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Research Article

Investigation of an impulsive Sturm-Liouville operator on semi axis

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Abstract

The objective of this work is to investigate some spectral properties of an impulsive Sturm-Liouville boundary value problem on the semi axis. By the help of analytical properties of the Jost solution and asymptotic properties of a transfer matrix M, we examine the existence of the spectral singularities and eigenvalues of the impulsive operator generated by the Sturm-Liouville equation.

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1. Introduction

As is well known in spectral theory, eigenvalues and spectral singularities are of a great importance and investigation of these notions have a long history in mathematics, theoretical physics and quantum mechanics. Let us shortly give an overview on the existing literature of this subject. Study of the spectral analysis of operators with continuous and discrete spectrum was began by Naimark [18]. Naimark discovered a part of continuous spectrum which is a mathematical obstruction for the completeness of the eigenvectors called a *spectral singularity*. Then various mathematicians studied spectral singularities in 1960s [20, 22]. Schwartz studied the spectral singularities of a certain class of abstract linear operators in a Hilbert space and proved that self-adjoint operators have no spectral singularity. Marchenko investigated the Sturm-Liouville equation

$$-y'' + q(x)y = \lambda^2 y, \quad 0 \le x < \infty, \tag{1.1}$$

where q is a real valued function and λ is a spectral parameter. The equation (1.1) has a bounded solution satisfying the condition

$$\lim_{x\to\infty} e(x,\lambda)e^{-i\lambda x} = 1,$$

where

$$\lambda \in \overline{\mathbb{C}}_+ := \{\lambda \in \mathbb{C} : Im\lambda \ge 0\}.$$

$e(x,\lambda)$ is called the Jost solution of (1.1) and it has an integral represantation

$$e(x,\lambda) = e^{i\lambda x} + \int_x^\infty K(x,t)e^{i\lambda t}dt, \quad \lambda \in \overline{\mathbb{C}}_+, \tag{1.2}$$

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under the condition

$$\int_0^\infty x|q(x)|dx < \infty, \tag{1.3}$$

where $K(x,t)$ is defined by the potential function q [13]. So far, there exist various studies on the concept of spectral singularities of differential equations [1,3,5,6,8,9,11].

On the other hand, during some physical and chemical phonemena, some discontinuities can be seen which change the state of the process. Even if the duration of this rapid change is negligible, it may cause some effects in scientific facts. The theory of impulsive differential equations was studied in applied mathematics in detail [4,21]. Besides, impulsive equations have been motivated by a number of problem in spectral theory [2,7,10,16,17,23]. Especially, the mechanism by which spectral singularities spoil the completeness of the eigenfunctions and their difference with exceptional points is discussed in [15]. In [14], the author provided the physical meanings of eigenvalues and spectral singularities of a general point interaction of Schrödinger equation at a single point. But most of these studies based on impulsive differantial equations were given on the whole axis.

In this paper, we are concerned with impulsive Sturm-Liouville operator on the semi axis and obtain some spectral properties.

2. Asymptotic properties

Let L denote the Sturm-Liouville operator on the semi axis in $L^2[0,\infty)$ by the equations

$$-y'' + q(x)y = \lambda^2 y, \quad x \in [0,1) \cup (1,\infty), \tag{2.1}$$

$$y(0) = 0 \tag{2.2}$$

with impulsive condition

$$\begin{bmatrix} y_+(1) \\ y'_+(1) \end{bmatrix} = B \begin{bmatrix} y_-(1) \\ y'_-(1) \end{bmatrix}, \quad B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \tag{2.3}$$

where a, b, c, d are complex numbers, λ is a spectral parameter and the complex valued q is supposed to satisfy the condition

$$\int_0^\infty x|q(x)|dx < \infty. \tag{2.4}$$

Furthermore, y_- and y_+ be respectively the restrictions of the solution of (2.1) to the interval $[0,1)$ and $(1,\infty)$, i.e.

$$\begin{cases} y_-(x) := y(x), & 0 \le x < 1 \\ y_+(x) := y(x), & x > 1. \end{cases}$$

Clearly, $x = 1$ is the interaction point and B is used to continue the solution of (2.1) from $[0,1)$ to $(1,\infty)$. It is well known that $S(x,\lambda^2)$ and $C(x,\lambda^2)$ are the fundamental solutions of (2.1) in the interval $[0,1)$ satisfying the initial conditions

$$S(0,\lambda^2) = 0, \qquad S'(0,\lambda^2) = 1$$

and

$$C(0,\lambda^2) = 1, \qquad C'(0,\lambda^2) = 0,$$

respectively. The solutions $S(x,\lambda^2)$ and $C(x,\lambda^2)$ are entire functions of λ and

$$W[S(x,\lambda^2), C(x,\lambda^2)] = -1, \quad \lambda \in \mathbb{C},$$

where $W[y_1,y_2]$ denotes the wronskian of the solutions y_1 and y_2 of the equation (2.1). Besides the solutions $S(x,\lambda^2)$ and $C(x,\lambda^2)$ have integral representations

$$S(x,\lambda^2) = \frac{\sin\lambda x}{\lambda} + \int_0^x \varphi(x,t)\frac{\sin\lambda t}{\lambda}dt \tag{2.5}$$

and

$$C(x,\lambda^2) = \cos\lambda x + \int_0^x P(x,t)\cos\lambda t dt, \tag{2.6}$$

where the kernels $\varphi(x,t)$ and $P(x,t)$ can be expressed in terms of the potential function q [12].

Furthermore, under the condition (2.4), $\hat{e}(x,\lambda)$ is an unbounded solution of (2.1) in the interval $(1,\infty)$ satisfying the condition

$$\lim_{x\to\infty} \hat{e}(x,\lambda)e^{i\lambda x} = 1,\ \lambda \in \overline{\mathbb{C}}_+ \tag{2.7}$$

and it has an integral representation [19]

$$\hat{e}(x,\lambda) = e^{-i\lambda x} + \frac{1}{2i\lambda}\int_{x_0}^{x} e^{i(x-t)\lambda} q(t)\hat{e}(t,\lambda)dt + \frac{1}{2i\lambda}\int_x^\infty e^{i(t-x)}q(t)\hat{e}(t,\lambda)dt, \tag{2.8}$$

where x_0 is a positive real number. From (1.2) and (2.8), it is easy to verify

$$W[e(x,\lambda),\hat{e}(x,\lambda)] = -2i\lambda, \quad x\in(1,\infty), \quad \lambda\in\overline{C}_+.$$

Now by the help of linearly independent solutions of the equation (2.1) in the intervals $[0,1)$ and $(1,\infty)$, the general solution of (2.1) can be indicated as

$$\begin{cases} y_-(x,\lambda) = A_-C(x,\lambda^2) + B_-S(x,\lambda^2), & 0\le x<1 \\ y_+(x,\lambda) = A_+e(x,\lambda) + B_+\hat{e}(x,\lambda), & x>1, \end{cases} \tag{2.9}$$

where $A_\pm$ and $B_\pm$ are λ dependent coefficients. Next, we set

$$y_\pm(1) = \lim_{x\to1^\pm} y_\pm(x)$$

and then from (1.2), (2.5), (2.6) and (2.8), we obtain easily

$$S(1,\lambda^2) = \frac{\sin\lambda}{\lambda} + \int_0^1 \varphi(1,t)\frac{\sin\lambda t}{\lambda}dt,$$

$$C(1,\lambda^2) = \cos\lambda + \int_0^1 P(1,t)\cos\lambda t dt,$$

$$e(1,\lambda) = e^{i\lambda} + \int_1^\infty K(1,t)e^{i\lambda t}dt,$$

$$\hat{e}(1,\lambda) = e^{-i\lambda} + \frac{1}{2i\lambda}\int_{x_0}^{1} e^{i(1-t)\lambda}q(t)\hat{e}(t,\lambda)dt + \frac{1}{2i\lambda}\int_1^\infty e^{i(t-1)}q(t)\hat{e}(t,\lambda)dt.$$

Combining the last equalities, we find $y_-(1,\lambda)$, $y_-'(1,\lambda)$, $y_+(1,\lambda)$ and $y_+'(1,\lambda)$. Using these solutions and the impulsive condition (2.3), we have

$$\begin{bmatrix} A_+ \\ B_+ \end{bmatrix} = M \begin{bmatrix} A_- \\ B_- \end{bmatrix}, \tag{2.10}$$

where

$$M := \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} = N^{-1}BD \tag{2.11}$$

such that

$$D := \begin{bmatrix} C(1,\lambda^2) & S(1,\lambda^2) \\ C'(1,\lambda^2) & S'(1,\lambda^2) \end{bmatrix} \tag{2.12}$$

and

$$N := \begin{bmatrix} N_{11} & N_{12} \\ N_{21} & N_{22} \end{bmatrix}. \tag{2.13}$$

Besides, components of N are given by

$$N_{11} = e^{i\lambda} + \int_1^\infty K(1,t)e^{i\lambda t}dt,$$

$$N_{12} = e^{-i\lambda} + \frac{1}{2i\lambda}\left(\int_{x_0}^1 e^{i(1-t)\lambda}q(t)\hat{e}(t,\lambda)dt + \int_1^\infty e^{i(t-1)}q(t)\hat{e}(t,\lambda)dt\right),$$

$$N_{21} = i\lambda e^{i\lambda} - K(1,1)e^{i\lambda} + \int_1^\infty K_x(1,t)e^{i\lambda t}dt,$$

$$N_{22} = -i\lambda e^{-i\lambda} + \frac{1}{2}\int_{x_0}^1 e^{i(1-t)\lambda}q(t)\hat{e}(t,\lambda)dt - \frac{1}{2\lambda}\int_1^\infty e^{i(t-1)}q(t)\hat{e}(t,\lambda)dt.$$

Now, let consider the Jost solution of boundary value problem (2.1)-(2.3) and denote by f. Since $\hat{e}(x,\lambda) \notin L^2(1,\infty)$, we can write $f(x,\lambda) \to e(x,\lambda)$, $x \to \infty$. Thus, from (2.9), the coefficients A_+ and B_+ turn into 1 and 0, respectively. Moreover, using the impulsive condition (2.3), we also get A_- and B_- uniquely. By (2.10), we find

$$A_- = \frac{M_{22}(\lambda)}{detM}, \quad B_- = -\frac{M_{21}(\lambda)}{detM}.$$

Therefore,

$$f(x,\lambda) = \begin{cases} \dfrac{M_{22}(\lambda)}{detM}C(x,\lambda^2) - \dfrac{M_{21}(\lambda)}{detM}S(x,\lambda^2), & 0 \le x < 1 \\ e(x,\lambda), & 1 < x < \infty \end{cases} \tag{2.14}$$

is called the Jost solution of impulsive boundary value problem (2.1)-(2.3).

On the other hand, if we consider another solution of (2.1)-(2.3) and denote by g, since the boundary condition (2.2) is satisfied, we easily express the solution g by the help of (2.9) and (2.10) as

$$g(x,\lambda) = \begin{cases} hS(x,\lambda^2), & 0 \le x < 1 \\ hM_{12}(\lambda)e(x,\lambda) + hM_{22}(\lambda)\hat{e}(x,\lambda), & 1 < x < \infty, \end{cases} \tag{2.15}$$

where h is an arbitrary non-zero complex number.

Now, we can give the following theorem.

Theorem 2.1. *The following equations hold:*

$$W[f,g](x,\lambda) = -2i\lambda h M_{22}(\lambda), \quad x \to \infty, \tag{2.16}$$

$$W[f,g](x,\lambda) = \frac{hM_{22}(\lambda)}{detM}, \quad x \to 0^+. \tag{2.17}$$

Proof. (2.14) and (2.15) can be used to calculate the Wronskian of the solutions for $x \to \infty$ and for $x \to 0^+$. Clearly for $x \to \infty$, we see

$$\begin{aligned} W[f,g](x,\lambda) &= e(x,\lambda)\left\{hM_{12}(\lambda)e'(x,\lambda) + hM_{22}(\lambda)\hat{e}'(x,\lambda)\right\} \\ &\quad - e'(x,\lambda)\left\{hM_{12}(\lambda)e(x,\lambda) + hM_{22}(\lambda)\hat{e}(x,\lambda)\right\} \\ &= -2i\lambda h M_{22}(\lambda), \end{aligned}$$

due to the fact that $W[e(x,\lambda),\hat{e}(x,\lambda)] = -2i\lambda$ independently of x. Next, for $x \to 0^+$, we find similarly

$$\begin{aligned} W[f,g](x,\lambda) &= \left\{\frac{M_{22}(\lambda)}{detM}C(x,\lambda^2) - \frac{M_{21}(\lambda)}{detM}S(x,\lambda^2)\right\}hS'(x,\lambda^2) \\ &\quad - \left\{\frac{M_{22}(\lambda)}{detM}C'(x,\lambda^2) - \frac{M_{21}(\lambda)}{detM}S'(x,\lambda^2)\right\}hS(x,\lambda^2) \\ &= \frac{hM_{22}(\lambda)}{detM}. \end{aligned}$$

It completes the proof. □

Note that the Wronskians of the solutions of (2.1) in the interval $(0,1)$ and $(1,\infty)$ are independent of x, but they are not equal because of the characteristic feature of impulsive equations.

Let y_1 is an arbitrary solution of (2.1) satisfying the boundary condition (2.2) and y_2 is the Jost solution of (2.1). In this situation, we have spectral singularities which are the real (non-vanishing) zeros of $W[y_1, y_2]$ [9].

Now using Theorem (2.1), we can give the following result.

Corollary 2.2. *A necessary and sufficient condition to investigate the eigenvalues and spectral singularities of the impulsive Sturm-Liouville operator L is to investigate the zeros of the function M_{22}.*

Thus, spectral singularities of operator L are given by λ^2, where λ is a (non-vanishing) real zero of M_{22} and eigenvalues of L are given by λ^2, where λ is a zero of M_{22} with $Im\lambda > 0$.

Theorem 2.3. *Under the condition* (2.4)*, the function M_{22} satisfies the following asymptotic equation for $\lambda \in \overline{\mathbb{C}}_+$, $|\lambda| \to \infty$*

$$M_{22}(\lambda) = \frac{b}{4}\left[1 + O\left(\frac{1}{\lambda}\right)\right]. \tag{2.18}$$

Proof. Since $\det N = -2i\lambda$, it is easy to obtain from (2.11)-(2.13) that

$$\begin{aligned} M_{22}(\lambda) \;\; = \;\; & \frac{i}{2\lambda}\Big\{-e'(1,\lambda)\left[aS\left(1,\lambda^2\right) + bS'\left(1,\lambda^2\right)\right] \\ & + e(1,\lambda)\left[cS\left(1,\lambda^2\right) + dS'\left(1,\lambda^2\right)\right]\Big\}. \end{aligned} \tag{2.19}$$

The derivative of $e(x,\lambda)$ satisfies the following asymptotic equation

$$e'(x,\lambda) = e^{i\lambda x}[i\lambda + O(1)], \quad x \in [0,\infty),\ \lambda \in \overline{C}_+,\ |\lambda| \to \infty. \tag{2.20}$$

From (2.19) and (2.20), we can express the function M_{22} for $\lambda \in \overline{C}_+$ and $|\lambda| \to \infty$,

$$\begin{aligned} M_{22}(\lambda) \;\; = \;\; & \Big\{-\frac{ia}{2}\left[\frac{e'(1,\lambda)\,e^{-i\lambda}}{\lambda}\right]\left[S\left(1,\lambda^2\right)e^{i\lambda}\right] - \frac{ib}{2}\left[\frac{e'(1,\lambda)\,e^{-i\lambda}}{\lambda}\right]\left[S'\left(1,\lambda^2\right)e^{i\lambda}\right] \\ & + \frac{ic}{2\lambda}\left[e(1,\lambda)\,e^{-i\lambda}\right]\left[S\left(1,\lambda^2\right)e^{i\lambda}\right] + \frac{id}{2\lambda}\left[e(1,\lambda)\,e^{-i\lambda}\right]\left[S'\left(1,\lambda^2\right)e^{i\lambda}\right]\Big\} \\ = \;\; & \Big\{\frac{a}{4i}\left[\frac{e^{2i\lambda}}{\lambda} - \frac{1}{\lambda} + \frac{1}{\lambda}\int_0^1 \varphi(1,t)\left[e^{i\lambda(1+t)} - e^{i\lambda(1-t)}\right]dt\right] \\ & + \frac{b}{4}\Big[e^{2i\lambda} + 1 + \varphi(1,1)\frac{e^{2i\lambda}}{i\lambda} - \frac{\varphi(1,1)}{i\lambda} \\ & + \frac{1}{i\lambda}\int_0^1 \varphi_x(1,t)\left[e^{i\lambda(1+t)} - e^{i\lambda(1-t)}\right]dt\Big] \\ & + \frac{c}{4}\left[\frac{e^{2i\lambda}}{\lambda^2} - \frac{1}{\lambda^2} + \frac{1}{\lambda^2}\int_0^1 \varphi(1,t)\left[e^{i\lambda(1+t)} - e^{i\lambda(1-t)}\right]dt\right] \\ & + \frac{id}{4}\Big[\frac{e^{2i\lambda}}{\lambda} + \frac{1}{\lambda} + \varphi(1,1)\frac{e^{2i\lambda}}{i\lambda^2} - \frac{\varphi(1,1)}{i\lambda^2} \\ & + \frac{1}{i\lambda^2}\int_0^1 \varphi_x(1,t)\left[e^{i\lambda(1+t)} - e^{i\lambda(1-t)}\right]dt\Big]\Big\} \\ = \;\; & \left\{O\left(\frac{1}{\lambda}\right) + \frac{b}{4} + O\left(\frac{1}{\lambda}\right) + O\left(\frac{1}{\lambda^2}\right) + O\left(\frac{1}{\lambda}\right)\right\} \\ = \;\; & \frac{b}{4}\left[1 + O\left(\frac{1}{\lambda}\right)\right]. \end{aligned}$$

This completes the proof. □

3. Conclusion and an example

In this paper, we have established an impulsive Sturm-Liouville boundary value problem under a certain condition. Then, we investigated eigenvalues and spectral singularities of this problem. The biggest difference from other papers is that we use an another method to make spectral analysis of the problem by determining a transfer matrix. Finally, we give a detailed example to illustrate our results.

Example 3.1. Let us consider the following impulsive Sturm-Liouville boundary value problem

$$\begin{cases} -y'' = \lambda^2 y, \qquad x \in [0,1) \cup (1,\infty), \\ y(0) = 0, \\ \begin{bmatrix} y_+(1) \\ y'_+(1) \end{bmatrix} = B \begin{bmatrix} y_-(1) \\ y'_-(1) \end{bmatrix}, \quad B = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}, \end{cases} \tag{3.1}$$

where $a, d \in \mathbb{C}$ and $ad \neq 0$. It is easy to see from (1.2), (2.5), (2.6) and (2.8) that

$$S(x,\lambda^2) = \frac{\sin \lambda x}{\lambda}, \quad C(x,\lambda^2) = \cos \lambda x, \quad e(x,\lambda) = e^{i\lambda x}, \quad \hat{e}(x,\lambda) = e^{-i\lambda x}.$$

Also, it follows from (2.19) that

$$M_{22}(\lambda) = \frac{ie^{i\lambda}}{2\lambda^2} \left(-ia\lambda \sin \lambda - ib\lambda^2 \cos \lambda + c \sin \lambda + d\lambda \cos \lambda \right). \tag{3.2}$$

In order to find eigenvalues and spectral singularities of (3.1), we examine the zeros of $M_{22}(\lambda)$. For this purpose, we obtain from (3.2) that

$$e^{2i\lambda} = \frac{a+d}{a-d}.$$

Using the last equation, we find

$$\lambda_k = -\frac{i}{2} \ln \left| \frac{A+1}{A-1} \right| + \frac{1}{2} Arg \left(\frac{A+1}{A-1} \right) + k\pi, \quad k \in \mathbb{Z},$$

where $A = \dfrac{a}{d}$.

Case 1: Let $A = \dfrac{e^{i\theta}+1}{e^{i\theta}-1}$ for $\theta \in \mathbb{R}$. In this case $Arg\left(\dfrac{A+1}{A-1}\right) = \theta$ and we obtain

$$\lambda_k = \frac{\theta}{2} + k\pi, \quad k \in \mathbb{Z}.$$

Since $\lambda_k \in \mathbb{R}$, $\mu_k = \lambda_k^2$, $k \in \mathbb{Z}$ are the spectral singularities of the impulsive boundary value problem (3.1).

Case 2: Let $ImA \neq 0$.

If A be pure imaginer, that is $ReA = 0$, we get $\left|\frac{A+1}{A-1}\right| = 1$ and so

$$\lambda_k = \pi \left(k + \frac{1}{2} \right), \quad k \in \mathbb{Z}.$$

That means, the numbers $\mu_k = \lambda_k^2$ are the spectral singularities of (3.1).

If $ReA < 0$, we have eigenvalues $\mu_k = \lambda_k^2$ with

$$\lambda_k = -\frac{i}{2} \ln \left| \frac{A+1}{A-1} \right| + \frac{1}{2} Arg \left(\frac{A+1}{A-1} \right) + k\pi, \quad k \in \mathbb{Z},$$

since $\lambda_k \in \mathbb{C}_+$.

Case 3: Let A be real. Then, we need to investigate some special cases.

(i): If $0 < A < 1$, we find

$$\lambda_k = -\frac{i}{2}\ln\left|\frac{A+1}{A-1}\right| + \pi\left(k+\frac{1}{2}\right), \quad k \in \mathbb{Z}.$$

Since $\lambda_k \in \mathbb{C}_- := \{\lambda \in \mathbb{C} : Im\lambda < 0\}$, in this case, the spectral singularities and eigenvalues of (3.1) are not existing.

(ii): If $1 < A < \infty$, we see that

$$\lambda_k = -\frac{i}{2}\ln\frac{A+1}{A-1} + k\pi, \quad k \in \mathbb{Z},$$

for $\lambda_k \in \mathbb{C}_-$, so similarly to the Case 3 (i), (3.1) has no spectral singularities and eigenvalues.

(iii): If $-1 < A < 0$, we obtain that

$$\lambda_k = \frac{i}{2}\ln\left|\frac{A-1}{A+1}\right| + \pi\left(k+\frac{1}{2}\right), \quad k \in \mathbb{Z}.$$

Since $\lambda_k \in \mathbb{C}_+$, the numbers $\mu_k = \lambda_k^2$, $k \in \mathbb{Z}$ are the eigenvalues of (3.1) and in this case, (3.1) has no spectral singularities.

(iv): If $-\infty < A < -1$, we find

$$\lambda_k = \frac{i}{2}\ln\frac{A-1}{A+1} + k\pi$$

for $\lambda_k \in \mathbb{C}_+$ and similar to the Case 3 (iii), $\mu_k = \lambda_k^2$, $k \in \mathbb{Z}$ are the eigenvalues of (3.1) and in this case, (3.1) has no spectral singularities as well.

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