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Research Article

Proximately chain refinable functions

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Abstract

We define proximately chain refinable functions as a generalization of refinable maps and investigate some of their properties for Hausdorff paracompact spaces. We prove that the proximate fixed point property is preserved by proximate near homeomorphisms in paracompact Hausdorff spaces. This generalizes a previous result of E. Grace.

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1. Introduction

Refinable maps, defined by Ford in [3], are shown to be a very fruitful class of continuous functions. By changing some of the requirements, Grace in papers [4–6] defined weakly refinable and proximately refinable functions and he obtained various properties of these maps. Main theorems usually work for compacta, and in some cases, for continua.

One of the main properties of proximately refinable functions is the preservation of proximate fixed point property for some types of functions and spaces.

In this paper we define a weaker class than proximately refinable functions, we investigate some of their properties and we prove a theorem about the proximate fixed point property in the domain of Hausdorff paracompact spaces.

2. Definitions and notations

Along this paper by a covering we mean an open covering of the space.

If $\mathcal{U}, \mathcal{V}$ are two coverings of the space X, then $\mathcal{V}$ is refinement of $\mathcal{U}$ if for every $V \in \mathcal{V}$ there exists $U \in \mathcal{U}$ such that $V \subseteq U$. We write $\mathcal{V} \prec \mathcal{U}$.

If $U \in \mathcal{U}$, then the star of U is the set $St(U, \mathcal{U}) = \{W \in \mathcal{U} | W \cap U \neq \emptyset\}$ and by $St\mathcal{U}$ will be denoted the collection of all $St(U, \mathcal{U}), U \in \mathcal{U}$.

By $f : X \to Y$ we denote a function (not necessarily continuous) from X to Y and $f : X \twoheadrightarrow Y$ means that f is a surjective function from X to Y.

Suppose $\mathcal{F}$ is an open family of subsets of X and x and y are two points in X. A *chain* in $\mathcal{F}$ from x and y is a finite sequence $F_1, F_2, ..., F_n$ of members of $\mathcal{F}$ such that $x \in F_1, y \in F_n$ and $F_i \cap F_{i+1} \neq \emptyset$, for $1 \leq i \leq n-1$. We say that the points x and y are *chain*$\mathcal{F}-$near.

If $F', F'' \in \mathcal{F}$, then a *chain* from F' to F'' is a finite sequence $F_1, F_2, ..., F_n$ of members

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of $\mathcal{F}$ such that $F' \cap F_1 \neq \emptyset, F'' \cap F_n \neq \emptyset$ and $F_i \cap F_{i+1} \neq \emptyset$, for $1 \leq i \leq n-1$.
The collection $\mathcal{F}$ is said to be *connected* if for every pair F', F'' of elements of $\mathcal{F}$ there exists a chain from F' to F''. For every collection $\mathcal{F}$ the *components* of $\mathcal{F}$ are defined as maximal connected subcollections of $\mathcal{F}$.

Definition 2.1. A covering of the space X is said to be *finite component covering* if all of its components are finite.

Definition 2.2. A space X is said to be *superparacompact* if every covering $\mathcal{U}$ of the space X has a finite component refinement $\mathcal{V}$.

For arbitrary covering $\mathcal{V}$ of the topological space X and for $V \in \mathcal{V}$ by *chain*V we denote the set $\bigcup\{W \in \mathcal{V} |$ there exists e chain in $\mathcal{V}$ from V to $W\}$.
By *chain*$\mathcal{V}$ we denote the covering $\{chainV |\ V \in \mathcal{V}\}$.

Let $f : X \to Y$ be a function and let $\mathcal{V}$ be a covering of Y. We say that $g : X \to Y$ is $\mathcal{V}-$near to f if for every $x \in X$, $f(x)$ and $g(x)$ lie in the same member of $\mathcal{V}$.

Definition 2.3. Let X, Y be spaces, and $\mathcal{V}$ a covering of Y . The function $f : X \to Y$ is $\mathcal{V}-$*continuous*, if for any $x \in X$, there exists a neighborhood U of x, such that $f(U) \subseteq V$ for some member $V \in \mathcal{V}$.
(The family of all such U form a covering $\mathcal{U}$ of X. Shortly, we say that $f : X \to Y$ is $\mathcal{V}-$continuous, if there exists such an $\mathcal{U}$ satisfying $f(\mathcal{U}) \prec \mathcal{V}$.)

Since refinable maps are investigated in the class of compact metric spaces, the domain best suited to a generalization of refinable maps seems to be the class of paracompact Hausdorff spaces. These spaces, as a class between compact and normal spaces, fulfill nice covering properties and provide a rich foundation for future applications. In the following definitions all spaces are assumed to be Hausdorff and paracompact.

Definition 2.4. Let $\mathcal{U}$ be a covering of the space X. The function $f : X \twoheadrightarrow Y$ is *strong* $\mathcal{U}-$*function* if for every $y \in Y$ there exists a neighborhood D of y in Y such that $f^{-1}(D)$ is contained in some member of $\mathcal{U}$.

Definition 2.5. The function $f : X \twoheadrightarrow Y$ is *proximately chain refinable* if for every covering $\mathcal{V}$ of Y and for every covering $\mathcal{U}$ for X, there exists $\mathcal{V}-$continuous strong $\mathcal{U}-$function $g : X \twoheadrightarrow Y$ which is *chain*$\mathcal{V}-$near to f.
We say that g is chain $(\mathcal{U}, \mathcal{V})$ refinement of f.

Definition 2.6. Let $\mathcal{U}$ be a covering of X and $\mathcal{V}$ a covering of Y. The bijective function $f : X \to Y$ is $(\mathcal{U}, \mathcal{V})-$*homeomorphism* if f and f^{-1} are $\mathcal{V}, \mathcal{U}-$continuous, respectively.

Definition 2.7. The function $f : X \to Y$ is *proximate (chain) near homeomorphism* if for every covering $\mathcal{U}$ of X and $\mathcal{V}$ of Y there exists $(\mathcal{U}, \mathcal{V})-$homeomorphism $g : X \to Y$ which is $\mathcal{V}$ (*chain*$\mathcal{V}$) near to f.

Definition 2.8. The function $f : X \to Y$ is *strong chain homeomorphism* if for every covering $\mathcal{V}$ of Y there exists a homeomorphism $g : X \to Y$ which is *chain*$\mathcal{V}-$near to f.

It is clear that every proximate chain near homeomorphism is a proximately chain refinable function.

For arbitrary covering of a connected space, from [9] any two points could be connected by finite chains, so by considering the papers [5, p.330] and [9, Proposition 1.1], it is always possible to construct a proximate near chain homeomorphism from an arc X to a continuum (compact connected metric space) Y, actually every surjection from an arc to continuum is proximate chain near homeomorphism. As a consequence, proximately chain refinable functions and proximate chain near homeomorphisms don't need to be continuous in all cases.

3. Properties of proximate chain refinable functions

First, we will show that composition of a proximately chain refinable function with a strong chain homeomorphism is proximately chain refinable.

The following theorem given in [7] as Proposition 1.3. is an easy consequence of the definitions of continuous functions over a covering.

Theorem 3.1. *Let X, Y and Z be topological spaces and let $\mathcal{W}$ be a covering of Y. For every $\mathcal{W}-$continuous function $g : Y \to Z$, there exists a covering $\mathcal{V}$ of Y, such that $g(\mathcal{V}) \prec W$ and for every $\mathcal{V}-$continuous function $f : X \to Y$ the composition $g \circ f$ is $\mathcal{W}-$continuous.*

Theorem 3.2. *Let X, Y and Z be Hausdorff paracompact spaces. If $f : X \twoheadrightarrow Y$ is proximately chain refinable and $g : Y \twoheadrightarrow Z$ is strong chain homeomorphism, then the composition $h = g \circ f : X \twoheadrightarrow Z$ is proximately chain refinable.*

Proof. Take $\mathcal{W}$ an arbitrary covering of Z and $\mathcal{U}$ of X. From Theorem 3.1. for every $\mathcal{W}-$continuous function $G : Y \to Z$ there exists a covering $\mathcal{V}_G$ of Y, such that for every $\mathcal{V}_G-$continuous function $F : X \to Y$ the composition $G \circ F$ is $\mathcal{W}-$continuous and $G(\mathcal{V}_G) \prec W$.

Choose $g_1 : Y \to Z$ to be a homeomorphism which is $chain\mathcal{W}$ near to g. There exists a covering $\mathcal{V}$ of Z such that $g_1(\mathcal{V}) \prec \mathcal{W}$.

Now, since f is proximately chain refinable, there exists a $\mathcal{V}-$continuous $\mathcal{U}-$ strong function $f_1 : X \twoheadrightarrow Y$ which is $chain\mathcal{V}$ near to f. Considering the fact that f_1 is $\mathcal{U}-$strong, there exists a covering $\mathcal{V}_2$ of Y such that for every $V \in \mathcal{V}_2$ we have $f_1^{-1}(V) \subseteq U$ for some set U of the covering $\mathcal{U}$.

It is clear that the composition $g_1 \circ f_1$ is $\mathcal{W}-$continuous.
From the fact that g_1^{-1} is $\mathcal{V}_2-$continous it implies that $g_1 \circ f_1$ is $\mathcal{U}-$strong.

Since g and g_1 are $chain\mathcal{W}-$near, the pairs of points $g(f_1(x))$, $g_1(f_1(x))$ and $g_1(f(x))$, $g(f(x))$ are connected by finite chains from the covering $\mathcal{W}$. On the other hand, from $g_1(\mathcal{V}) \prec \mathcal{W}$ and considering the fact that $f(x), f_1(x)$ are $chain\mathcal{V}-$near it implies that the points $g_1(f(x))$, $g_1(f_1(x))$ are connected by a finite chain from $\mathcal{W}$. This shows that the composition $g_1 \circ f_1$ is $chain\mathcal{W}-$near to $g \circ f$.

We have shown that the composition $g_1 \circ f_1$ is $\mathcal{W}-$continuous $\mathcal{U}-$strong function which is $chain\mathcal{W}$ near to $g \circ f$, so $g \circ f$ is proximately chain refinable. □

Theorem 3.3. *Let X be a paracompact Hausdorff and Y a superparacompact Hausdorff space. If $f : X \twoheadrightarrow Y$ is proximately chain refinable, then for every covering $\mathcal{V}$ of Y f is $chain\mathcal{V}-$continuous.*

Proof. Let $\mathcal{V}$ be an arbitrary covering of the space Y. From the superparacompactness of Y by [1, Theorem 2.2] there exists a disjoint refinement $\mathcal{W}$ of $chain\mathcal{V}$. Let $f_1 : X \to Y$ be a $\mathcal{W}-$continuous function which is $chain\mathcal{W}-$near to f. If $x \in X$, then there exists a neighborhood U of x such that $f_1(U) \subseteq W \subseteq chainV$ for some $V \in \mathcal{V}$ and $W \in \mathcal{W}$. We have $f(U) \subseteq W$, because otherwise some point from U will be mapped by f and f_1 to different sets of $\mathcal{W}$ and that is impossible since f and f_1 are $chain\mathcal{W} = \mathcal{W}$ near. So, $f(U) \subseteq chainV$, hence f is $chain\mathcal{V}-$continuous. □

Remark. In the previous theorem we could not assume that every proximately chain refinable function is $\mathcal{V}-$continuous since the function $f : [0, 1] \to [0, 1]$ defined by $f(x) = x$ for $x \in [0, 1/2]$ and $f(x) = -x + 3/2$ for $x \in (1/2, 1]$ is proximately chain refinable, but it is not $\mathcal{V}-$continuous for

$$\mathcal{V} = \{(0, 1/3), (1/4, 3/4), (2/3, 1)\}.$$

Theorem 3.4. *Let X be a Hausdorff paracompact and Y a Hausdorff superparacompact space. If $f : X \twoheadrightarrow Y$ is proximately chain refinable, then X has an isolated point if and only if Y does.*

Proof. If x is an isolated point of X, then for the covering $\mathcal{U} = \{x\} \cup (X \setminus \{x\})$ there exists a $(\mathcal{U}, \{Y\})$ proximately chain refinement g of f. For the point $g(x) \in Y$ there exists a neighborhood $V_{g(x)}$ such that $g^{-1}(V_{g(x)}) \subseteq \{x\}$, so $V_{g(x)} = \{g(x)\}$, which implies that $g(x)$ is an isolated point of Y.

For the opposite, let y be an isolated point of Y and let $f(x) = y$. Take a covering $\mathcal{V} = \{y\} \cup \mathcal{V}'$ such that $y \notin V'$ for all $V' \in \mathcal{V}'$. From Theorem 3.3 f is *chain*$\mathcal{V}-$continuous so there exists a neighborhood $U \subseteq X$ of x such that $f(U) \subseteq \{y\}$. If we suppose that $x' \neq x$ and $x' \in U$ then from the fact that X is Hausdorff there exist neighborhoods $U_x, U_{x'}$ of x, x' respectively, such that $U_x \cap U_{x'} = \emptyset$. If we fix $a \in X \setminus \{x, x'\}$, again from the fact that X is Hausdorff, we could find neighborhood U_a of a such that $x, x' \notin U_a$. If we take the covering $\mathcal{U} = \{U_x\} \cup \{U_{x'}\} \cup \{U_a | x, x' \notin U_a\}$ of X, then for the $g(\mathcal{U}, \mathcal{V})$-proximately chain refinement of f we have $g(U) \subseteq \{y\}$. On the other hand, $x, x' \in g^{-1}(\{y\})$, but in the covering $\mathcal{U}$ we could not find a member containing in the same time the points x, x', so $x = x'$ i.e. $U = \{x\}$. Hence x is an isolated point of X. $\square$

4. Proximate near homeomorphisms and the proximate fixed point property

Definition 4.1. An Hausdorff paracompact space X has the *proximate fixed point property* if for every covering $\mathcal{U}$ of X there exists a covering $\mathcal{V}$ of X with the property: for every $\mathcal{V}-$continuous function $f : X \to X$ there is a point $x \in X$ such that x and $f(x)$ lie together in some member of $\mathcal{U}$.(In this case we write x is a $\mathcal{U}$ invariant point for f).
We say that X has the p.f.p.p..

Example 4.2. The space of integers $\mathbb{Z}$ with discrete topology doesn't have the p.f.p.p..

Proof. If we take the covering $\mathcal{U} = \{\{z\} | z \in \mathbb{Z}\}$ and define the function $f : \mathbb{Z} \to \mathbb{Z}$ by $f(z) = z + 1$, then f is continuous so it is $\mathcal{V}-$continuous for every covering $\mathcal{V}$ of $\mathbb{Z}$. On the other hand for every $z \in \mathbb{Z}$, the points z and $f(z) = z + 1$ are not connected by a set from $\mathcal{U}$, so $\mathbb{Z}$ doesn't have the p.f.p.p. $\square$

Every arc, as an absolute retract, has the proximate fixed point property [8, Theorem 6]. On the other hand, there exists a continuum without the p.f.p.p.[4, p.295]. From the previous discussion there exists a proximate chain near homeomorphism from the arc onto every continuum, so the proximate fixed point property could not be preserved by proximate chain near homeomorphisms.

In [4] E. Grace proved that the p.f.p.p. is preserved by proximate near homeomorphisms for the class of continua.

We will investigate the preservation of p.f.p.p. by proximate near homeomorphisms in the class of paracompact connected Hausdorff spaces, but first we need the following lemma.

Lemma 4.3. *Let X, Y be Hausdorff paracompact spaces and $f : X \twoheadrightarrow Y$. If for every covering $\mathcal{V}$ of Y, there exists a $\mathcal{V}-$continuous function $g : X \twoheadrightarrow Y$ which is $\mathcal{V}$ near to f, then f is continuous.*

Proof. Let the assumtions be fulfilled. First we will prove that f is $\mathcal{W}-$continuous for every covering $\mathcal{W}$ of Y.

From paracompactness of Y there exists a covering $\mathcal{W}_1$ of Y such that $st\mathcal{W}_1 \prec \mathcal{W}$ [2, Theorem 5.1.12]. Take $g : X \twoheadrightarrow Y$ to be a $\mathcal{W}_1-$continuous function $\mathcal{W}_1$ near f, then for arbitrary point x of X there exists a neighborhood U_x of x such that $g(U_x) \subseteq W_1$ for some $W_1 \in \mathcal{W}_1$. On the other hand for $x \in U_x$ the points $f(x)$ and $g(x)$ lie in the same set W_2 from $\mathcal{W}_1$. There exists a set $W \in \mathcal{W}$ such that $stW_1 \subseteq W$. From $W_1 \cap W_2 \neq \emptyset$ it implies that $W_2 \subseteq stW_1 \subseteq W$. Hence $f(U_x) \subseteq W$, so f is $\mathcal{W}-$continuous.

Now, we will prove the continuity of f. Let $x \in X$, $f(x) = y$. Take an open neighborhood V of y. For every $y' \in Y$, $y \neq y'$ there exists a neighborhood V'_y such that $y \notin V'_y$. Now, take the covering $\mathcal{V} = \{V'_y | y' \neq y\} \cup \{V\}$ of Y. The function f is $\mathcal{V}-$continuous so there exists a neighborhood U_x of x and a set O from $\mathcal{V}$ such that $f(U_x) \subseteq O$. But, V is the unique set from $\mathcal{V}$ that contains y, so $O = V$. We showed that $f(U_x) \subseteq V$, hence f is continuous. □

Theorem 4.4. *Let X, Y be Hausdorff paracompact connected spaces and let $f : X \twoheadrightarrow Y$ be a proximate near homeomorphism. If X has the p.f.p.p., then Y also has the p.f.p.p..*

Proof. Let $\mathcal{W}$ be a covering of Y. Similarly as in the previous proof there exists a covering $\mathcal{V}$ of Y such that $st\mathcal{V} \prec \mathcal{W}$. From Lemma 4.3. f is continuous so $f(\mathcal{U}_1) \prec \mathcal{V}$ for some covering $\mathcal{U}_1$ of X. From the fact that X has the p.f.p.p. there exists a covering $\mathcal{U}_2$ of X such that every $\mathcal{U}_2-$continuous function has an $\mathcal{U}_1$ invariant point. There exists a bijection $g : X \to Y$ which is $\mathcal{V}$ near f, g^{-1} is $\mathcal{U}_2-$continuous and $g^{-1}(\mathcal{V}_1) \prec \mathcal{U}_2$ for some covering $\mathcal{V}_1$ of Y.

On the other hand, let $G : Y \to Y$ be an arbitrary $\mathcal{V}_1-$continuous function. We will show that G has $\mathcal{W}$ invariant point.

Taking into consideration Theorem 3.1 and the continuity of f, the composition $G \circ f$ is $\mathcal{V}_1-$continuous. Now, the function $F = g^{-1} \circ G \circ f : X \to X$ is $\mathcal{U}_2-$continuous, so it has an $\mathcal{U}_1$ invariant point x.

From $F(x), x \in U_1 \in \mathcal{U}_1$, the points $f(F(x)), f(x)$ lie in the same set V from the covering $\mathcal{V}$. From the fact that g is $\mathcal{V}$ near to f we have that the points $f(F(x)) = f(g^{-1}(G(f(x)))), G(f(x)) = g(g^{-1}(G(f(x))))$ lie in the same set $V' \in \mathcal{V}$.

From $V \cap V' \neq \emptyset$, we have $f(x), G(f(x)) \in stV' \in st\mathcal{V}$ which proves that $f(x)$ is $st\mathcal{V}$ invariant point for G. From $st\mathcal{V} \prec \mathcal{W}$ we have that $f(x)$ is $\mathcal{W}$ invariant point for G. Hence Y has the p.f.p.p.. □

5. Conclusion

For arbitrary coverings of paracompact Hausdorff nonconnected spaces, by investigating finite chains between images of points, we obtained proximately chain refinable functions as a new class of noncontinuous functions.

The definition of continuity over a covering allowed us to generalize a result about the proximate fixed point property from compact metric spaces to paracompact Hausdorff spaces.

We are curious if the following questions will be affirmative:

Question 1. If $f : X \twoheadrightarrow Y$ is proximately chain refinable and X superparacompact, does Y needs to be superparacompact?

Question 2. Could we drop the assumption of superparacompactness in Theorem 3.4?

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