Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 48 (5) (2019), 1398 – 1408
DOI : 10.15672/HJMS.2018.581

Research Article

Common fixed point theorems for (ψ, φ)-weak contractive conditions in metric spaces

Y. Mahendra Singh*[1], G. A. Hirankumar Sharma[1], M. R. Singh[2]

[1] *Department of Humanities and Applied Sciences, Manipur Institute of Technology, Takyelpat–795004, India*

[2] *Department of Mathematics, Manipur University, Canhipur–795003, India*

Abstract

The intent of this paper is to prove common fixed point theorems under (ψ, φ) -weak contractive conditions satisfying the common limit range property and the common property-(E.A) of four self mappings in the setting of metric spaces. Illustrative examples are also furnished to justify the validity of our results.

Mathematics Subject Classification (2010). 47H10, 54H25

Keywords. altering distance function, (ψ, φ)-weak contractive conditions, weakly compatible, property-(E.A), common limit range property, common fixed point

1. Introduction and Preliminaries

Let (X, d) be a metric space. A mapping $T : X \to X$, is called contraction if for each $x, y \in X$, there exists a constant $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y). \tag{1.1}$$

Alber and Guerre- Delabriere [5] introduced the concept of weak contractive mapping on a Hilbert space and proved the existence of fixed point of such mapping. Rhoades [23] showed that the results of Alber and Guerre-Delabriere [5] are also valid for any Banach space. A mapping $T : (X, d) \to (X, d)$ is called a weakly contractive (in the sense of Rhoades [23]), if for each $x, y \in X$

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) \tag{1.2}$$

where, $\varphi : [0, \infty) \to [0, \infty)$ is continuous, non-decreasing such that $\varphi(t) = 0$ if and only if $t = 0$. If $\varphi(t) = (1 - k)t$, then (1.2) reduces to (1.1).

In 2008, Dutta and Choudhury [10] introduced a generalization of Banach contraction mapping principle which includes weak contractive condition(φ-weak contraction). Song [27] and Zhang and Song [28] generalized φ-weak contractive condition in two mappings and proved existence of common fixed points. Doric [9], introduced the concept of (ψ, φ)-weak contractive mappings which generalized the contractive principle of Dutta

*Corresponding Author.
Email addresses: ymahenmit@rediffmail.com (Y. M. Singh), hguruaribam@gmail.com (G. A. H. Sharma), mranjitmu@rediffmail.com (M. R. Singh)
Received: 27.10.2016; Accepted: 10.04.2018

and Choudhury [10]. In the recent years, many authors used varieties of weak contractive conditions to prove the existence of fixed point theorems (e.g. [2–4, 6–8], [21, 22, 24]).

Let A and S be two self mappings on a metric space (X, d).

Definition 1.1 ([13]). Mappings A and S are said to be *commuting*, if $ASx = SAx$, for all $x \in X$.

Definition 1.2 ([14]). Mappings A and S are said to be *compatible*, if

$$\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$$

whenever, $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$.

Definition 1.3 ([19, 20]). Mappings A and S are said to be *non-compatible*, if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$, but $\lim_{n\to\infty} d(ASx_n, SAx_n)$ is either non-zero or non-existent.

Definition 1.4 ([1, 25]). Mappings A and S are said to satisfy *property-(E. A)* (or, *tangential mappings*), if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n$.

Definition 1.5 ([15]). Mappings A and S are said to be *weakly compatible* if $ASx = SAx$, whenever $Ax = Sx$, for some $x \in X$.

Definition 1.6 ([26]). Mappings A and S are said to satisfy the *common limit range property* with respect to S, denoted by (CRL_S), if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \in SX$, for some $t \in X$.

The concept of *property-(E.A)*(or, *tangential mappings*) is contained in the classes of *compatible* and *non-compatible* mappings. Furthermore, the *property-(E.A)*(or, *tangential mappings*) is also contained in the class of *common limit range property*. Therefore, every pair (A, S) of mappings which satisfies the *common limit range property* (i.e., (CRL_A) or (CRL_S)) also satisfies *property-(E.A)*. We denote $\mathbb{N}$, the set of natural numbers.

Example 1.7. Let $X = [0, 1)$ with usual metric d. Define self mappings A and S on X by

$$Ax = \begin{cases} 0, & 0 \le x \le \frac{1}{2} \\ 1-x, & \frac{1}{2} < x < 1 \end{cases} \quad ; \quad Sx = \begin{cases} \frac{3}{4}, & 0 \le x \le \frac{1}{2} \\ x, & \frac{1}{2} < x < 1. \end{cases}$$

Consider a sequence $\{x_n\} = \{\frac{1}{2} + \frac{1}{n}\}_{n\in\mathbb{N}}$, then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \frac{1}{2} = t$. Since, $AX = [0, \frac{1}{2})$ and $SX = (\frac{1}{2}, 1)$ where $\frac{1}{2} = t \notin AX \cap SX$. Therefore, the pair (A, S) satisfies the property-(E.A) but it satisfies neither the (CRL_A)-property nor the (CRL_S)-property.

Example 1.8. Let $X = [0, 1]$ with usual metric d. Define self mappings A and S on X by

$$Ax = \begin{cases} \frac{1}{3}, & 0 \le x \le \frac{1}{2} \\ 1-x, & \frac{1}{2} < x \le 1 \end{cases} \quad ; \quad Sx = \begin{cases} \frac{3}{4}, & 0 \le x < \frac{1}{2} \\ x, & \frac{1}{2} \le x \le 1. \end{cases}$$

Since, $AX = [0, \frac{1}{2})$ and $SX = [\frac{1}{2}, 1]$. Consider a sequence $\{x_n\} = \{\frac{1}{2} + \frac{1}{n}\}_{n\in\mathbb{N}}$, then $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \frac{1}{2} = t$, where $t \in SX$ i.e., the pair (A, S) satisfies the (CRL_S)-property. Therefore, the pair (A, S) is also satisfied property-(E.A).

Let A, B, S, T be four self mappings defined on a metric space (X, d).

Definition 1.9 ([17]). Pairs of mappings (A, S) and (B, T) are said to satisfy *common property-(E.A)*, if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n$.

Definition 1.10 ([11,12])**.** Pairs of mappings (A, S) and (B, T) are said to satisfy *common limit range property* with respect to S and T, denoted by (CRL_{ST}), if there exists two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t \in SX \cap TX$, for some $t \in X$.

Example 1.11. Let $X = [0, 12)$ with usual metric d. Define self mappings A, B, S and T by

$$Ax = \begin{cases} 6, & 0 \le x \le 6 \\ 9, & 6 < x < 12 \end{cases} \quad ; \quad Bx = \begin{cases} 0, & 0 \le x < 6 \\ 6, & 6 \le x < 12 \end{cases};$$

$$Sx = \begin{cases} 6, & 0 \le x \le 6 \\ 3, & 6 < x < 12 \end{cases} \quad ; \quad Tx = \begin{cases} 4, & 0 \le x < 6 \\ 12 - x, & 6 \le x < 12. \end{cases}$$

Consider two sequences $\{x_n\} = \{\frac{1}{n}\}_{n\in\mathbb{N}}$ and $\{y_n\} = \{6 + \frac{1}{n}\}_{n\in\mathbb{N}}$ in X. Since, $SX = \{3, 6\}$ and $TX = (0, 6]$. Also, $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 6 \in SX$ and $\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 6 \in TX$ i.e., $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 6 = t \in SX \cap TX$. Therefore, the pairs (A, S) and (B, T) satisfy (CRL_{ST})-property.

Khan et al. [16] introduced a new class of fixed point results considering with the help of control function which they called an altering distance function.

Definition 1.12 ([16])**.** A function $\psi : [0, \infty) \to [0, \infty)$ is called an *altering distance* if ψ is continuous, monotonically non-decreasing and $\psi(0) = 0$.

Let $\varphi : [0, \infty) \to [0, \infty)$ be a lower semi-continuous function with $\varphi(t) = 0$ if and only if $t = 0$. We denote Ψ and Φ, the sets of all real functions satisfying the definitions of ψ and φ respectively. Let A, B, S and T be self mappings on a metric space (X, d). Motivated by the recent work of Murthy et al. [18], we define

$$M(x, y) = \max\Big\{d(Sx, Ty), \frac{1}{2}\Big(d(Sx, Ax) + d(Sx, By)\Big), \frac{1}{2}\Big(d(Ty, By) + d(Ty, Ax)\Big)\Big\}$$

$$m(x, y) = \min\Big\{d(Sx, Ty), \frac{1}{2}\Big(d(Sx, Ax) + d(Sx, By)\Big), \frac{1}{2}\Big(d(Ty, By) + d(Ty, Ax)\Big)\Big\}$$

$$N(x, y) = \max\Big\{d(Sx, Ty), \frac{1}{2}\Big(d(Sx, Ax) + d(Ty, By)\Big), \frac{1}{2}\Big(d(Sx, By) + d(Ty, Ax)\Big)\Big\}$$

and

$$n(x, y) = \min\Big\{d(Sx, Ty), \frac{1}{2}\Big(d(Sx, Ax) + d(Ty, By)\Big), \frac{1}{2}\Big(d(Sx, By) + d(Ty, Ax)\Big)\Big\}.$$

The utility and pivoting role of *common limit range property* for two pairs of mappings are shown in Imdad et al.([11,12]) in order to prove the existence of common fixed points without considering the inclusion of one range set to the other range set and the closeness of the underlying subspaces. In this paper, we prove common fixed point theorems under (ψ, φ)-weak contractive conditions satisfying the *common limit range property* and the *property-(E.A)* of four self mappings in the setting of metric spaces.

2. Main results

Before we start to prove our main theorems, we discuss two lemmas by employing the *property-(E.A)* and the *common limit range property.*

Lemma 2.1. *Let $A, B, S and T$ be self mappings on a metric space (X, d). Suppose that*
(i) $AX \subset TX$;
(ii) the pair (A, S) satisfies the property-(E.A);
(iii) there exists $\varphi \in \Phi$, $\psi \in \Psi$ such that

$$\psi\Big(d(Ax, By)\Big) \leq \psi\Big(M(x, y)\Big) - \varphi\Big(m(x, y)\Big), \forall x, y \in X$$

Then, the pairs (A, S) and (B, T) share the common property-$(E.A)$.

Proof. Since the pair (A, S) satisfies property-(E.A), there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$. Since $AX \subset TX$, there exists a sequence $\{y_n\}$ in X such that $Ax_n = Ty_n$. Therefore, $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Ty_n = t$. Now, we claim that $\lim_{n\to\infty} By_n = t$. By triangular inequality, $d(By_n, t) \leq d(Ax_n, By_n) + d(Ax_n, t)$, therefore in order to prove our claim it is sufficient to prove $d(Ax_n, By_n) \to 0$ as $n \to \infty$. Suppose not, there exists $\epsilon > 0$ such that for some positive integer $k \geq n$ and $d(Ax_{n_k}, By_{n_k}) \geq \epsilon$. From (iii) with $x = x_{n_k}, y = y_{n_k}$, we obtain

$$\psi\Big(d(Ax_{n_k}, By_{n_k})\Big) \leq \psi\Big(M(x_{n_k}, y_{n_k})\Big) - \varphi\Big(m(x_{n_k}, y_{n_k})\Big) \tag{2.1}$$

where

$$M(x_{n_k}, y_{n_k}) = \max\Big\{d(Sx_{n_k}, Ty_{n_k}), \frac{1}{2}\Big(d(Sx_{n_k}, Ax_{n_k}) + d(Sx_{n_k}, By_{n_k})\Big), \frac{1}{2}\Big(d(Ty_{n_k}, By_{n_k}) + d(Ty_{n_k}, Ax_{n_k})\Big)\Big\}$$

and

$$m(x_{n_k}, y_{n_k}) = \min\Big\{d(Sx_{n_k}, Ty_{n_k}), \frac{1}{2}\Big(d(Sx_{n_k}, Ax_{n_k}) + d(Sx_{n_k}, By_{n_k})\Big), \frac{1}{2}\Big(d(Ty_{n_k}, By_{n_k}) + d(Ty_{n_k}, Ax_{n_k})\Big)\Big\}.$$

Letting $k \to \infty$ in (2.1) and using the definitions of ψ and φ, we thus obtain

$$\lim_{k\to\infty} \psi\Big(d(Ax_{n_k}, By_{n_k})\Big) \leq \lim_{k\to\infty} \psi\Big(M(x_{n_k}, y_{n_k})\Big) - \lim_{k\to\infty} \varphi\Big(m(x_{n_k}, y_{n_k})\Big)$$

$$\psi(\epsilon) \leq \psi\Big(\lim_{k\to\infty} M(x_{n_k}, y_{n_k})\Big) - \lim_{k\to\infty} \varphi\Big(m(x_{n_k}, y_{n_k})\Big) \tag{2.2}$$

where

$$\lim_{k\to\infty} M(x_{n_k}, y_{n_k}) = \max\Big\{0, \frac{\epsilon}{2}, \frac{\epsilon}{2}\Big\} = \frac{\epsilon}{2}$$

and

$$\lim_{k\to\infty} m(x_{n_k}, y_{n_k}) = \min\Big\{0, \frac{\epsilon}{2}, \frac{\epsilon}{2}\Big\} = 0.$$

Also,

$$\varphi(0) \leq \lim_{k\to\infty} \inf \varphi\Big(m(x_{n_k}, y_{n_k})\Big). \tag{2.3}$$

From (2.2) and (2.3), we obtain

$$\begin{aligned}\psi(\epsilon) \leq& \psi\Big(\frac{\epsilon}{2}\Big) - \lim_{k\to\infty} \varphi\Big(m(x_{n_k}, y_{n_k})\Big)\\ \leq& \psi\Big(\frac{\epsilon}{2}\Big)\end{aligned}$$

a contradiction with the definition of ψ and hence, $\lim_{n\to\infty} By_n = t$. Therefore, the pairs (A, S) and (B, T) share the common property-(E.A). □

On the same line of the above Lemma 2.1, we establish the following.

Lemma 2.2. *Let $A, B, S and T$ be self mappings on a metric space (X, d) satisfying conditions (i) and (iii) of Lemma 2.1, but condition (ii) is replaced by*
(ii)^o the pair (A, S) satisfies the (CRL_S)-property;
(iv) TX is closed.
Then, the pairs (A, S) and (B, T) share the (CRL_{ST})-property.

Proof. Since the pair (A, S) satisfies (CRL_S)-property, there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \in SX$, for some $t \in X$ i.e., the pair (A, S) satisfies property-(E.A). Now, the pairs (A, S) and (B, T) satisfy all the conditions of Lemma 2.1. Therefore, by Lemma 2.1 and the closeness of TX, the pairs (A, S) and (B, T) share the (CRL_{ST})-property. □

Now, we prove the following theorems for two pairs of self mappings satisfying the common property-(E.A) and the (CRL_{ST})-property).

Theorem 2.3. *Let A, B, S and T be self mappings on a metric space (X, d) satisfying the inequality (iii) of Lemma 2.1. If the pairs (A, S) and (B, T) satisfy the (CRL_{ST})-property, then A, B, S and T have a unique common fixed point in X, provided both the pairs (A, S) and (B, T) are weakly compatible.*

Proof. Suppose that the pairs (A, S) and (B, T) share the (CRL_{ST})-property, there exist the sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t$$

where $t \in SX \cap TX$, for some $t \in X$. Therefore, $t \in SX \cap TX$ implies that $t \in SX$ and $t \in TX$. First, we consider the case $t \in SX$, then there exists $u \in X$ such that $Su = t$. Now, we claim that $Au = Su$, otherwise using inequality (iii) taking with $x = u$, $y = y_n$, we obtain

$$\psi\Big(d(Au, By_n)\Big) \leq \psi\Big(M(u, y_n)\Big) - \varphi\Big(m(u, y_n)\Big) \tag{2.4}$$

where

$$M(u, y_n) = \max\Big\{d(Su, Ty_n), \frac{1}{2}\Big(d(Su, Au) + d(Su, By_n)\Big), \frac{1}{2}\Big(d(Ty_n, By_n) + d(Ty_n, Au)\Big)\Big\}$$

and

$$m(u, y_n) = \min\Big\{d(Su, Ty_n), \frac{1}{2}\Big(d(Su, Au) + d(Su, By_n)\Big), \frac{1}{2}\Big(d(Ty_n, By_n) + d(Ty_n, Au)\Big)\Big\}.$$

Letting $n \to \infty$ in (2.4) and using the definitions of ψ and φ, we thus obtain

$$\lim_{n\to\infty} \psi\Big(d(Au, By_n)\Big) \leq \lim_{n\to\infty} \psi\Big(M(u, y_n)\Big) - \lim_{n\to\infty} \varphi\Big(m(u, y_n)\Big)$$

$$\psi\Big(d(Au, t)\Big) \leq \psi\Big(\lim_{n\to\infty} M(u, y_n)\Big) - \lim_{n\to\infty} \varphi\Big(m(u, y_n)\Big) \tag{2.5}$$

where

$$\lim_{n\to\infty} M(u, y) = \max\Big\{0, \frac{1}{2}d(t, Au), \frac{1}{2}d(t, Au)\Big\} = \frac{1}{2}d(Au, t)$$

and

$$\lim_{n\to\infty} m(u, y) = \min\Big\{0, \frac{1}{2}d(t, Au), \frac{1}{2}d(t, Au)\Big\} = 0$$

also,

$$\varphi(0) \le \lim_{n\to\infty} \inf \varphi\Big(m(u,y_n)\Big). \tag{2.6}$$

From (2.5) and (2.6), we obtain

$$\begin{aligned}\psi\Big(d(Au,t)\Big) &\le \psi\Big(\frac{1}{2}d(Au,t)\Big) - \lim_{n\to\infty} \varphi\Big(m(u,y_n)\Big)\\ &\le \psi\Big(\frac{1}{2}d(Au,t)\Big)\end{aligned}$$

a contradiction with the definition of ψ, which in turn gives $d(Au,t) = 0$ i.e., $Au = t$. Therefore, $Au = Su = t$ i.e., A and S have a coincidence point. Secondly, we consider the case $t \in TX$, there exists $v \in X$ such that $Tv = t$. Now, we show that $Bv = Tv$, otherwise using inequality (iii) taking with $x = x_n$ and $y = v$, we obtain

$$\psi\Big(d(Ax_n,Bv)\Big) \le \psi\Big(M(x_n,v)\Big) - \varphi\Big(m(x_n,v)\Big) \tag{2.7}$$

where

$$\begin{aligned}M(x_n,v) = \max\Big\{&d(Sx_n,Tv), \frac{1}{2}\Big(d(Sx_n,Ax_n)+d(Sx_n,Bv)\Big),\\ &\frac{1}{2}\Big(d(Tv,Bv)+d(Tv,Ax_n)\Big)\Big\}\end{aligned}$$

and

$$\begin{aligned}m(x_n,v) = \min\Big\{&d(Sx_n,Tv), \frac{1}{2}\Big(d(Sx_n,Ax_n)+d(Sx_n,Bv)\Big),\\ &\frac{1}{2}\Big(d(Tv,Bv)+d(Tv,Ax_n)\Big)\Big\}.\end{aligned}$$

Letting $n \to \infty$ in (2.7), and using definition of ψ and φ, we obtain

$$\lim_{n\to\infty} \psi\Big(d(Ax_n,Bv)\Big) \le \lim_{n\to\infty} \psi\Big(M(x_n,v)\Big) - \lim_{n\to\infty} \varphi\Big(m(x_n,v)\Big)$$

$$\psi\Big(d(t,Bv)\Big) \le \psi\Big(\lim_{n\to\infty} M(x_n,v)\Big) - \lim_{n\to\infty} \varphi\Big(m(x_n,v)\Big) \tag{2.8}$$

where

$$\lim_{n\to\infty} M(x_n,v) = \max\Big\{0, \frac{1}{2}d(t,Bv), \frac{1}{2}d(t,Bv)\Big\} = \frac{1}{2}d(t,Bv)$$

and

$$\lim_{n\to\infty} m(x_n,v) = \min\Big\{0, \frac{1}{2}d(t,Bv), \frac{1}{2}d(t,Bv)\Big\} = 0$$

also,

$$\varphi(0) \le \lim_{n\to\infty} \inf \varphi\Big(m(x_n,v)\Big). \tag{2.9}$$

From (2.8) and (2.9), we thus obtain

$$\begin{aligned}\psi\Big(d(t,Bv)\Big) &\le \psi\Big(\frac{1}{2}d(t,Bv)\Big) - \lim_{n\to\infty} \varphi\Big(m(x_n,v)\Big)\\ &\le \psi\Big(\frac{1}{2}d(t,Bv)\Big)\end{aligned}$$

a contradiction with the definition of ψ, which in turn yields $d(t,Bv) = 0$ i.e., $t = Bv = Tv$ so, B and T have a coincidence point. Therefore, for some $t \in SX \cap TX$, we thus obtain $Au = Su = Bv = Tv = t$. Since, the pairs (A,S) and (B,T) are weakly compatible, so that $ASu = SAu$ i.e., $At = St$ and $BTv = TBv$ i.e., $Bt = Tt$. Now, we show that $At = St = Bt = Tt$. If not, putting $x = y = t$ in (iii), we obtain

$$\psi\Big(d(At, Bt)\Big) \leq \psi\Big(M(t,t)\Big) - \varphi\Big(m(t,t)\Big) \tag{2.10}$$

where

$$\begin{aligned} M(t,t) =& max\Big\{d(St, Tt), \frac{1}{2}\Big(d(St, At) + d(Tt, Bt)\Big), \\ & \frac{1}{2}\Big(d(St, Bt) + d(Tt, At)\Big)\Big\} \\ =& d(At, Bt) \end{aligned}$$

and

$$\begin{aligned} m(t,t) =& min\Big\{d(St, Tt), \frac{1}{2}\Big(d(St, At) + d(St, Bt)\Big), \\ & \frac{1}{2}\Big(d(Tt, Bt) + d(Tt, At)\Big)\Big\} \\ =& \frac{1}{2}d(At, Bt). \end{aligned}$$

Therefore, (2.10) becomes

$$\psi\Big(d(At, Bt)\Big) \leq \psi\Big(d(At, Bt)\Big) - \varphi\Big(\frac{1}{2}d(At, Bt)\Big)$$

a contradiction if $\varphi\Big(\frac{1}{2}d(At, Bt)\Big) \neq 0$, which in turn yields $d(At, Bt) = 0$ i.e., $At = Bt$. Therefore, $At = St = Bt = Tt$ i.e., the mappings A, B, S and T have a common coincidence point. Now, we claim that $At = t$. For this, putting $x = t$ and $y = y_n$ in (iii), we obtain

$$\psi\Big(d(At, By_n)\Big) \leq \psi\Big(M(t, y_n)\Big) - \varphi\Big(m(t, y_n)\Big) \tag{2.11}$$

where

$$M(t, y_n) = \max\Big\{d(St, Ty_n), \frac{1}{2}\Big(d(St, At) + d(St, By_n)\Big), \frac{1}{2}\Big(d(Ty_n, By_n) + d(Ty_n, At)\Big)\Big\}$$

and

$$m(t, y_n) = \min\Big\{d(St, Ty_n), \frac{1}{2}\Big(d(St, At) + d(St, By_n)\Big), \frac{1}{2}\Big(d(Ty_n, By_n) + d(Ty_n, At)\Big)\Big\}.$$

Letting $n \to \infty$ in (2.11), we obtain

$$\lim_{n\to\infty} \psi\Big(d(At, By_n)\Big) \leq \lim_{n\to\infty} \psi\Big(M(t, y_n)\Big) - \lim_{n\to\infty} \varphi\Big(m(t, y_n)\Big)$$

$$\psi\Big(d(At, t)\Big) \leq \psi\Big(\lim_{n\to\infty} M(t, y_n)\Big) - \lim_{n\to\infty} \varphi\Big(m(t, y_n)\Big) \tag{2.12}$$

where

$$\lim_{n\to\infty} M(t, y_n) = \max\Big\{d(At, t), \frac{1}{2}d(At, t), \frac{1}{2}d(t, At)\Big\} = d(At, t)$$

and

$$\lim_{n\to\infty} m(t, y_n) = \min\Big\{d(At, t), \frac{1}{2}d(At, t), \frac{1}{2}d(t, At)\Big\} = \frac{1}{2}d(At, t)$$

also,

$$\varphi\Big(\frac{1}{2}d(At, t)\Big) \leq \lim_{n\to\infty} \inf \varphi\Big(m(t, y_n)\Big). \tag{2.13}$$

From (2.12) and (2.13), we obtain

$$\begin{aligned}\psi\Big(d(At,t)\Big) \leq & \psi\Big(d(At,t)\Big) - \lim_{n\to\infty}\varphi\Big(m(t,y_n)\Big)\\ \leq & \psi\Big(d(At,t)\Big) - \varphi\Big(\frac{1}{2}d(At,t)\Big)\end{aligned}$$

a contradiction if $\varphi\Big(\frac{1}{2}d(At,t)\Big) \neq 0$, which in turn gives $(d(At,t) = 0$ i.e., $At = t$. Since, $At = St = Bt = Tt$. Therefore, $At = St = Bt = Tt = t$ i.e., t is a common fixed point of A, B, S and T. Now, it remains to show that z is the uniqueness common fixed point of A, B, S and T. If not, there is another point $z \in X$ such that $Az = Sz = Bz = Tz = z$. For this, using (iii) with $x = t$ and $y = z$, we obtain

$$\psi\Big(d(t,z)\Big) = \psi\Big(d(At,Bz)\Big) \leq \psi\Big(M(t,z)\Big) - \varphi\Big(m(t,z)\Big)$$

where

$$M(t,z) = \max\Big\{d(t,z), \frac{1}{2}d(t,z), \frac{1}{2}d(z,t)\Big\} = d(t,z)$$

and

$$m(t,z) = \min\Big\{d(t,z), \frac{1}{2}d(t,z), \frac{1}{2}d(z,t)\Big\} = \frac{1}{2}d(t,z).$$

Therefore,

$$\psi\Big(d(t,z)\Big) \leq \psi\Big(d(t,z)\Big) - \varphi\Big(\frac{1}{2}d(t,z)\Big)$$

a contradiction if $\varphi\Big(\frac{1}{2}d(t,z)\Big) \neq 0$, which yields $d(t,z) = 0$ i.e., $t = z$. This completes the proof. □

The following example shows the validity of Theorem 2.3.

Example 2.4. Let $X = [2,11)$ with usual metric d. Define A, B, S and T on X by

$$Ax = \begin{cases} 3, & x \in [2,3] \cup (5,11) \\ 2, & x \in (3,5] \end{cases}; \qquad Bx = \begin{cases} 3, & x \in [2,3] \cup (5,11) \\ 5, & x \in (3,5] \end{cases}$$

$$Sx = \begin{cases} 3, & x \in [2,3] \\ 6, & x \in (3,5] \\ \frac{x+4}{3}, & x \in (5,11) \end{cases}; \qquad Tx = \begin{cases} 3, & x \in [2,3] \\ 10, & x \in (3,5] \\ x-2, & x \in (5,11). \end{cases}$$

Since, $AX = \{2,3\} \nsubseteq TX = [3,9) \cup \{10\}$ and $BX = \{3,5\} \nsubseteq [3,5) \cup \{6\} = SX$. Consider two sequences $\{x_n\} = \{3\}, \{y_n\} = \{5 + \frac{1}{n}\}_{n\in\mathbb{N}}$(or,$\{x_n\} = \{5 + \frac{1}{n}\}_{n\in\mathbb{N}}$, $\{y_n\} = \{3\}$). Then, $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = 3 = t \in SX \cap TX$ i.e., the pairs (A,S) and (B,T) satisfy the (CRL_{ST})-property.
Now, taking in particular

$$\psi(t) = 2t, \quad \varphi(t) = \begin{cases} \frac{1}{t}, & t > 0 \\ 0, & t = 0, \end{cases}$$

then, one can verify that the contractive condition (iii) is satisfied for every $x, y \in X$. Thus, all the conditions of Theorem 2.3 are satisfied and hence, 3 is the unique common fixed point of A, B, S and T.

Theorem 2.5. *Let A, B, S and T be self mappings on a metric space (X,d) satisfying any one of the conditions $(i^o), (2^o)$ and (3^o) of the following:*
(1^o) *(a) all the conditions of Lemma 2.1,*
(b) SX is closed in X;
(2^o) *(a) Inequality (iii) of Lemma 2.1,*

(b) SX and TX are closed in X,
(c) the pairs (A, S) and (B, T) satisfy common property-(E.A);
(3^o) *all the conditions of Lemma 2.2;*
Then, A, B, S and T have a unique common fixed point in X, provided both the pairs (A, S) and (B, T) are weakly compatible.

Theorem 2.6. *Let A, B, S and T be self mappings on a metric space (X, d) satisfying the inequality*

$$\psi\Big(d(Ax, By)\Big) \leq \psi\Big(N(x, y)\Big) - \varphi\Big(m(x, y)\Big), \forall x, y \in X$$

where $\psi \in \Psi, \varphi \in \Phi$. If the pairs (A, S) and (B, T) satisfy the (CRL_{ST})-property, then A, B, S and T have a unique common fixed point in X, provided both the pairs (A, S) and (B, T) are weakly compatible.

Taking $\psi(t) = t$ in Theorem 2.3 and Theorem 2.6, we obtain the following corollaries.

Corollary 2.7. *Let A, B, S and T be self mappings on a metric space (X, d) satisfying the inequality*

$$d(Ax, By) \leq M(x, y) - \varphi\Big(m(x, y)\Big), \forall x, y \in X$$

where $\varphi \in \Phi$. If the pairs (A, S) and (B, T) satisfy the (CRL_{ST})-property, then A, B, S and T have a unique common fixed point in X, provided both the pairs (A, S) and (B, T) are weakly compatible.

Corollary 2.8. *Let A, B, S and T be self mappings on a metric space (X, d) satisfying the inequality*

$$d(Ax, By) \leq N(x, y) - \varphi\Big(m(x, y)\Big), \forall x, y \in X$$

where $\varphi \in \Phi$. If the pairs (A, S) and (B, T) satisfy the (CRL_{ST})-property, then A, B, S and T have a unique common fixed point in X provided, both the pairs (A, S) and (B, T) are weakly compatible.

One can check the validity of Theorem 2.6, Corollaries 2.7 and 2.8 with Example 2.4.

Discussion

We write the following inequality

$$\psi\Big(d(Ax, By)\Big) \leq \psi\Big(N(x, y)\Big) - \varphi\Big(n(x, y)\Big), \forall x, y \in X \tag{2.14}$$

in lieu of inequality (iii) and retain the remaining conditions of Theorem 2.6. Taking two points $z, z' \in X$, where $z \neq z'$ such that $Az = Bz = Sz = Tz = z$ and $Az' = Bz' = Sz' = Tz' = z'$. Using the above inequality (2.14), we have

$$\begin{aligned}
\psi\Big(d(z, z')\Big) =& \psi\Big(d(Az, Bz')\Big) \\
\leq& \psi\Big(N(z, z')\Big) - \varphi\Big(n(z, z')\Big) \\
=& \psi\Big(d(z, z')\Big) - \varphi(0) \\
=& \psi\Big(d(z, z')\Big)
\end{aligned}$$

In view of above, inequality (2.14) does not work in order to obtain a unique common fixed point of the pairs (A, S) and (B, T), so the inequality (2.14) is not applicable in Theorem 2.6. It notices that Theorem 2.1 of Murthy et al.[18] does not give a unique common fixed point. Therefore, the revised version of Theorem 2.1[18] may be stated as follows:

Theorem 2.9. *Let $A, B, S and T$ be self mappings on a complete metric space (X, d). Suppose that*
(i) $AX \subset TX$ *and* $BX \subset SX$*;*
(ii) *there exists* $\varphi \in \Phi$, $\psi \in \Psi$ *such that*

$$\psi\Big(d(Ax, By)\Big) \leq \psi\Big(M(x, y)\Big) - \varphi\Big(m(x, y)\Big), \forall x, y \in X$$

Then, the pairs (A, S) and (B, T) have a unique common fixed point in X, provided both the pairs (A, S) and (B, T) are weakly compatible.

Note that the above theorem is a corollary of Theorem 2.3.

Acknowledgment. The authors are thankful to the anonymous referee for his/her valuable constructive comments and suggestions to improve the quality of the manuscript.

References

[1] M. Aamri and D. EL Moutawakil, *Some new common fixed points under strict contractive conditions*, J. Math. Anal. Appl. **270**, 181-188, 2002.
[2] M. Abbas and D. Doric, *Common fixed point theorem for four mappings satisfying generalized weak contractive conditions*, Filomat **24** (2), 1-10, 2010.
[3] M. Abbas and M.A. Khan, *Common fixed point theorems of two mappings satisfying a generalized weak contractive condition*, Int. J. Math. Math. **2009**, 9 pages, 2009.
[4] A.P. Agarwal, M.A. Alghamdi and N. Shahzad, *Fixed point theory for cyclic genralized contractions in partial metric spaces*, Fixed Point Theory Appl. **2012**, 11 pages, 2012.
[5] I.Y. Alber and S. Guerre-Delabriere, *Principle of weakly contractive maps in Hilbert space*, in: New Results in Operator Theory and its Appl. **98**, 7-22, Birkhäuser, Basel, Switzerland, 1997.
[6] I. Beg and M. Abbas, *Coincidence point and invariant approximation for mappings satisfying generalized weak contractive condition*, Fixed Point Theory Appl. **2006**, Article ID 74503, 7 pages, 2006.
[7] B.S. Choudhury, P. Konar, B.E. Rhoades and N. Metiya, *Fixed point theorems for generalized weakly contractive mapppings*, Nonlinear Anal.: Theory Appl. **74** (6), 2116-2126, 2011.
[8] H.S. Ding, Z. Kadelburg, E. Karapinar and S. Radenovic, *Common fixed points of weak contractions in cone metric spaces*, Abstract Appl. Anal. **2012**, 18 pages, 2012.
[9] D. Doric, *Common fixed point for generalized (ψ,φ)-weak contractions*, Appl. Math. Lett. **22**, 1896-1900, 2009.
[10] P.N. Dutta and B.S. Choudhury, *A generalization of contraction principle in metric spaces*, Fixed Point Theory Appl. **2008**, 8 pages, 2008.
[11] M. Imdad, S. Chouhan and Z. Kadelburg, *Fixed point theorems for mappings with common limit range property satisfying generalized (ψ,φ)-weak contractive conditions*, Math. Sci., **7** (16), 1-8, 2013.
[12] M. Imdad, B.D. Pant and S. Chouhan, *Fixed point theorems in Menger spaces using the (CRL_ST) property and applications*, J. Nonlinear Anal. Optm. **3** (2), 225-237, 2012.
[13] G. Jungck, *Commuting mappings and fixed points*, Amer. Math. Monthly **83**, 261-263, 1976.
[14] G. Jungck, *Compatible mappigs and common fixed points*, Int. J. Math. Math. Sci. **9**, 771-779, 1986.
[15] G. Jungck and B.E. Rhoades, *Fixed points for set-valued functions without continuity*, Indian J. Pure Appl. Math. **29** (3), 227-238, 1998.
[16] M.S. Khan, M. Swaleh and S. Sessa, *Fixed point theorems by altering distances between the points*, Bull. Aust. Math. Soc. **30** (1), 1-9, 1981.

[17] Y. Liu, J. Wu and Z. Li, *Common fixed points of single-valued and multi-valued maps*, Int. J. Math. Math. Sci. **19**, 3045-3055, 2005.

[18] P.P. Murthy, K. Tas and U.D. Patel, *Common fixed point theorems for generalized* (ψ, φ)*-weak contraction condition in complete metric spaces*, J. Ineq. Appl., **2015**, Article 139, 2015.

[19] R.P. Pant, *Common fixed points of contractive maps*, J. Math. Anal. Appl., **226**, 251-258, 1998.

[20] R.P. Pant, *R-weakly commutativity and common fixed points*, Soochow J. Math. **25**, 37-42, 1999.

[21] S. Radenovic and Z. Kadelburg, *Generalized weak contractions in partially ordered metric spaces*, Computers Math. Appl. **60**, 1176-1783, 2010.

[22] S. Radenovic, Z. Kadelburg, D. Jandrlic and A Jandrlic, *Some results on weakly contractive maps*, Bull. Iranian Math. Soc. **38** (3), 625-645, 2012.

[23] B. E. Rhoades, *Some theorems on weakly contractive maps*, Nonlinear Anal.: Theory, Method and Appl. **47** (4), 2683-2693, 2001.

[24] B.D. Rouhani and S. Moradi, *Common fixed point of multivalued g eneralized* φ*-weak contractive mappings*, Fixed Point Theory Appl. **2010**, 13 pages, 2010.

[25] K.P.R. Sastry and I.R.S. Krishna Murthy, *Common fixed points of two partially commuting tangential selfmaps on a metric space*, J. Math. Anal. Appl. **250**, 731-734, 2000.

[26] W. Sintunavarat and P Kumam, *Common fixed point theorems for a pair of weakly compatible mappings in Fuzzy metric spaces*, J. Appl. Math. **2011**, 14 pages, 2011.

[27] Y. Song, *Coincidence Points for Non commuting* f *-Weakly Contractive Mappings*, Int. J. Computational Appl. Math. **2** (1), 51-57, 2007.

[28] Q. Zhang, Y. Song, *Fixed point theory for generalized* φ*-Weak Contractions*, Appl. Math. Lett. **22**, 75-78, 2009.