu-\frac{1}{2}}J_{\mu-\frac{1}{2}}(rx)\,dx. \tag{3.8}$$

Let

$$p(x)=e^{-\frac{x}{2}},\quad f(x)=\frac{x}{e^{\frac{x}{2}}+e^{-\frac{x}{2}}},\quad g(x)=x^{\mu-\frac{1}{2}}J_{\mu-\frac{1}{2}}(rx) \tag{3.9}$$

then from (2.2)

$$P=\int_0^\infty p(x)\,dx=\int_0^\infty e^{-\frac{x}{2}}dx=2, \tag{3.10}$$

$$P\cdot\mathcal{M}(f;p)=\int_0^\infty e^{-\frac{x}{2}}\cdot\frac{x}{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}dx=\int_0^\infty\frac{x}{e^x+1}dx=\frac{\zeta(2)}{2}=\frac{\pi^2}{12} \tag{3.11}$$

and

$$P\cdot\mathcal{M}(g;p)=\int_0^\infty e^{-\frac{x}{2}}\cdot x^{\mu-\frac{1}{2}}J_{\mu-\frac{1}{2}}(rx)\,dx=\frac{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu)}{\sqrt{\pi}\left(\left(\frac{1}{2}\right)^2+r^2\right)^\mu}, \tag{3.12}$$

where we have used the fact that [13]

$$\left(1-2^{-p}\right)\cdot\Gamma(p+1)\,\zeta(p+1)=\int_0^\infty\frac{x^p}{e^x+1}dx\ , \tag{3.13}$$

to procure (3.11) and from Watson [32, p. 386]

$$\int_0^\infty e^{-\alpha x}\cdot x^\nu J_\nu(\beta x)\,dx=\frac{(2\beta)^\nu}{\sqrt{\pi}}\cdot\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{\left(\alpha^2+\beta^2\right)^{\nu+\frac{1}{2}}},\quad \mathrm{Re}(\nu)>\frac{1}{2},\quad \mathrm{Re}(\alpha)>|\mathrm{Im}(\beta)|,$$

with $\alpha=\frac{1}{2}$, $\nu=\mu-\frac{1}{2}$, $\beta=r$ to obtain (3.12).

Now, from (2.1) – (2.4) we have on using (3.9) – (3.12)

$$\begin{aligned}\tilde{\chi}_\mu(r)&-\frac{1}{2}\cdot\frac{(2r)^{\mu-\frac{1}{2}}}{\sqrt{\pi}}\cdot\frac{\Gamma(\mu)}{\left(r^2+\frac{1}{4}\right)^\mu}\cdot\frac{\pi^2}{12}\\&=\int_0^\infty e^{-\frac{x}{2}}\left(x^{\mu-\frac{1}{2}}J_{\mu-\frac{1}{2}}(rx)-K\right)\left(\frac{x}{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}-\frac{\pi^2}{24}\right)dx.\end{aligned} \tag{3.14}$$

Further, using the Cauchy-Buniakowsky-Schwartz inequality, we have from (3.14)

$$\begin{aligned}&\left|\tilde{\chi}_\mu(r)-\frac{1}{2}\cdot\frac{(2r)^{\mu-\frac{1}{2}}}{\sqrt{\pi}}\cdot\frac{\Gamma(\mu)}{\left(r^2+\frac{1}{4}\right)^\mu}\cdot\frac{\pi^2}{12}\right|\\&\qquad\le\left(\int_0^\infty e^{-\frac{x}{2}}\left(x^{\mu-\frac{1}{2}}J_{\mu-\frac{1}{2}}(rx)-K\right)^2dx\right)^{\frac{1}{2}}\\&\qquad\qquad\times\left(\int_0^\infty e^{-\frac{x}{2}}\left(\frac{x}{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}-\frac{\pi^2}{24}\right)^2dx\right)^{\frac{1}{2}}.\end{aligned} \tag{3.15}$$

As mentioned in Section 2, the appropriate choice of K is the weighted integral mean as given from (3.12), namely

$$K=K_*=\frac{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu)}{2\sqrt{\pi}\left(r^2+\frac{1}{4}\right)^\mu}. \tag{3.16}$$

It may be easily shown by expansion that

$$\int_a^b p(t)\left[h(t)-\mathcal{M}(h;p)\right]^2dt=\int_a^b p(t)\,h^2(t)\,dt-P\cdot\mathcal{M}^2(h;p). \tag{3.17}$$

The result (3.17) was a by product of the proof of Lemma 2.2.

This result will be utilized to evaluate the two expressions on the right hand side of (3.15).

Thus from (3.15) we have

$$\int_0^\infty e^{-\frac{x}{2}}\left(\frac{x}{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}-\frac{\pi^2}{24}\right)^2 dx=\int_0^\infty e^{-\frac{x}{2}}\left(\frac{x}{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}\right)^2 dx-2\left(\frac{\pi^2}{24}\right)^2. \tag{3.18}$$

Now, allowing for the permissible interchange of integration and summation, we have

$$\begin{aligned}\int_0^\infty e^{-\frac{x}{2}}\left(\frac{x}{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}\right)^2 dx &=\int_0^\infty e^{-\frac{3x}{2}}\left(\frac{x}{1+e^{-x}}\right)^2 dx \\ &=\int_0^\infty e^{-\frac{3x}{2}}x^2\left(\sum_{n=1}^\infty(-1)^{n-1}ne^{-nx}\right)dx \\ &=\sum_{n=1}^\infty(-1)^{n-1}n\int_0^\infty e^{-\left(\frac{2n+1}{2}\right)x}x^2dx \\ &=\sum_{n=1}^\infty\frac{(-1)^{n-1}n\Gamma(3)}{\left(\frac{2n+1}{2}\right)^3}=\sum_{n=1}^\infty\frac{(-1)^{n-1}2n}{\left(n+\frac{1}{2}\right)^3} \\ &=2^3\sum_{n=1}^\infty(-1)^{(n-1)}\left(\frac{1}{(2n+1)^2}-\sum_{n=1}^\infty\frac{1}{(2n+1)^3}\right) \\ &=\frac{\pi^3}{4}-8\cdot G.\end{aligned} \tag{3.19}$$

where $G=\sum_{n=0}^\infty\frac{(-1)^n}{(2n+1)^2}=0.9159655941772$ ([24, p. 610]), Catalan's constant and $\sum_{n=0}^\infty\frac{(-1)^n}{(2n-1)^3}=\frac{\pi^3}{32}$ (see [24, Chapters 24-25]).

In (3.19) we have used the fact that

$$\int_0^\infty e^{-\alpha x}x^pdx=\frac{\Gamma(p+1)}{\alpha^{p+1}}.$$

Hence, from (3.18) and (3.19) we have

$$\left[\int_0^\infty e^{-\frac{x}{2}}\left(\frac{x}{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}-\frac{\pi^2}{24}\right)^2 dx\right]^{\frac{1}{2}}=\left[\frac{\pi^3}{4}-8\cdot G-2\cdot\left(\frac{\pi^2}{24}\right)^2\right]^{\frac{1}{2}}. \tag{3.20}$$

Now, for the first expression on the right hand side of (3.15), we have, on using (3.16) and (3.17)

$$\int_0^\infty e^{-\frac{x}{2}}\left(x^{\mu-\frac{1}{2}}J_{\mu-\frac{1}{2}}(rx)-K_*\right)^2dx=\int_0^\infty e^{-\frac{x}{2}}x^{2\mu-1}J^2_{\mu-\frac{1}{2}}(rx)\,dx-2K_*^2. \tag{3.21}$$

A result in Watson [32, p. 290] states that

$$\int_0^\infty e^{-2at}J_\alpha(\gamma t)\,J_\beta(\gamma t)\,t^{\alpha+\beta}dt$$
$$=\frac{\Gamma\left(\alpha+\beta+\frac{1}{2}\right)}{\pi^{\frac{3}{2}}}\gamma^{\alpha+\beta}\int_0^{\frac{\pi}{2}}\frac{\cos^{\alpha+\beta}\phi\cos(\alpha-\beta)\,\phi}{(a^2+\gamma^2\cos^2\phi)^{\alpha+\beta+\frac{1}{2}}}d\phi \tag{3.22}$$

and so taking $a = \frac{1}{4}$, $\alpha = \beta = \mu - \frac{1}{2}$ and $\gamma = r$ in (3.22) gives

$$\int_0^\infty e^{-\frac{x}{2}} x^{2\mu-1} J^2_{\mu-\frac{1}{2}}(rx)\,dx = \frac{\Gamma\left(2\mu-\frac{1}{2}\right) r^{2\mu-1}}{\pi^{\frac{3}{2}}} \int_0^{\frac{\pi}{2}} \frac{\cos^{2\mu-1}\phi}{\left(\left(\frac{1}{4}\right)^2 + r^2\cos^2\phi\right)^{2\mu-\frac{1}{2}}} d\phi \quad (3.23)$$

That is,

$$\left[\int_0^\infty e^{-\frac{x}{2}} \left(x^{\mu-\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx) - K_*\right)^2 dx\right]^{\frac{1}{2}} = \left[\frac{\Gamma\left(2\mu-\frac{1}{2}\right)}{\pi^{\frac{3}{2}}} r^{2\mu-1} \int_0^{\frac{\pi}{2}} \frac{\cos^{2\mu-1}\phi}{\left[\left(\frac{1}{4}\right)^2 + r^2\cos^2\phi\right]^{2\mu-\frac{1}{2}}} d\phi - 2K_*^2\right]^{\frac{1}{2}}. \quad (3.24)$$

Placing (3.24) and (3.20) into (3.15) produces the stated result (3.6). □

Theorem 3.3. *For $\mu > 0$ and $r > 0$, the alternating generalized Mathieu series $\tilde{S}_\mu(r)$ satisfies the following bounds,*

$$\left|\tilde{S}_\mu(r) - \frac{\pi^2}{24\mu\left(r^2+\frac{1}{4}\right)^\mu}\right| \quad (3.25)$$

$$\le \tilde{\kappa}\left[\frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma\left(2\mu-\frac{1}{2}\right)}{2^{2\mu-1}\Gamma^2(\mu+1)} \int_0^{\frac{\pi}{2}} \frac{\cos^{2\mu-1}\phi}{\left[\left(\frac{1}{4}\right)^2 + r^2\cos^2\phi\right]^{2\mu-\frac{1}{2}}} d\phi - \frac{1}{2\mu^2\left(r^2+\frac{1}{4}\right)^{2\mu}}\right]^{\frac{1}{2}}$$

$$\le \tilde{\kappa}\left[\frac{1}{2^{3\mu-1}\mu^2(\mu-\frac{1}{2})B(\mu,\mu-\frac{1}{2})} - \frac{1}{2\mu^2\left(r^2+\frac{1}{4}\right)^{2\mu}}\right]^{\frac{1}{2}},$$

where $\tilde{\kappa}$ is as given by (3.7).

Proof. From (3.2) we have, since $C_\mu(r)$, as defined by (1.8), is positive so that using Lemma 3.2 readily produces the above results (3.25) upon simplification. The coarser bound results on using Lemma 3.1. □

Corollary 3.4. *The alternating Mathieu series. $\tilde{S}(r)$ satisfies the result*

$$\left|\sum_{n=1}^\infty \frac{(-1)^{n-1}\cdot 2n}{(n^2+r^2)^2} - \frac{\pi^2}{24\left(r^2+\frac{1}{4}\right)}\right| \le 2\sqrt{2}\cdot\tilde{\kappa}\left\{\frac{2}{1+(4r)^2} - \frac{1}{\left[1+(2r)^2\right]^2}\right\}^{\frac{1}{2}}, \quad (3.26)$$

where $\tilde{\kappa}$ is as given by (3.7).

Proof. Let $\mu = 1$ in (3.25) and using (1.1) and (1.5) gives the above result (3.26), on noting that

$$2^6 \int_0^{\frac{\pi}{2}} \frac{\cos\phi}{\left[1+(4r\cos\phi)^2\right]^{\frac{3}{2}}} d\phi = \frac{64}{1+(4r)^2}$$

and after some simplification. □

Remark 3.5. The result of Theorem 3.3 holds for any $\mu > 0$ and $r > 0$ whereas those obtained in [8] were valid for $\mu > \frac{1}{2}$.

Remark 3.6. From (3.27) we may infer,since $\tilde{\kappa} > 0$,

$$G < \frac{\pi 3}{32}\left(1 - \frac{\pi}{72}\right) = 0.9266678949\ldots. \tag{3.27}$$

4. Odd and even generalized Mathieu series

Using the generalized Mathieu series, $S_\mu(r)$ as given in (1.1) and (1.7)-(1.8) together with the *alternating* generalized Mathieu series $\tilde{S}_\mu(r)$ as given in (1.22)-(1.23) we introduce the *odd generalized Mathieu series*, $\phi_\mu(r)$ and the *even generalized Mathieu series,* $\psi_\mu(r)$. These are given by

$$\begin{aligned}
\phi_\mu(r) &:= \frac{S_\mu(r) + \tilde{S}_\mu(r)}{2} \qquad (4.1)\\
&= \sum_{n=1}^{\infty} \frac{2(2n-1)}{\left((2n-1)^2 + r^2\right)^{1+\mu}}\\
&= C_\mu(r) \cdot 2\int_0^\infty \frac{x^{\mu+\frac{1}{2}}}{e^x - e^{-x}} J_{\mu-\frac{1}{2}}(rx)\,dx, \quad r,\mu > 0,
\end{aligned}$$

and

$$\begin{aligned}
\psi_\mu(r) &:= \frac{S_\mu(r) - \tilde{S}_\mu(r)}{2} \qquad (4.2)\\
&= \sum_{n=1}^{\infty} \frac{2\cdot(2n)}{\left((2n)^2 + r^2\right)^{1+\mu}}\\
&= C_\mu(r) \cdot 2\int_0^\infty \frac{x^{\mu+\frac{1}{2}}}{e^{2x} - 1} J_{\mu-\frac{1}{2}}(rx)\,dx, \qquad r,\mu > 0,
\end{aligned}$$

where $C_\mu(r)$ is positive as defined in (1.8).

Remark 4.1. It may be noticed that if we have *identities* for *any two* of the generalized Mathieu type series $S_\mu(r)$, $\tilde{S}_\mu(r)$, $\phi_\mu(r)$, $\psi_\mu(r)$ then we may deduce the other two. In particular $S_\mu(r) = \frac{\phi_\mu(r) + \psi_\mu(r)}{2}$ and $\tilde{S}_\mu(r) = \frac{\phi_\mu(r) - \psi_\mu(r)}{2}$.This however, is **not** the case with regards to inequalities or bounds since, recourse to the triangle inequality would result in a coarser bound. We may further notice that their integral representation may be given by

$$2C_\mu(r)\int_0^\infty H(x)\cdot x^{\mu-\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx)\,dx, \qquad r,\mu > 0 \tag{4.3}$$

where $C_\mu(r)$ is positive as defined in (1.8)and $H(x)$ is one of the following

$$H_M(x) = \frac{x}{e^x - 1},\ H_A(x) = \frac{x}{e^x + 1},\ H_O(x) = \frac{x}{e^x - e^{-x}},\ H_E(x) = \frac{x}{e^{2x} - 1}, \tag{4.4}$$

where the subscripts relate to the generalized Mathieu , alternating Mathieu, odd Mathieu and even Mathieu series integral representations, respectively.

Remark 4.2. It should be emphasized that the $H_\cdot(\cdot)$ in (4.4) represent the weights associated with the integral representation of the generalized Mathieu and its companions. They satisfy the following conditions

$$\begin{aligned}
H_A(x) &< H_E(x) < H_O(x) < H_M(x) \quad, x < \ln(2) \qquad (4.5)\\
H_E(x) &< H_A(x) < H_O(x) < H_M(x) \quad, x > \ln(2).
\end{aligned}$$

We will now follow closely the approach of Section 3 in obtaining results pertaining to the odd and even generalized Mathieu series. The subscripts of O and E will be used to denote the cases related to $\phi_\mu(r)$ and $\psi_\mu(r)$ respectively.

We note from (4.1) that

$$\frac{\phi_\mu(r)}{2C_\mu(r)} = \int_0^\infty H_O(x)\cdot x^{\mu-\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx)\,dx, \qquad r,\mu>0 \tag{4.6}$$

where from (4.4)

$$H_O(x) = \frac{x}{e^x - e^{-x}}. \tag{4.7}$$

Theorem 4.3. *For $\mu > 0$ and $r > 0$ the odd generalized Mathieu series $\phi_\mu(r)$ satisfies the following relationship, namely,*

$$\left|\phi_\mu(r) - \frac{\pi^2}{4\mu\,(r^2+1)^\mu}\right| \tag{4.8}$$

$$\le \kappa_O \left[\frac{4\Gamma\left(2\mu-\frac{1}{2}\right)}{2^{2\mu-1}\sqrt{\pi}\,\Gamma^2(\mu+1)} \int_0^{\frac{\pi}{2}} \frac{\cos^{2\mu-1}\phi}{\left[(1)^2 + r^2\cos^2\phi\right]^{2\mu-\frac{1}{2}}}d\phi - \frac{4}{\mu^2\,(1^2+r^2)^{2\mu}}\right]^{\frac{1}{2}}$$

$$\le \kappa_O \left[\frac{1}{4^{\mu-1}\mu^2(\mu-\frac{1}{2})\cdot B\left(\mu,\mu-\frac{1}{2}\right)} - \frac{4}{\mu^2\,(1^2+r^2)^{2\mu}}\right]^{\frac{1}{2}},$$

where,

$$\kappa_O = \left[\frac{\pi^2}{8}(1-\frac{\pi^2}{8}) + \frac{7}{8}\zeta(3)\right]^{\frac{1}{2}}$$

and $B(x,y)$ is the Euler beta function given by (3.3).

Proof. We notice that $\frac{\phi_\mu(r)}{2C_\mu(r)}$ from (4.6) and (4.7) may be written in the form

$$\frac{\phi_\mu(r)}{2C_\mu(r)} = \int_0^\infty e^{-x}\cdot\frac{x}{1-e^{-2x}}\cdot x^{\mu-\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx)\,dx, \qquad r,\mu>0 \tag{4.9}$$

If we now let

$$p_O(x) = e^{-x}, \quad f_O(x) = \frac{x}{1-e^{-2x}}, \quad g(x) = x^{\mu-\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx) \tag{4.10}$$

then from (2.2)

$$P_O = \int_0^\infty p_O(x)\,dx = \int_0^\infty e^{-x}dx = 1, \tag{4.11}$$

$$P_O\cdot\mathcal{M}(f_O;p) = \int_0^\infty e^{-x}\cdot\frac{x}{1-e^{-2x}}dx = (1-2^{-2})\cdot\zeta(2) = \frac{\pi^2}{8} \tag{4.12}$$

and

$$P_O\cdot\mathcal{M}(g;p) = \int_0^\infty e^{-x}\cdot x^{\mu-\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx)\,dx = \frac{(2r)^{\mu-\frac{1}{2}}\,\Gamma(\mu)}{\sqrt{\pi}\,(1^2+r^2)^\mu}, \tag{4.13}$$

where we have used the fact that [13]

$$(1-2^{-(p+1)})\cdot\Gamma(p+1)\,\zeta(p+1) = \int_0^\infty \frac{x^p}{e^x-e^{-x}}dx \tag{4.14}$$

to procure (4.12),and from Watson [32, p. 386]

$$\int_0^\infty e^{-\alpha x}\cdot x^\nu J_\nu(\beta x)\,dx = \frac{(2\beta)^\nu}{\sqrt{\pi}}\cdot\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{(\alpha^2+\beta^2)^{\nu+\frac{1}{2}}}, \quad \mathrm{Re}(\nu) > \frac{1}{2}, \quad \mathrm{Re}(\alpha) > |\mathrm{Im}(\beta)|,$$

with $\alpha = 1$, $\nu = \mu - \frac{1}{2}$, $\beta = r$ to obtain (4.13).

Now, from (2.1) – (2.4) we have on using (3.9) – (3.12)

$$\frac{\phi_\mu(r)}{2C_\mu(r)} - \frac{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu)}{\sqrt{\pi}(1^2+r^2)^\mu} \cdot \frac{\pi^2}{8} = \int_0^\infty e^{-x}\left(x^{\mu-\frac{1}{2}}J_{\mu-\frac{1}{2}}(rx) - K\right)\left(\frac{x}{1-e^{-2x}} - \frac{\pi^2}{8}\right)dx. \tag{4.15}$$

Further, using the Cauchy-Buniakowsky-Schwartz inequality, we have from (3.14)

$$\left|\frac{\phi_\mu(r)}{2C_\mu(r)} - \frac{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu)}{\sqrt{\pi}(1^2+r^2)^\mu} \cdot \frac{\pi^2}{8}\right| \leq \left(\int_0^\infty e^{-x}\left(x^{\mu-\frac{1}{2}}J_{\mu-\frac{1}{2}}(rx) - K\right)^2 dx\right)^{\frac{1}{2}} \times \left(\int_0^\infty e^{-x}\left(\frac{x}{1-e^{-2x}} - \frac{\pi^2}{8}\right)^2 dx\right)^{\frac{1}{2}}. \tag{4.16}$$

As mentioned in Section 2, the appropriate choice of K is the weighted integral mean as given from (4.13), namely

$$K = K_* = \frac{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu)}{\sqrt{\pi}(1^2+r^2)^\mu}. \tag{4.17}$$

Now using the result

$$\int_a^b p(t)\left[h(t) - \mathcal{M}(h;p)\right]^2 dt = \int_a^b p(t)\, h^2(t)\, dt - P \cdot \mathcal{M}^2(h;p). \tag{4.18}$$

to evaluate the two expressions on the right hand side of (4.16) produces; firstly,

$$\int_0^\infty e^{-x}\left(\frac{x}{1-e^{-2x}} - \frac{\pi^2}{8}\right)^2 dx = \int_0^\infty e^{-x}\left(\frac{x}{1-e^{-2x}}\right)^2 dx - 1 \cdot \left(\frac{\pi^2}{8}\right)^2. \tag{4.19}$$

and secondly, allowing for the permissible interchange of integration and summation, we have

$$\begin{aligned}
\int_0^\infty e^{-x}\left(\frac{x}{1-e^{-2x}}\right)^2 dx &= \int_0^\infty e^{-x}\left(\frac{x}{1-e^{-2x}}\right)^2 dx \\
&= \int_0^\infty e^{-x}x^2 \cdot e^{2x}\left(\sum_{n=1}^\infty ne^{-2nx}\right)dx \\
&= \sum_{n=1}^\infty n\int_0^\infty e^{-(2n-1)x}x^2 dx \\
&= \sum_{n=1}^\infty \frac{n\Gamma(3)}{(2n-1)^3} = \sum_{n=1}^\infty \frac{2n}{(2n-1)^3} \\
&= \sum_{n=0}^\infty\left(\frac{1}{(2n+1)^2} + \sum_{n=1}^\infty \frac{1}{(2n+1)^3}\right) \\
&= \frac{\pi^2}{8} + \frac{7}{8}\zeta(3)
\end{aligned} \tag{4.20}$$

where $\sum_{n=0}^\infty \frac{1}{(2n+1)^n} = (1-2^{-n})\zeta(n)$ ([24, p. 602]) and, in (4.20) we have used the fact that

$$\int_0^\infty e^{-\alpha x}x^p dx = \frac{\Gamma(p+1)}{\alpha^{p+1}}.$$

Hence, from (4.19) and (4.20) we have

$$\left[\int_0^\infty e^{-x}\left(\frac{x}{1-e^{-2x}}-\frac{\pi^2}{8}\right)^2 dx\right]^{\frac{1}{2}}=\left[\frac{\pi^2}{8}(1-\frac{\pi^2}{8})+\frac{7}{8}\zeta\left(3\right)\right]^{\frac{1}{2}}. \tag{4.21}$$

Now, for the first expression on the right hand side of (4.16), we have, on using (4.17) and (4.18)

$$\int_0^\infty e^{-x}\left(x^{\mu-\frac{1}{2}}J_{\mu-\frac{1}{2}}\left(rx\right)-K_*\right)^2 dx=\int_0^\infty e^{-x}x^{2\mu-1}J^2_{\mu-\frac{1}{2}}\left(rx\right)dx-1\cdot K_*^2. \tag{4.22}$$

A result in Watson [32, p. 290] states that

$$\int_0^\infty e^{-2at}J_\alpha\left(\gamma t\right)J_\beta\left(\gamma t\right)t^{\alpha+\beta}dt$$
$$=\frac{\Gamma\left(\alpha+\beta+\frac{1}{2}\right)}{\pi^{\frac{3}{2}}}\gamma^{\alpha+\beta}\int_0^{\frac{\pi}{2}}\frac{\cos^{\alpha+\beta}\phi\cos\left(\alpha-\beta\right)\phi}{\left(a^2+\gamma^2\cos^2\phi\right)^{\alpha+\beta+\frac{1}{2}}}d\phi \tag{4.23}$$

and so taking $a=\frac{1}{2}$, $\alpha=\beta=\mu-\frac{1}{2}$ and $\gamma=r$ in (4.23) gives

$$\int_0^\infty e^{-x}x^{2\mu-1}J^2_{\mu-\frac{1}{2}}\left(rx\right)dx=\frac{\Gamma\left(2\mu-\frac{1}{2}\right)r^{2\mu-1}}{\pi^{\frac{3}{2}}}\int_0^{\frac{\pi}{2}}\frac{\cos^{2\mu-1}\phi}{\left(\left(\frac{1}{2}\right)^2+r^2\cos^2\phi\right)^{2\mu-\frac{1}{2}}}d\phi \tag{4.24}$$

That is, from (4.22) and (4.24) we have

$$\left[\int_0^\infty e^{-x}\left(x^{\mu-\frac{1}{2}}J_{\mu-\frac{1}{2}}\left(rx\right)-K_*\right)^2 dx\right]^{\frac{1}{2}}$$
$$=\left[\frac{\Gamma\left(2\mu-\frac{1}{2}\right)}{\pi^{\frac{3}{2}}}r^{2\mu-1}\int_0^{\frac{\pi}{2}}\frac{\cos^{2\mu-1}\phi}{\left[\left(\frac{1}{2}\right)^2+r^2\cos^2\phi\right]^{2\mu-\frac{1}{2}}}d\phi-1\cdot K_*^2\right]^{\frac{1}{2}}. \tag{4.25}$$

Placing (4.25) and (4.21) into (4.16) produces the stated result (4.8) upon multiplication by $2C_\mu\left(r\right)$ and using (1.8).

For the coarser bound in (4.8) we have from (3.4) of Lemma 3.1

$$\int_0^{\frac{\pi}{2}}\frac{\cos^{2\mu-1}\phi}{\left[1^2+r^2\cos^2\phi\right]^{2\mu-\frac{1}{2}}}d\phi\le\frac{1}{2}B(\frac{1}{2},\mu)=\frac{\sqrt{\pi}\Gamma\left(\mu\right)}{2\Gamma\left(\mu+\frac{1}{2}\right)}$$

and so on substitution into the first result in (4.8) produces the second, upon some simplification. □

Theorem 4.4. *For $\mu>0$ and $r>0$ the even generalized Mathieu series $\psi_\mu(r)$ satisfies the following relationship*

$$\left|\psi_\mu(r)-\frac{\pi^2}{6\mu\left(r^2+2^2\right)^\mu}\right| \tag{4.26}$$

$$\leq \kappa_E \left[\frac{4\Gamma\left(2\mu - \frac{1}{2}\right)}{2^{2\mu-1}\sqrt{\pi}\Gamma^2(\mu+1)} \int_0^{\frac{\pi}{2}} \frac{\cos^{2\mu-1}\phi}{\left[(1)^2 + r^2\cos^2\phi\right]^{2\mu-\frac{1}{2}}} d\phi - \frac{8}{\mu^2(1^2+r^2)^{2\mu}} \right]^{\frac{1}{2}}$$

$$\leq \kappa_E \left[\frac{1}{4^{\mu-1}\mu^2(\mu-\frac{1}{2})\cdot B\left(\mu, \mu-\frac{1}{2}\right)} - \frac{8}{\mu^2(1^2+r^2)^{2\mu}} \right]^{\frac{1}{2}},$$

where,

$$\kappa_E = \left[\frac{\pi^2}{24}(1-\frac{\pi^2}{12})\right]^{\frac{1}{2}} \tag{4.27}$$

and $B(x,y)$ is the Euler beta function given by (3.3).

Proof. (Sketch) The proof follows that of the previous theorem. The subscript E represents terms related to the *even generalized Mathieu series.*

Some of the terms are : $p_E(x) = e^{-2x},\ f_E(x) = \frac{x}{1-e^{-2x}}$ and $g(x) = x^{\mu-1}J_{\mu-\frac{1}{2}}(rx)$.

Further, $P_E = \int_0^\infty p_E(x)dx = \frac{1}{2},\ P_E \cdot \mathcal{M}(f_E; p_E) = \int_0^\infty e^{-2x}\frac{x}{1-e^{-2x}}dx = \int_0^\infty \frac{x}{e^{2x}-1}dx = \frac{\pi^2}{24}$ and $P_E \cdot \mathcal{M}(g; p_E) = \int_0^\infty e^{-2x}\cdot x^{\mu-\frac{1}{2}}J_{\mu-\frac{1}{2}}(rx)\,dx = \frac{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu)}{\sqrt{\pi}(2^2+r^2)^{\mu}}$.

We omit further details. □

Lemma 4.5. *The companion generalised Mathieu series may be expressed in terms of the generalised Mathieu series, namely,*

$$\begin{aligned} \tilde{S}_\mu(r) &= S_\mu(r) - 4^{-\mu}S_\mu\left(\frac{r}{2}\right) \\ \phi_\mu(r) &= S_\mu(r) - 2^{-2\mu-1}S_\mu\left(\frac{r}{2}\right) \\ \psi_\mu(r) &= 2^{-2\mu-1}S_\mu\left(\frac{r}{2}\right). \end{aligned} \tag{4.28}$$

Proof. From the generalised Mathieu series (1.5) it may be shown that

$$S_\mu\left(\frac{r}{2}\right) = 2^{2\mu+1}\sum_{n=1}^\infty \frac{2\cdot(2n)}{((2n)^2+r^2)^{1+\mu}} \tag{4.29}$$

and so from (4.2) and (4.29)gives $\psi_\mu(r) = 2^{-2\mu-1}S_\mu\left(\frac{r}{2}\right)$,the third result. Further, the first result of (4.28) readily follows on noting that $2\cdot\psi_\mu(r) = S_\mu(r) - \tilde{S}_\mu(r)$. The second result is procured from 4.1), $2\cdot\phi_\mu(r) = S_\mu(r) + \tilde{S}_\mu(r)$ and substituting the first result for $\tilde{S}_\mu(r)$. □

Remark 4.6. It is important to emphasize, as mentioned earlier, that obtaining bounds for the companions in terms of those of the generalised Mathieu series would produce inferior bounds *from using* the triangle inequality required for the first two results in (4.28).

Theorem 4.7. *The following relationship holds,*

$$S_\mu(r) = 2\phi_\mu(r) - \tilde{S}_\mu(r). \tag{4.30}$$

Proof. The relationship (4.30) follows easily from (4.28) by subtracting the first equation from twice the second . □

Remark 4.8. The equation (4.30) recaptures, on allowing $r- > 0$, the well known result involving the Zeta function $\zeta(x)$

$$\zeta(x) = 2\lambda(x) - \eta(x) \tag{4.31}$$

where $\lambda(x)$ is the odd zeta and $\eta(x)$ is the alternating zeta and $x = 2\mu + 1$.This demonstrates that (4.30) is an extention of the Zeta expression (4.31) through the variable r of Mathieu type functions.

Remark 4.9. The Čebyšev Functional bounds have been used to procure bounds for the Mathieu family of special functions. Much effort has been expended in the literature as to various ways of bounding the Mathieu series (1.1). The accuracy of bounds over perticular regions of parameters cannot be determined **a priori** (see also for example [22],[2] A comparison of the bounds using (1.2) and (3.26) demonstates that the upper bound for the Mathieu series is better for $0 < r < 0.855662$ and for the lower bound, better over $0 < r < 1.206377$ for (3.26) and better for the remainder of r for the bounds (1.2). It must be remembered however that (3.26)is valid for the more general result involving parameters r and μ .

5. Some properties of the generalized Mathieu series and its companions

Let

$$G_\mu(r;H) = \gamma_\mu \int_0^\infty H(x)x^{\mu-\frac{1}{2}} \cdot \frac{J_{\mu-\frac{1}{2}}(rx)}{r^{\mu-\frac{1}{2}}}dx, \qquad r,\mu > 0 \tag{5.1}$$

where, from (4.4)

$$\gamma_\mu = \begin{cases} C_\mu = \frac{\sqrt{\pi}}{2^{\mu-\frac{1}{2}}\Gamma(\mu+1)}, \text{for } H_M(\cdot) \text{ and } H_A(\cdot) \\ 2C_\mu \ , \qquad\qquad\qquad \text{for } H_O(\cdot) \text{ and } H_E(\cdot) \end{cases} . \tag{5.2}$$

The following proposition determines the moments of (5.1). See also [9] where a Mellin transform has been treated only for the generalised Mathieu series.

Proposition 5.1. *The moments of $G_\mu(r;H)$ from (5.1) and (5.2)given by*

$$\begin{aligned} M^{(k)} &= \int_0^\infty r^k G_\mu(r;H)dr \\ &= [1or2]\,\frac{B(\frac{k}{2}+\frac{1}{2},\mu-\frac{k}{2}+\frac{1}{2})}{\Gamma\left(2\mu-k\right)}\int_0^\infty x^{2\mu-k-2}H(x)dx \end{aligned} \tag{5.3}$$

where $B(x,y)$ is the Euler beta function given by (3.3) and,

$$[1or2] = \begin{cases} 1, & \textit{for } H_M(\cdot) \textit{ and } H_A(\cdot) \\ 2, & \textit{for } H_O(\cdot) \textit{ and } H_E(\cdot) \end{cases} . \tag{5.4}$$

Proof. From from (5.1), (5.2) and (5.3) we have

$$M^{(k)} = \gamma_\mu \int_0^\infty x^{\mu-\frac{1}{2}}H(x)\int_0^\infty r^k \cdot \frac{J_{\mu-\frac{1}{2}}(rx)}{r^{\mu-\frac{1}{2}}}drdx \tag{5.5}$$

and so the substitution of $\omega = rx$ produces

$$\begin{aligned} \int_0^\infty r^k \cdot \frac{J_{\mu-\frac{1}{2}}(rx)}{r^{\mu-\frac{1}{2}}}dr &= x^{\mu-k-\frac{3}{2}}\int_0^\infty \omega^{k-(\mu-\frac{1}{2})} \cdot J_{\mu-\frac{1}{2}}(\omega)\,d\omega \\ &= x^{\mu-k-\frac{3}{2}} \cdot 2^{k-\mu+\frac{1}{2}}\frac{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)}{\Gamma\left(\mu-\frac{k}{2}\right)} \end{aligned} \tag{5.6}$$

where we have used the result

$$\int_0^\infty \omega^{\lambda-\nu} \cdot J_\nu(\omega)\,d\omega = 2^{\lambda-\nu}\frac{\Gamma\left(\frac{\lambda}{2}+\frac{1}{2}\right)}{\Gamma(\nu-\frac{\lambda}{2}+\frac{1}{2})}$$

with $\nu = \mu - \frac{1}{2}$ and $\lambda = k$.Thus, substituting (5.6) into (5.5) gives

$$M^{(k)} = \gamma_\mu \; \delta_{\mu,k} \int_0^\infty x^{2\mu-k-2} H(x)dx \tag{5.7}$$

where,

$$\delta_{\mu,k} = 2^{k-\mu+\frac{1}{2}} \frac{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)}{\Gamma\left(\mu-\frac{k}{2}\right)}. \tag{5.8}$$

Further, using the duplication formula for the gamma function, $\sqrt{\pi}\Gamma(2z) = 2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$ with $z = \mu - \frac{k}{2}$ we have $\Gamma\left(\mu - \frac{k}{2}\right) = \frac{\sqrt{\pi}\Gamma(2\mu-k)}{2^{2\mu-k-1}\Gamma\left(\mu-\frac{k}{2}+\frac{1}{2}\right)}$ and so, from (5.2) and (5.8), we have

$$\gamma_\mu \; \delta_{\mu,k} = [1or2] \frac{\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)\Gamma\left(\mu-\frac{k}{2}+\frac{1}{2}\right)}{\Gamma(\mu+1)\Gamma(2\mu-k)} \tag{5.9}$$

Substitution of (5.9) into (5.7) produces the statement of the proposition. □

The following Corollary gives the moments for the generalized Mathieu series and its companions.

Corollary 5.2. *Let the subscript of M, A, O, E indicate the generalized :Mathieu, Alternating, Odd and Even series moments respectively. These are then given by*

$$\begin{aligned}
M_M^{(k)} &= B(\frac{k}{2}+\frac{1}{2},\mu-\frac{k}{2}+\frac{1}{2})\zeta(2\mu-k)\,, \qquad 2\mu-k>1 \qquad (5.10)\\
M_A^{(k)} &= B(\frac{k}{2}+\frac{1}{2},\mu-\frac{k}{2}+\frac{1}{2})(1-2^{-(2\mu-k-1)})\zeta(2\mu-k)\,,\ 2\mu-k>0\\
M_O^{(k)} &= 2B(\frac{k}{2}+\frac{1}{2},\mu-\frac{k}{2}+\frac{1}{2})(1-2^{-(2\mu-k)})\zeta(2\mu-k)\,,\ 2\mu-k>1\\
M_E^{(k)} &= 2B(\frac{k}{2}+\frac{1}{2},\mu-\frac{k}{2}+\frac{1}{2})2^{-(2\mu-k)}\zeta(2\mu-k)\,,\ 2\mu-k>1
\end{aligned}$$

where $B(x,y)$ is the Euler beta function given by (3.3).

Proof. From (5.3) we require to evaluate the integral for the various $H(x)$ representing each of the generalized Mathieu functions. That is we require to evaluate

$$M(q;H) = \int_0^\infty x^{q-1} H(x)dx \tag{5.11}$$

where $q = 2\mu - k - 1$ and for each of $H(x)$ as given in (4.4).

Now, for the generalized Mathieu series we have from (4.4)

$$\begin{aligned}
M(q;H_M) &= \int_0^\infty \frac{x^q}{e^x-1}dx,\ q = 2\mu-k-1 \qquad (5.12)\\
&= \Gamma(2\mu-k)\zeta(2\mu-k)
\end{aligned}$$

where we have used the result from ([24, p.604, 25.5.1])

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x-1}dx,\ \mathrm{Re}s>1. \tag{5.13}$$

Substituting (5.12) into (5.3) and noting (5.4) gives the first result.

For the alternating generalized Mathieu series we have from (4.4)

$$\begin{aligned}
M(q;H_A) &= \int_0^\infty \frac{x^q}{e^x+1}dx,\ q = 2\mu-k-1 \qquad (5.14)\\
&= \Gamma(2\mu-k)(1-2^{1-(2\mu-k)})\zeta(2\mu-k)
\end{aligned}$$

where we have used the result from ([24, pg.604, 25.5.3])

$$\zeta(s)=\frac{1}{(1-2^{1-s})\Gamma(s)}\int_0^\infty \frac{x^{s-1}}{e^x+1}dx,\ \operatorname{Re}s>0. \tag{5.15}$$

Substituting (5.14) into (5.3) and noting (5.4) gives the second result.

Further for the odd generalized Mathieu series we have from (4.4)

$$\begin{aligned} M(q;H_O) &= \int_0^\infty \frac{x^q}{e^x-e^{-x}}dx,\ q=2\mu-k-1 \\ &= \Gamma(2\mu-k)(1-2^{-(2\mu-k)})\zeta(2\mu-k) \end{aligned} \tag{5.16}$$

where we have used the result,

$$\begin{aligned} \int_0^\infty e^{-x}\frac{x^q}{1-e^{-2x}}dx &= \int_0^\infty e^{-x}\frac{x^q}{1-e^{-2x}}dx \\ &= \int_0^\infty e^{-x}x^q\left(\sum_{n=0}^\infty e^{-2nx}\right)dx \\ &= \sum_{n=0}^\infty\int_0^\infty e^{-(2n+1)x}x^q dx \\ &= \sum_{n=0}^\infty \frac{\Gamma(q+1)}{(2n+1)^{q+1}} \\ &= \Gamma(q+1)(1-2^{-(q+1)})\zeta(q+1) \end{aligned}$$

where $\sum_{n=0}^\infty \frac{1}{(2n+1)^s}=(1-2^{-s})\zeta(s)$, $\operatorname{Re}s>1$. ([24, p.602, 25.2.2]) and we have allowed the permissible interchange of integration and summation.

Substituting (5.16) into (5.3) and noting (5.4) gives the third result.

Finally, for the even generalized Mathieu series we have from (4.4)

$$\begin{aligned} M(q;H_E) &= \int_0^\infty \frac{x^q}{e^{2x}-1}dx,\ q=2\mu-k-1 \\ &= \Gamma(2\mu-k)2^{-(2\mu-k)}\zeta(2\mu-k) \end{aligned} \tag{5.17}$$

where we have from (5.13)

$$\begin{aligned} \int_0^\infty \frac{x^q}{e^{2x}-1}dx &= 2^{-(q+1)}\int_0^\infty \frac{u^q}{e^u-1}du \\ &= \Gamma(q+1)\,2^{-(q+1)}\zeta(q+1) \end{aligned}$$

giving (5.17).

Substituting (5.17) into (5.3) and noting (5.4) gives the last result and thus completing the proof. □

The generalized Mathieu series $S_\mu(r)$ is a positive, decreasing function of both μ and r for $\mu>0$, $r>0$.

The following interesting results hold (see also [8]).

Corollary 5.3. *The generalized Mathieu series as defined in (1.5) satisfies the identity*

$$\int_0^\infty S_\mu(r)\,dr=\sqrt{\pi}\cdot\frac{\Gamma\left(\mu+\frac{1}{2}\right)}{\mu\Gamma(\mu)}\zeta(2\mu),\quad \mu>0. \tag{5.18}$$

For m a positive integer, then

$$\int_0^\infty S_m(r)\,dr=\frac{(-1)^{m-1}2^{2m-1}\pi^{2m+\frac{1}{2}}}{m!\,(2m)!}\Gamma\left(m+\frac{1}{2}\right)B_{2m}, \tag{5.19}$$

where B_k are the Bernoulli numbers defined by

$$\frac{x}{e^x-1}=\sum_{k=0}^{\infty}\frac{x^k}{k!}B_k,\quad |x|<2\pi.$$

Proof. From (5.10) we have

$$M_M^{(0)}=\int_0^\infty S_\mu(r)\,dr=B(\frac{1}{2},\mu+\frac{1}{2})\zeta(2\mu)$$

where $B(\frac{1}{2},\mu+\frac{1}{2})=\frac{\Gamma(\frac{1}{2})\Gamma(\mu+\frac{1}{2})}{\Gamma(\mu+1)}$ and $\Gamma(\mu+1)=\mu\Gamma(\mu)$, $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$.
Taking $\mu=m\in\mathbb{N}$ in (5.18) gives

$$\int_0^\infty S_m(r)\,dr=\frac{\sqrt{\pi}\Gamma\left(m+\frac{1}{2}\right)}{m!}\zeta(2m).\tag{5.20}$$

Now, a 1748 result of Euler states that for $m\in\mathbb{N}$

$$\zeta(2m)=(-1)^{m-1}\frac{2^{2m-1}\pi^{2m}}{(2m)!}B_{2m}.\tag{5.21}$$

Substitution of (5.21) in (5.20) readily produces the result (5.19). □

Remark 5.4. If we take $\mu=1$ in (5.18) (or alternatively, $m=1$ in (5.20)), then we recapture the result of Guo [15], namely

$$\int_0^\infty S_1(r)\,dr=\frac{\pi^3}{12}.\tag{5.22}$$

Remark 5.5. An alternative representation to (5.19) is given in a 1999 paper by Lin in Chinese (see [17]), namely,

$$\zeta(2m)=A_m\pi^{2m},\tag{5.23}$$

where A_m satisfies the recurrence relation

$$A_m=(-1)^{m-1}\cdot\frac{m}{(2m+1)!}+\sum_{j=1}^{m-1}\frac{(-1)^{j-1}}{(2j+1)!}A_{m-j}\tag{5.24}$$

and by convention the sum is neglected for $m=1$ so that $A_1=\frac{1}{3!}$.
Thus an equivalent result to (5.19) may be obtained as, from (5.20) and (5.23),

$$\int_0^\infty S_m(r)\,dr=\frac{\Gamma\left(m+\frac{1}{2}\right)}{m!}\pi^{2m+\frac{1}{2}}\cdot A_m$$

with A_m being given by (5.24).

Remark 5.6. Similar results to the above Corollary may be obtained for the companion zeroth moments by taking k =0 in (5.10). These are obviously related, for example,

$$\begin{aligned}M_A^{(0)}&=M_M^{(0)}-M_E^{(0)}\\ M_O^{(0)}&=2M_M^{(0)}-M_E^{(0)}.\end{aligned}\tag{5.25}$$

Remark 5.7. The moments may be used to approximate the class of generalized Mathieu series and obtain bounds for the remainders. Further, the current paper has aimed at investigating odd and even members of generalized Mathieu series, which it is believed not to have been treated in the literature. Their relationship to the Zeta function has also been highlighted throughout the paper and in particular in in Theorem 4.7 and Remark 4.8.

Remark 5.8. The main thrust of the paper has been focused on generalizations based around the Mathieu series. We may also introduce the **alternating odd generalized Mathieu series**

$$\begin{aligned}\tilde{\phi}_{\mu}(r) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}\, 2(2n-1)}{\left((2n-1)^2+r^2\right)^{1+\mu}} \\ &= C_{\mu}(r)\cdot 2\int_0^{\infty} \frac{x^{\mu+\frac{1}{2}}}{e^x+e^{-x}} J_{\mu-\frac{1}{2}}(rx)\,dx, \quad r,\mu>0.\end{aligned} \tag{5.26}$$

This is, in part, inspired by the alternating odd zeta function, $\beta(s)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)^s}$ which has explicit closed form solution in terms Euler polynomials for $s=2m+1$ whereas $\zeta(2m)$ for $m\in\mathbb{N}$, is explicitly given in terms of Bernoulli polynomials as seen in (5.21).This is so since using a limiting argument $\tilde{\phi}_{\mu}(0)=4\beta(2\mu+1)$.

This, however, will not be elaborated upon further given space considerations.

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