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Research Article

Group-2-groupoids and 2G-crossed modules

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Abstract

In this paper, we introduce the notion of a group-2-groupoid as a group object in the category of 2-groupoids. We also obtain a 2G-crossed module by using the structure of a group-2-groupoid. Then we prove that the category Gp2Gd of group-2-groupoids and the category 2GXMod of 2G-crossed modules are equivalent.

Mathematics Subject Classification (2010). 18D05, 18D35, 20J15, 20L05

Keywords. group-2-groupoid, 2G-crossed module

1. Introduction

A *groupoid* is a small category whose all morphisms are invertible [7, 9]. A groupoid can be thought of as a group with many objects and also a group is a groupoid with a single object [6]. A group object in the category of groupoids is called a 2-group [4], (resp. "$\mathcal{G}$-groupoid" in [7] and "group-groupoid" in [11]). For further information on the 2-group, see [3–5, 7, 12, 14]. This definition was generalized to ring-groupoid in [11] and to R-Module groupoid in [1]. Recently the concepts of normal and quotient objects in the category of 2-groups have been obtained by Mucuk et al. [13].

Crossed modules defined by Whitehead can be viewed as 2-dimensional groups [16, 17]. In [7], Brown and Spencer proved that the category of 2-groups is equivalent to the category of crossed modules of groups. And so a crossed module is essentially the same thing as a 2-group. This result was generalized to the crossed modules and internal groupoids in some algebraic categories including groups in [15]. Also this result was proved for the category of topological 2-groups and the category of topological crossed modules in [5].

A 2-group can be thought of as a 2-category with one object in which all 1-morphisms and 2-morphisms are invertible [3, 14]. The 2-categorical approach to 2-groups is a powerful conceptual tool. However, for explicit calculations it is often useful to treat 2-groups as crossed modules[3].

In Section 3, we have inspired by the work of Brown and Spencer [7], and then we define the group-2-groupoid as a group object in the category of 2-categories. The main goal of this paper is to investigate how a group-2-groupoid corresponds to an algebraic structure similar to crossed modules. For this purpose, we first introduce *2G-crossed*

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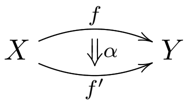
modules as an analogue of crossed modules given in [7]. Then we also define morphisms of group-2-groupoids and 2G-crossed modules. Finally, we prove that the category GP2GD of group-2-groupoids and the category 2GXMOD of 2G-crossed modules are equivalent.

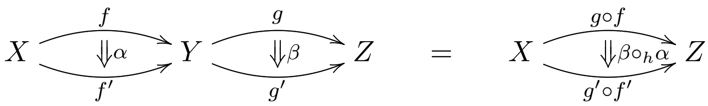
2. Preliminaries

The following definition is given in [3].

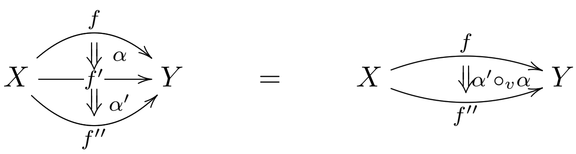
Definition 2.1. A 2-category $\mathcal{C}$ consists of

- objects $X, Y, Z, \ldots$
- 1-morphisms: $X \xrightarrow{f} Y$
- 2-morphisms: 

1-morphisms can be composed as in a category, and 2-morphisms can be composed in two distinct ways: horizontally:



and vertically:

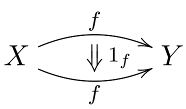


A few simple axioms must hold for this to be a 2-category:

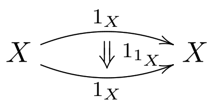
- Composition of 1-morphisms must be associative, and every object X must have a 1-morphism

$$X \xrightarrow{1_X} X$$

  serving as an identity for composition, just as in an ordinary category.
- Vertical composition must be associative, and every 1-morphism $X \xrightarrow{f} Y$ must have a 2-morphism



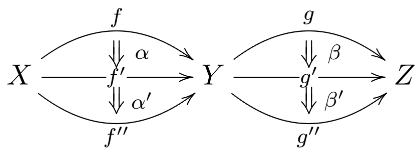
  serving as an identity for vertical composition.
- Horizontal composition must be associative, and the 2-morphism



  must serve as an identity for horizontal composition.
- Vertical composition and horizontal composition of 2-morphisms must satisfy the following *interchange law:*

$$(\beta' \circ_v \beta) \circ_h (\alpha' \circ_v \alpha) = (\beta' \circ_h \alpha') \circ_v (\beta \circ_h \alpha).$$

so that diagrams of the form



define unambiguous 2-morphisms.

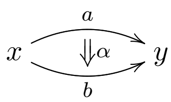
Here are some examples of 2-categories.

- The category of small categories CAT is a 2-category whose objects are small categories, 1-morphisms are functors and 2-morphisms are natural transformations between functors [2].
- The category of topological spaces TOP form a 2-category with homotopies between maps as 2-morphisms [2].
- Every category is a 2-category whose 2-morphisms are identity [14].

A *2-functor* $F\colon \mathcal{C} \to \mathcal{D}$ between two 2-categories $\mathcal{C}$ and $\mathcal{D}$ is a triple of functions sending objects 1-morphisms and 2-morphisms of $\mathcal{C}$ to items of the same types in $\mathcal{D}$ so as to preserve all the categorical structures (source, target, identities, and composites) [10].

Thus, small 2-categories and 2-functors between them form a category which is denoted by 2CAT [14].

A *2-groupoid* is a 2-category $\mathcal{G}$ in which every 1-morphism and every 2-morphism have inverses [14]. So a 2-groupoid $\mathcal{G} = (G_0, G_1, G_2)$ has a set G_0 of objects, a set G_1 of 1-morphisms and a set G_2 of 2-morphisms together with the source and target maps

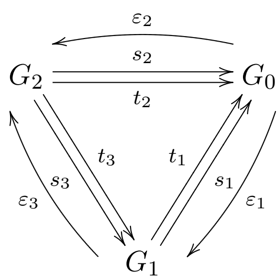


$$s_1, t_1\colon G_1 \longrightarrow G_0, \quad s_1(a) = x, \quad t_1(a) = y,$$
$$s_2, t_2\colon G_2 \longrightarrow G_0, \quad s_2(\alpha) = x, \quad t_2(\alpha) = y,$$
$$s_3, t_3\colon G_2 \longrightarrow G_1, \quad s_3(\alpha) = a, \quad t_3(\alpha) = b,$$

and the identity maps

$$\varepsilon_1\colon G_0 \longrightarrow G_1, \quad \varepsilon_1(x) = 1_x,$$
$$\varepsilon_2\colon G_0 \longrightarrow G_2, \quad \varepsilon_2(x) = 1_{1_x},$$
$$\varepsilon_3\colon G_1 \longrightarrow G_2, \quad \varepsilon_3(a) = 1_a,$$

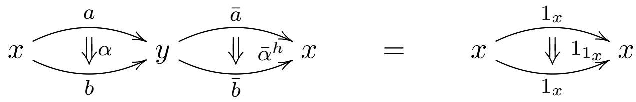
such that the following diagram commute for all objects

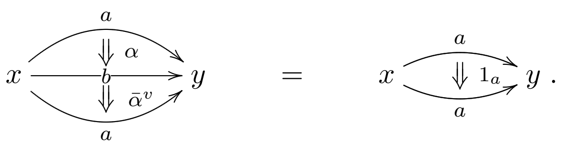


If $a, b \in G_1$, $s_1(b) = t_1(a)$ and $\alpha, \alpha', \beta \in G_2$, $s_2(\beta) = t_2(\alpha)$ and $s_3(\alpha') = t_3(\alpha)$ then the composition maps

$$\circ\colon G_1 \;{}_{s_1}\!\times_{t_1} G_1 \longrightarrow G_1,$$
$$\circ_h\colon G_2 \;{}_{s_2}\!\times_{t_2} G_2 \longrightarrow G_2,$$
$$\circ_v\colon G_2 \;{}_{s_3}\!\times_{t_3} G_2 \longrightarrow G_2,$$

exist such that, $s_1(b \circ a) = s_1(a)$, $t_1(b \circ a) = t_1(b)$, $s_2(\beta \circ_h \alpha) = s_2(\alpha)$, $t_2(\beta \circ_h \alpha) = s_2(\beta)$, $s_3(\alpha' \circ_v \alpha) = s_3(\alpha)$ and $t_3(\alpha' \circ_v \alpha) = t_3(\alpha')$. Further, these partial compositions are associative, for $x \in G_0$ and $a \in G_1$ the elements $\varepsilon_1(x) = 1_x$, $\varepsilon_2(x) = 1_{1_x}$ and $\varepsilon_3(a) = 1_a$ act as the identities and each 1-morphism a has an inverse $\bar{a}$ and each 2-morphism α has a horizontal inverse $\bar{\alpha}^h$ and a vertical inverse $\bar{\alpha}^v$ such that





The maps

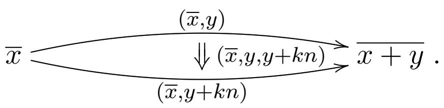
$$\eta_1 \colon G_1 \longrightarrow G_1, \quad \eta_1(a) = \bar{a},$$

$$\eta_2 \colon G_2 \longrightarrow G_2, \quad \eta_2(\alpha) = \bar{\alpha}^h,$$

$$\eta_3 \colon G_2 \longrightarrow G_2, \quad \eta_3(\alpha) = \bar{\alpha}^v$$

are called the inversions.

Example 2.2. Let G_0, G_1 and G_2 be the sets $\mathbb{Z}_n$, $\mathbb{Z}_n \times \mathbb{Z}$ and $\mathbb{Z}_n \times \mathbb{Z} \times \mathbb{Z}$, respectively. We assume that both pairs $(\overline{x}, y)$ and $(\overline{x}, y + kn)$ are 1-morphisms from $\overline{x}$ to $\overline{x + y}$ (for $k \in \mathbb{Z}$) and the triple $(\overline{x}, y, y + kn)$ is a 2-morphism from $(\overline{x}, y)$ to $(\overline{x}, y + kn)$ as follows:



Then we can define the composition of 1-morphisms by

$$(\overline{x + y}, z) \circ (\overline{x}, y) = (\overline{x}, y + z),$$

the vertical composition of 2-morphisms (for any $k_i \in \mathbb{Z}$) by

$$(\overline{x}, y + k_2 n, y + k_3 n) \circ_v (\overline{x}, y + k_1 n, y + k_2 n) = (\overline{x}, y + k_1 n, y + k_3 n),$$

and the horizontal composition by

$$(\overline{x + y}, z, z + k_2 n) \circ_h (\overline{x}, y, y + k_1 n) = \Big(\overline{x}, y + z, y + z + (k_1 + k_2) n\Big).$$

It is easy to prove that the vertical and horizontal compositions satisfy interchange law. For $\overline{x} \in G_0$ and $(\overline{x}, y) \in G_1$, the identity morphisms are $1_{\overline{x}} = (\overline{x}, 0)$ and $1_{1_{\overline{x}}} = (\overline{x}, 0, 0)$ and $1_{(\overline{x}, y)} = (\overline{x}, y, y)$. The inverse of $(\overline{x}, y)$ under $\circ$ is $\overline{(\overline{x}, y)} = (\overline{x + y}, -y)$, the inverse of $(\overline{x}, y, y + kn)$ under $\circ_h$ is $\overline{(\overline{x}, y, y + kn)}^h = (\overline{x + y}, -y, -y - kn)$ and under $\circ_v$ is $\overline{(\overline{x}, y, y + kn)}^v = (\overline{x}, y + kn, y)$. Thus the triple (G_0, G_1, G_2) is a 2-groupoid.

A *morphism of 2-groupoids* is simply a 2-functor between the underlying 2-categories. Hence small 2-groupoids and their morphisms form a category which is denoted by 2Gpd [14].

3. Group-2-groupoids and 2G-crossed modules

We now define the group object in 2Cat similar to group object in Cat as follows:

Definition 3.1. A *group object* $\mathcal{G}$ in 2Cat is a small 2-category $\mathcal{G}$ equipped with the following 2-functors satisfying group axioms

(1) the product $m\colon \mathcal{G}\times\mathcal{G}\longrightarrow\mathcal{G}$,

$$x \underset{b}{\overset{a}{\Downarrow\alpha}} y\ ,\ x' \underset{b'}{\overset{a'}{\Downarrow\alpha'}} y' \ \mapsto\ xx' \underset{bb'}{\overset{aa'}{\Downarrow\alpha\alpha'}} yy'$$

(2) the inverse $inv\colon \mathcal{G}\longrightarrow\mathcal{G}$,

$$x \underset{b}{\overset{a}{\Downarrow\alpha}} y \ \mapsto\ x^{-1} \underset{b^{-1}}{\overset{a^{-1}}{\Downarrow\alpha^{-1}}} y^{-1}$$

(3) the unit $\varepsilon\colon\{*\}\longrightarrow\mathcal{G}$ (where $\{*\}$ is the terminal object in 2Cat).

Remark 3.2. The one-object discrete category (i.e. every morphism is an identity) is the terminal object of the category of small categories Cat [8]. Similarly, the category $\{*\}$ which is defined as terminal object of 2Cat above, is the one-object discrete 2-category (i.e. every 1-morphism and every 2-morphism is an identity).

In terms of group object in 2Cat, a group-2-groupoid can be obtained in the following way:

Proposition 3.3. *A group object $\mathcal{G}$ in* 2Cat *is a 2-groupoid.*

Proof. Let $\mathcal{G}$ be a group object in 2Cat. Then 2-functors $m\colon\mathcal{G}\times\mathcal{G}\to\mathcal{G}$ called product, $inv\colon\mathcal{G}\to\mathcal{G}$ called inverse and $\varepsilon\colon\{*\}\to\mathcal{G}$ (where $\{*\}$ is the terminal object in 2Cat) called unit satisfying the usual group axioms. The product of $x \underset{b}{\overset{a}{\Downarrow\alpha}} y$ and $x' \underset{b'}{\overset{a'}{\Downarrow\alpha'}} y'$ is written as $xx' \underset{bb'}{\overset{aa'}{\Downarrow\alpha\alpha'}} yy'$, the inverse of $x \underset{b}{\overset{a}{\Downarrow\alpha}} y$ is written as $x^{-1} \underset{b^{-1}}{\overset{a^{-1}}{\Downarrow\alpha^{-1}}} y^{-1}$.

Let $\circ$, $\circ_h$ and $\circ_v$ be the composition of 1-morphisms, the horizontal composition and the vertical compositions of 2-morphisms in $\mathcal{G}$, respectively. To prove $\mathcal{G}$ is a 2-groupoid, we have to show that all 1-morphisms and 2-morphisms in $\mathcal{G}$ have inverses for compositions $\circ$, $\circ_h$ and $\circ_v$.

The 2-functor m gives interchange laws

$$(c\circ a)(c'\circ a') = (cc')\circ(aa'),$$

$$(\beta\circ_h\alpha)(\beta'\circ_h\alpha') = (\beta\beta')\circ_h(\alpha\alpha')$$

$$(\delta\circ_v\alpha)(\delta'\circ_v\alpha') = (\delta\delta')\circ_v(\alpha\alpha')$$

whenever $c\circ a$, $c'\circ a'$, $\beta\circ_h\alpha$, $\beta'\circ_h\alpha'$, $\delta\circ_v\alpha$ and $\delta'\circ_v\alpha'$ are defined.

In [7], it was proved that $c\circ a = a1_y^{-1}c = c1_y^{-1}a$ and $\overline{a} = 1_x a^{-1} 1_y$ is the inverse of a under $\circ$.

We also give the following relations for the horizontal and vertical composition of 2-morphisms just the same way as in [7];

For horizontal composition, we have

$$\beta \circ_h \alpha = (\beta 1_{1_e}) \circ_h (1_{1_y} 1_{1_y}^{-1} \alpha) = (\beta \circ_h 1_{1_y})(1_{1_e} \circ_h (1_{1_y}^{-1} \alpha)) = \beta 1_{1_y}^{-1} \alpha \tag{3.1}$$

and similarly

$$\beta \circ_h \alpha = \alpha 1_{1_y}^{-1} \beta. \tag{3.2}$$

So it is easy to see from (3.1) and (3.2) that $\overline{\alpha}^h = 1_{1_x} \alpha^{-1} 1_{1_y}$ is the inverse of α under $\circ_h$.

For the vertical composition, we have

$$\delta \circ_v \alpha = (\delta 1_{1_e}) \circ_v (1_b 1_b^{-1} \alpha) = (\delta \circ_v 1_b)(1_{1_e} \circ_v 1_b^{-1} \alpha) = \delta 1_b^{-1} \alpha \tag{3.3}$$

and

$$\delta \circ_v \alpha = \alpha 1_b^{-1} \delta. \tag{3.4}$$

And also it is easy to see from (3.3) and (3.4) that $\overline{\alpha}^v = 1_b \alpha^{-1} 1_a$ is the inverse of α under $\circ_v$.

Hence any group object in 2Cat is a 2-groupoid.

Furthermore, if $y = e$, then $\alpha\beta = \beta\alpha$; hence the elements of Kers_2 and Kert_2 commute under the group operation. In [7], it was proved that if $a, a_1 \in$ Kers_1 and $t_1(a) = x$, then

$$a a_1 a^{-1} = 1_x a_1 1_x^{-1}.$$

Similarly, we show that if $\alpha, \alpha_1 \in$ Kers_2 and $t_2(\alpha) = x$, then

$$\alpha \alpha_1 \alpha^{-1} = 1_{1_x} \alpha_1 1_{1_x}^{-1}. \tag{3.5}$$

□

Definition 3.4. A group object in the category of 2-groupoids is called a *group-2-groupoid.*

Example 3.5. $\mathcal{G} = (\mathbb{Z}_n, \mathbb{Z}_n \times \mathbb{Z}, \mathbb{Z}_n \times \mathbb{Z} \times \mathbb{Z})$ is a group-2-groupoid with the following 2-functors:

- $\oplus \colon \mathcal{G} \times \mathcal{G} \longrightarrow \mathcal{G}$, $(\overline{x_1}, y_1, z_1) \oplus (\overline{x_2}, y_2, z_2) = (\overline{x_1 + x_2}, y_1 + y_2, z_1 + z_2)$
- $inv \colon \mathcal{G} \longrightarrow \mathcal{G}$, $(\overline{x}, y, z)^{-1} = (\overline{n - x}, -y, -z)$
- $\varepsilon \colon \{*\} \longrightarrow \mathcal{G}$, $e = \overline{0}$, $1_e = (\overline{0}, 0)$, $1_{1_e} = (\overline{0}, 0, 0)$.

Definition 3.6. Let $\mathcal{G} = (G_0, G_1, G_2)$ and $\mathcal{H} = (H_0, H_1, H_2)$ be group-2-groupoids and let $F = (f_0, f_1, f_2) \colon \mathcal{G} \to \mathcal{H}$ be a 2-functor. If F preserves the group structures, then it is called a *morphism of group-2-groupoids.*

So group-2-groupoids and morphisms of them form a category which is denoted by Gp2Gd.

The following theorem was proved by Brown and Spencer in [7]:

Theorem 3.7. *The category of 2-groups and the category of crossed modules are equivalent.*

Remark 3.8. Let $\mathcal{G} = (G_0, G_1, G_2)$ be a group-2-groupoid and s_1, t_1 be the source and target maps from G_1 to G_0. Let $M =$Kers_1, $N = G_0$ and $\partial_1 = t_1|_M$. It was proved in Theorem 3.7 that $(M, N, \partial_1, \bullet)$ is a crossed module with the action $(x, a) \mapsto x \bullet a = 1_x a 1_x^{-1}$ of the group N on the group M and the map $\partial_1 = t_1|_M$.

Proposition 3.9. *Let $\mathcal{G} = (G_0, G_1, G_2)$ be a group-2-groupoid and s_2, t_2 be the source and target maps from G_2 to G_0. Then $(Kers_2, G_0, t_2|_L)$ is a crossed module.*

Proof. Let $L =$Kers_2, $N = G_0$. Then L, N inherit group structures from that of $\mathcal{G}$ and the map $\partial_2 = t_2|_L \colon L \to N$ is a morphism of groups. Further we have an action $(x, \alpha) \mapsto x \blacktriangleright \alpha$ of N on the group L given by $x \blacktriangleright \alpha = 1_{1_x} \alpha 1_{1_x}^{-1}$. It is easy to show that $\partial_2(x \blacktriangleright \alpha) = x \partial_2(\alpha) x^{-1}$ and $\partial_2(\alpha) \blacktriangleright \alpha_1 = \alpha \alpha_1 \alpha^{-1}$ by using (3.5). Thus $(L, N, \partial_2, \blacktriangleright)$ is a crossed module. □

Proposition 3.10. *Let $\mathcal{G} = (G_0, G_1, G_2)$ be a group-2-groupoid, t_3 be the target map from G_2 to G_1 and $(M, N, \partial_1, \bullet)$, $(L, N, \partial_2, \blacktriangleright)$ be crossed modules which corresponds to the group-2-groupoid $\mathcal{G}$ as above. Then $\partial_3 = t_3|_L \colon L \to M$ is a surjective morphism of groups which preserves actions of crossed modules.*

Proof. Since $\mathcal{G}$ is a group-2-groupoid, then t_3 is a morphism of groups. Therefore, the $\partial_3 = t_3|_L \colon L \to M$ which is the restriction of t_3, is also a morphism of groups. And for any 1-morphism $a \in M$, there is a 2-morphism $\alpha \in L$ such that $\partial_3(\alpha) = t_3|_L(a)$. So, the group morphism ∂_3 is surjective. It is clear that $t_2 = t_1 t_3$ and so $\partial_2 = \partial_1 \partial_3$. Since ∂_3 is a group morphism, we obtain

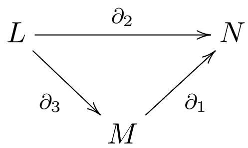
$$\partial_3(x \blacktriangleright \alpha) = \partial_3(1_{1_x} \alpha 1_{1_x}^{-1}) = \partial_3(1_{1_x})\partial_3(\alpha)\partial_3(1_{1_x}^{-1}) = 1_x \partial_3(\alpha) 1_x^{-1} = x \bullet \partial_3(\alpha).$$

□

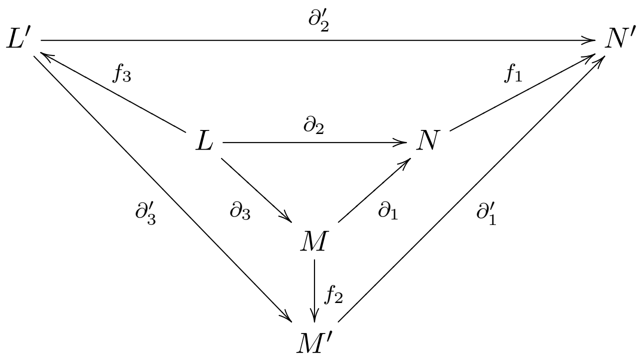
From Remark 3.8, Proposition 3.9 and Proposition 3.10, we can define a new structure of crossed modules which corresponds to group-2-groupoids as follows:

Definition 3.11. Let $(M, N, \partial_1, \bullet)$ and $(L, N, \partial_2, \blacktriangleright)$ be crossed modules. A *2G-crossed module* $(L, M, N, \partial_1, \partial_2, \partial_3, \bullet, \blacktriangleright)$ is a pair $(M, N, \partial_1, \bullet)$, $(L, N, \partial_2, \blacktriangleright)$ of crossed modules with a surjective morphism of groups $\partial_3 \colon L \to M$ which satisfies the following axioms:

(1) $\partial_2 = \partial_1 \partial_3$
(2) $\partial_3(n \blacktriangleright l) = n \bullet \partial_3(l)$, for $n \in N, l \in L$.



Definition 3.12. Let $K = (L, M, N, \partial_1, \partial_2, \partial_3)$ and $K' = (L', M', N', \partial'_1, \partial'_2, \partial'_3)$ be 2G-crossed modules. A morphism $(f_3, f_2, f_1) \colon K \to K'$ of 2G-crossed modules is a pair $(f_2, f_1) \colon (M, N, \partial_1, \bullet) \to (M', N', \partial'_1, \bullet')$, $(f_3, f_1) \colon (L, N, \partial_2, \blacktriangleright) \to (L', N', \partial'_2, \blacktriangleright')$ of morphisms of crossed modules such that $f_2 \partial_3 = \partial'_3 f_3$.



Therefore, 2G-crossed modules and morphisms between them form a category which is denoted by 2GXMod.

Definition 3.13. An *equivalence* between categories $\mathcal{C}$ and $\mathcal{D}$ is defined to be a pair of functors $S \colon \mathcal{C} \to \mathcal{D}$, $T \colon \mathcal{D} \to \mathcal{C}$ together with natural isomorphisms $1_{\mathcal{C}} \cong TS$, $1_{\mathcal{D}} \cong ST$, where $1_{\mathcal{C}}$ and $1_{\mathcal{D}}$ are the identity functors [10].

Theorem 3.14. *The category* Gp2Gd *of group-2-groupoids and the category* 2GXMod *of 2G-crossed modules are equivalent.*

Proof. A functor

$$\gamma\colon \textsc{Gp2Gd} \to \textsc{2GXMod}$$

is defined as follows: For a group-2-groupoid $\mathcal{G} = (G_0, G_1, G_2)$, by using Remark 3.8, Proposition 3.9 and Proposition 3.10, we can define a 2G-crossed module $\gamma(\mathcal{G})=K = (L, M, N, \partial_1, \partial_2, \partial_3)$ which corresponds to the group-2-groupoid $\mathcal{G}$.

Conversely, define a functor

$$\psi\colon \textsc{2GXMod} \to \textsc{Gp2Gd}$$

in the following way. Let $K = (L, M, N, \partial_1, \partial_2, \partial_3, \bullet, \blacktriangleright)$ be a 2G-crossed module. A group-2-groupoid $\psi(K)$ can be defined as follows. The group of objects of $\psi(K)$ is N. The group of 1-morphisms of $\psi(K)$ is the semi-direct product $N \ltimes M$ with the group structure

$$(n, m)(n', m') = (nn', m(n \bullet m')).$$

The source and target maps are defined by $s_1(n, m) = n$, $t_1(n, m) = \partial_1(m)n$, respectively and the identity 1-morphism of $\circ$ is (n, e_M), while the composition is defined by

$$(\partial_1(m)n, m_1) \circ (n, m) = (n, m_1m)$$

in [7]. Now the group of 2-morphisms of $\mathcal{G}$ can be defined the semi-direct product $N \ltimes M \ltimes L$ with the group structure

$$(n, m, l)(n', m', l') = (nn', m(n \bullet m'), l(n \blacktriangleright l')).$$

If $\partial_2(l) = \partial_2(k)$ then pairs $(n, \partial_3(l))$ and $(n, \partial_3(k))$ are 1-morphisms from n to $\partial_2(l)n$. Hence we can define 2-morphism $(n, \partial_3(l), k)$ from $(n, \partial_3(l))$ to $(n, \partial_3(k))$ as follows:

$$n \underset{(n,\partial_3(k))}{\overset{(n,\partial_3(l))}{\Longrightarrow}} \partial_2(l)n, \qquad \Downarrow (n,\partial_3(l),k).$$

The source and target maps of 2-morphisms can be defined by $s_2(n, \partial_3(l), k) = n$, $s_3(n, \partial_3(l), k) = (n, \partial_3(l))$, $t_2(n, \partial_3(l), k) = \partial_2(l)n$, $t_3(n, \partial_3(l), k) = (n, \partial_3(k))$, respectively and the identity 2-morphism of $\circ_h$ for $n \in N$ is (n, e_M, e_L), when the horizontal composition of 2-morphisms is defined by

$$(\partial_1(m)n, m_1, l_1) \circ_h (n, m, l) = (n, m_1m, l_1l).$$

If $\partial_2(l) = \partial_2(k) = \partial_2(h)$, then $(n, \partial_3(k), h)$ is 2-morphism from $(n, \partial_3(k))$ to $(n, \partial_3(h))$ and the vertical composition of 2-morphisms is defined by

$$(n, \partial_3(k), h) \circ_v (n, \partial_3(l), k) = (n, \partial_3(l), h).$$

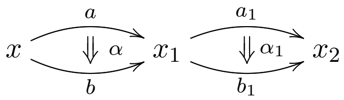
The identity 2-morphism of $\circ_v$ for $(n, \partial_3(l)) \in N \ltimes M$ is $(n, \partial_3(l), l)$ and the inverse $\overline{(n, \partial_3(l), k)}^{v} = (n, \partial_3(k), l)$. Thus $\psi(K) = (N, N \ltimes M, N \ltimes M \ltimes L)$ is a group-2-groupoid.

To define a natural isomorphism $S\colon \psi\gamma \to 1_{\textsc{Gp2Gd}}$, let $\mathcal{G}$ be a group-2-groupoid. A map $S_{\mathcal{G}}\colon \psi\gamma(\mathcal{G}) \to \mathcal{G}$ is defined to be the identity on objects, on 1-morphisms is given by $a \mapsto (x, a1_x^{-1})$ and on 2-morphisms is given by $\alpha \mapsto (x, a1_x^{-1}, \alpha 1_{1_x}^{-1})$.

$$x \underset{b}{\overset{a}{\Longrightarrow}} x_1 \ (\Downarrow \alpha) \quad \mapsto \quad x \underset{(x,b1_x^{-1})}{\overset{(x,a1_x^{-1})}{\Longrightarrow}} x_1 \ (\Downarrow (x,a1_x^{-1},\alpha 1_{1_x}^{-1})) .$$

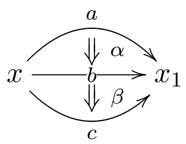
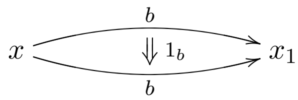
It is clear that $S_{\mathcal{G}}$ is bijective on 1-morphisms and 2-morphisms and also preserves the group operation and compositions as follows:

$$\begin{aligned} S_{\mathcal{G}}(\alpha)S_{\mathcal{G}}(\alpha') &= (x, a1_x^{-1}, \alpha 1_{1_x}^{-1})(x', a'1_{x'}^{-1}, \alpha' 1_{1_{x'}}^{-1}) \\ &= (xx', a1_x^{-1}(x \bullet a'1_{x'}^{-1}), \alpha 1_{1_x}^{-1}(x \blacktriangleright \alpha' 1_{1_{x'}}^{-1})) \\ &= (xx', a1_x^{-1}1_x a' 1_{x'}^{-1} 1_x^{-1}, \alpha 1_{1_x}^{-1} 1_{1_x} \alpha' 1_{1_{x'}}^{-1} 1_{1_x}^{-1}) \\ &= (xx', aa'1_{xx'}^{-1}, \alpha\alpha' 1_{1_{xx'}}^{-1}) \\ &= S_{\mathcal{G}}(\alpha\alpha'), \end{aligned}$$

for 

$$S_{\mathcal{G}}(a_1 \circ a) = S_{\mathcal{G}}(a_1 1_{x_1}^{-1} a) = (x, a_1 1_{x_1}^{-1} a 1_x^{-1}) = (x_1, a_1 1_{x_1}^{-1}) \circ (x, a1_x^{-1}) = S_{\mathcal{G}}(a_1) \circ S_{\mathcal{G}}(a),$$

$S_{\mathcal{G}}(\alpha_1 \circ_h \alpha) = S_{\mathcal{G}}(\alpha_1 1_{1_{x_1}}^{-1} \alpha) = (x, a_1 1_{x_1}^{-1} a 1_x^{-1}, \alpha_1 1_{1_{x_1}}^{-1} \alpha 1_{1_x}^{-1}) = S_{\mathcal{G}}(\alpha_1) \circ_h S_{\mathcal{G}}(\alpha)$ and for

 and 

$$\begin{aligned} S_{\mathcal{G}}(\beta \circ_v \alpha) = S_{\mathcal{G}}(\beta 1_b^{-1} \alpha) &= (x, a1_x^{-1}, \beta 1_b^{-1} \alpha 1_{1_x}^{-1}) \\ &= (x, a1_x^{-1}, \beta 1_{1_x}^{-1}(1_b 1_{1_x}^{-1})^{-1} \alpha 1_{1_x}^{-1}) \\ &= (x, b1_x^{-1}, \beta 1_{1_x}^{-1})(x, b1_x^{-1}, 1_b 1_{1_x}^{-1})^{-1}(x, a1_x^{-1}, \alpha 1_{1_x}^{-1}) \\ &= (x, b1_x^{-1}, \beta 1_{1_x}^{-1}) \circ_v (x, a1_x^{-1}, \alpha 1_{1_x}^{-1}) \\ &= S_{\mathcal{G}}(\beta) \circ_v S_{\mathcal{G}}(\alpha). \end{aligned}$$

Finally, we define a natural isomorphism $T\colon 1_{\mathrm{2GXMod}} \to \gamma\psi$, as follows: If $K = (L, M, N, \partial_1, \partial_2, \partial_3)$ is a 2G-crossed module, then T_K is the identity on N, on M is given by $m \mapsto (e_N, m)$ and on L is given by $l \mapsto (e_N, e_M, l)$. Clearly T_K is bijective and preserves the group operations as follows:

$$T_K(m)T_K(m') = (e_N, m)(e_N, m') = (e_N, m(e_N \bullet m')) = (e_N, mm') = T_K(mm'),$$

$$T_K(l)T_K(l') = (e_N, e_M, l)(e_N, e_M, l') = ((e_N, e_M, l(e_N \blacktriangleright l')) = (e_N, e_M, ll') = T_K(ll').$$

Hence, by Defination 3.13, the category GP2GD of group-2-groupoids and the category 2GXMOD of 2G-crossed modules are equivalent. □

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