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Research Article

Quantale-valued uniform convergence towers for quantale-valued metric spaces

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Abstract

We show that quantale-valued metric spaces and quantale-valued partial metric spaces allow natural quantale-valued uniform convergence structures. We furthermore characterize quantale-valued metric spaces and quantale-valued partial metric spaces by these quantale-valued uniform convergence structures. For special choices of the quantale, the results specialize to metric spaces and probabilistic metric spaces.

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1. Introduction

Quantale-valued metric spaces, first discussed in [14] in order to establish a categorical approach to metric spaces, generalize metric spaces by allowing the range of the metric to be a quantale, the values of which are interpreted as generalized distances between points. If we choose as quantale Lawvere's quantale, i.e. the extended half line $[0, \infty]$ ordered opposite to the natural order and with extended addition as quantale operation, then classical metric spaces are recovered. Choosing as quantale the completely distributive lattice of distance distribution functions with a sup-continuous triangle function as quantale operation, then we obtain probabilistic metric spaces [7].

Like metric spaces, also quantale-valued metric spaces allow the definition of underlying uniform structures and concepts like completeness can be studied [6]. A similar generalization for the case of partial metric spaces is e.g. developed in [13]. In this paper, rather than focusing on underlying uniform structures, we develop a framework that allows to *characterize* quantale-valued metric spaces by suitable uniform structures. To this end, we define for each element of the quantale a structure and show that the whole "tower" of these structures then can be used to characterize quantale-valued metric spaces. The resulting quantale-valued uniform convergence tower spaces are special instances of a more general concept introduced recently [11]. In this paper, we will show that the category of quantale-valued metric spaces can be coreflectively embedded into the category of quantale-valued uniform convergence tower spaces and we identify a subcategory that is isomorphic to the category of quantale-valued metric spaces. In this sense, quantale-valued metric spaces can be characterized by their quantale-valued uniform convergence towers. Corresponding results are also achieved for quantale-valued partial metric spaces.

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2. Preliminaries

Let L be a complete lattice with $\top \neq \bot$ for the top element $\top$ and the bottom element $\bot$. In any complete lattice L we can define the *well-below relation* $\alpha \lhd \beta$ if for all subsets $D \subseteq L$ such that $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. Then $\alpha \leq \beta$ whenever $\alpha \lhd \beta$ and, for a subset $B \subseteq L$, we have $\alpha \lhd \bigvee_{\beta \in B} \beta$ iff $\alpha \lhd \beta$ for some $\beta \in B$. Sometimes we need a weaker relation, the *way-below relation*, $\alpha \prec \beta$ if for all directed subsets $D \subseteq L$ such that $\beta \leq \bigvee D$ there is $\delta \in D$ such that $\alpha \leq \delta$. The properties of this relation are similar to the properties of the well-below relation, replacing arbitrary subsets by directed subsets. But we have $\alpha \vee \beta \prec \gamma$ whenever $\alpha, \beta \prec \gamma$, a property not enjoyed by the well-below relation. A complete lattice is completely distributive, if and only if we have $\alpha = \bigvee\{\beta : \beta \lhd \alpha\}$ for any $\alpha \in L$ and it is *continuous* if and only if we have $\alpha = \bigvee\{\beta : \beta \prec \alpha\}$ for any $\alpha \in L$, [9, 19]. Clearly $\alpha \lhd \beta$ implies $\alpha \prec \beta$ and hence every completely distributive lattice is also continuous. For more results on lattices we refer to [9].

The triple $\mathsf{L} = (L, \leq, *)$, where $(L, \leq)$ is a complete lattice, is called a *commutative and integral quantale* if $(L, *)$ is a commutative semigroup for which the top element of L acts as the unit, and $*$ is distributive over arbitrary joins, i.e. $(\bigvee_{i \in J} \alpha_i) * \beta = \bigvee_{i \in J}(\alpha_i * \beta)$.

Typical examples of such quantales are e.g. the unit interval $[0, 1]$ with a left-continuous *t-norm* [21]. Another important example is given by *Lawvere's quantale*, the interval $[0, \infty]$ with the opposite order and addition $\alpha * \beta = \alpha + \beta$ (extended by $\alpha + \infty = \infty + a = \infty$), see e.g. [7]. A further noteworthy example is the quantale of distance distribution functions. A *distance distribution function* $\varphi : [0, \infty] \longrightarrow [0, 1]$, satisfies $\varphi(x) = \sup_{y<x} \varphi(y)$ for all $x \in [0, \infty]$. The set of all distance distribution functions is denoted by Δ^+. With the pointwise order, the set Δ^+ then becomes a completely distributive lattice [7] with top-element ε_0. A quantale operation on Δ^+, $* : \Delta^+ \times \Delta^+ \longrightarrow \Delta^+$, is also called a *sup-continuous triangle function* [21].

We consider in the sequel only commutative and integral quantales $\mathsf{L} = (L, \leq, *)$ with underlying complete lattices that are completely distributive.

For a set X, we denote its power set by $P(X)$ and the set of all filters $\mathbb{F}, \mathbb{G}, ...$ on X by $\mathbb{F}(X)$. The set $\mathbb{F}(X)$ is ordered by set inclusion and maximal elements of $\mathbb{F}(X)$ in this order are called *ultrafilters*. The set of all ultrafilters on X is denoted by $\mathbb{U}(X)$. In particular, for each $x \in X$, the point filter $[x] = \{A \subseteq X : x \in A\} \in \mathbb{F}(X)$ is an ultrafilter. If $\mathbb{F} \in \mathbb{F}(X)$ and $f : X \longrightarrow Y$ is a mapping, then we define $f(\mathbb{F}) \in \mathbb{F}(Y)$ by $f(\mathbb{F}) = \{G \subseteq Y : f(F) \subseteq G \text{ for some } F \in \mathbb{F}\}$. For filters $\Phi, \Psi \in \mathbb{F}(X \times X)$ we define Φ^{-1} to be the filter generated by the filter base $\{F^{-1} : F \in \Phi\}$ where $F^{-1} = \{(x, y) \in X \times X : (y, x) \in F\}$ and $\Phi \circ \Psi$ to be the filter generated by the filter base $\{F \circ G : F \in \Phi, G \in \Psi\}$, whenever $F \circ G \neq \emptyset$ for all $F \in \Phi, G \in \Psi$, where $F \circ G = \{(x, y) \in X \times X : (x, s) \in F, (s, y) \in G \text{ for some } s \in X\}$.

For details and notation from category theory we refer to [1] and [18].

3. L-uniform convergence tower spaces, L-limit spaces and L-metric spaces

Let X be a set. A family $\overline{\Lambda} = (\Lambda_\alpha)_{\alpha \in L}$ with $\Lambda_\alpha \subseteq \mathbb{F}(X \times X)$ which satisfies the axioms

(LUC1) $[(x, x)] \in \Lambda_\alpha$ for all $x \in X, \alpha \in L$;
(LUC2) $\Psi \in \Lambda_\alpha$ whenever $\Phi \leq \Psi$ and $\Phi \in \Lambda_\alpha$;
(LUC3) $\Phi, \Psi \in \Lambda_\alpha$ implies $\Phi \wedge \Psi \in \Lambda_\alpha$;
(LUC4) $\Lambda_\beta \subseteq \Lambda_\alpha$ whenever $\alpha \leq \beta$;
(LUC5) $\Phi^{-1} \in \Lambda_\alpha$ whenever $\Phi \in \Lambda_\alpha$;
(LUC6) $\Phi \circ \Psi \in \Lambda_{\alpha * \beta}$ whenever $\Phi \in \Lambda_\alpha$, $\Psi \in \Lambda_\beta$ and $\Phi \circ \Psi$ exists;
(LUC7) $\Lambda_\bot = \mathbb{F}(X \times X)$

is called an L-*uniform convergence tower on* X and the pair $(X, \overline{\Lambda})$ is called an L-*uniform convergence tower space*. A mapping $f : (X, \overline{\Lambda}) \longrightarrow (X', \overline{\Lambda'})$ between L-uniform convergence tower spaces is called *uniformly continuous* if, for all $\Phi \in \mathbb{F}(X \times X)$, $(f \times f)(\Phi) \in \Lambda'_\alpha$ whenever $\Phi \in \Lambda_\alpha$. The category of L-uniform convergence tower spaces with uniformly continuous mappings as morphisms is denoted by L-UCTS. We note that an L-uniform convergence tower space is a stratified $\{0,1\}\{0,1\}L$-uniform convergence tower space in the definition of [11].

For $\mathsf{L} = (\{0,1\}, \leq, \wedge)$ we obtain uniform convergence spaces [18], for $\mathsf{L} = ([0,1], \leq, *)$ with a left-continuous t-norm we obtain probabilistic uniform convergence spaces in the definition of Nusser [17], for $\mathsf{L} = (\Delta^+, \leq, *)$ we obtain the probabilistic uniform convergence spaces in [2] and for $\mathsf{L} = ([0,\infty], \geq, +)$ we obtain the approach uniform convergence spaces of Lee and Windels [15]. It follows from [11] that the category L-UCTS is topological and Cartesian closed.

An L-uniform convergence tower space $(X, \overline{\Lambda})$ is called *principal* if $\bigwedge_{i \in I} \Phi_i \in \Lambda_\alpha$ whenever $\Phi_i \in \Lambda_\alpha$ for all $i \in I$ $(\alpha \in L)$. It is called *left-continuous* if for all subsets $M \subseteq L$ we have $\Phi \in \Lambda_{\bigvee M}$ whenever $\Phi \in \Lambda_\alpha$ for all $\alpha \in M$.

For $(X, \overline{\Lambda}) \in |\mathsf{L}\text{-}\mathsf{UCTS}|$, $\mathbb{F} \in \mathbb{F}(X)$, $x \in X$ and $\alpha \in L$ we define

$$x \in c_\alpha^{\overline{\Lambda}}(\mathbb{F}) \iff [x] \times \mathbb{F} \in \Lambda_\alpha.$$

It is then not difficult to show that $(X, \overline{c^{\overline{\Lambda}}} = (c_\alpha^{\overline{\Lambda}})_{\alpha \in L})$ is an L-*limit tower space*, i.e. satisfies the axioms (see [12])

(LC1) $x \in c_\alpha^{\overline{\Lambda}}([x])$ for all $x \in X, \alpha \in L$;
(LC2) $c_\alpha^{\overline{\Lambda}}(\mathbb{F}) \subseteq c_\alpha^{\overline{\Lambda}}(\mathbb{G})$ whenever $\mathbb{F} \leq \mathbb{G}$;
(LC3) $c_\alpha^{\overline{\Lambda}}(\mathbb{F} \wedge \mathbb{G}) = c_\alpha^{\overline{\Lambda}}(\mathbb{F}) \cap c_\alpha^{\overline{\Lambda}}(\mathbb{G})$;
(LC4) $c_\beta^{\overline{\Lambda}}(\mathbb{F}) \subseteq c_\alpha^{\overline{\Lambda}}(\mathbb{F})$ whenever $\alpha \leq \beta$;
(LC5) $x \in c_\perp^{\overline{\Lambda}}(\mathbb{F})$ for all $x \in X, \mathbb{F} \in \mathbb{F}(X)$.

For $\mathsf{L} = ([0,1], \leq, *)$ with a left-continuous t-norm, we obtain the probabilistic limit spaces of [17, 20]. For Lawvere's quantale, $\mathsf{L} = ([0,\infty], \geq, +)$, an L-limit tower space is a limit tower space in the definition of [5] and for $\mathsf{L} = (\Delta^+, \leq, *)$ we obtain the probabilistic convergence spaces in [10].

An L-*metric space* [7] is a pair (X, d) of a set X and a mapping $d : X \times X \longrightarrow L$ which satisfies the axioms

(LM1) $d(x,x) = \top$ for all $x \in X$;
(LM2) $d(x,y) = d(y,x)$ for all $x, y \in X$;
(LM3) $d(x,y) * d(y,z) \leq d(x,z)$ for all $x, y, z \in X$.

A mapping between two L-metric spaces, $f : (X, d_X) \longrightarrow (X', d')$ is called an L-*metric morphism* if $d(x_1, x_2) \leq d'(f(x_1), f(x_2))$ for all $x_1, x_2 \in X$. We denote the category of L-metric spaces with L-metric morphisms by L-MET.

If we leave away the *symmetry axiom* (LM2), then we shall speak of an L-*quasimetric space*. In case $L = (\{0,1\}, \leq, \wedge)$, an L-quasimetric space is a preordered set. If $\mathsf{L} = ([0,\infty], \geq, +)$ with the opposite order and extended addition as quantale operation, an L-metric space is a pseudometric space. If $\mathsf{L} = (\Delta^+, \leq, *)$, an L-metric space is a probabilistic pseudometric space, see [7].

4. Embedding L-MET into L-UCTS

Let $(X, d) \in |\mathsf{L}\text{-}\mathsf{MET}|$. Define, for $\alpha \in L$, $\Lambda_\alpha^d \subseteq \mathbb{F}(X \times X)$ by

$$\Phi \in \Lambda_\alpha^d \iff \bigvee_{F \in \Phi} \bigwedge_{(x,y) \in F} d(x,y) \geq \alpha.$$

Lemma 4.1. *Let* $(X,d) \in |\mathsf{L\text{-}MET}|$. *Then* $[(x,y)] \in \Lambda^d_\alpha$ *if and only if* $d(x,y) \geq \alpha$.

Proof. We have, choosing $F = \{(x,y)\} \in [(x,y)]$, $\bigvee_{F\in[(x,y)]} \bigwedge_{(u,v)\in F} d(u,v) \geq d(x,y)$ and on the other hand, as $(x,y) \in F$ for all $F \in [(x,y)]$ we get $\bigvee_{F\in[(x,y)]} \bigwedge_{(u,v)\in F} d(u,v) \leq \bigvee_{F\in[(x,y)]} d(x,y) = d(x,y)$. □

Proposition 4.2. *Let* $(X,d) \in |\mathsf{L\text{-}MET}|$. *Then* $(X, \overline{\Lambda^d}) \in |\mathsf{L\text{-}UCTS}|$, *and is left-continuous and principal.*

Proof. From $d(x,x) = \top \geq \alpha$, we obtain $[(x,x)] \in \Lambda^d_\alpha$ and (LUC1) is valid. (LUC2), (LUC3) and (LUC6) are obvious. (LUC4) follows from the symmetry of the L-metric.

For (LUC5) let $\Phi \in \Lambda^d_\alpha$ and $\Psi \in \Lambda^d_\beta$ and let $\Phi \circ \Psi$ exist. Let further $\alpha' \lhd \alpha$ and $\beta' \lhd \beta$. Then there is $F \in \Phi$ such that for all $(x,y) \in F$ we have $d(x,y) \geq \alpha'$. Likewise, there is $G \in \Psi$ such that for all $(u,v) \in G$ we have $d(u,v) \geq \beta'$. Let $(s,t) \in F \circ G$. Then there is $r \in X$ such that $(s,r) \in F$ and $(r,t) \in G$ and hence $d(s,t) \geq d(s,r) * d(r,t) \geq \alpha' * \beta'$. It follows $\bigvee_{H\in\Phi\circ\Psi} \bigwedge_{(s,t)\in H} d(s,t) \geq \bigwedge_{(s,t)\in F\circ G} d(s,t) \geq \alpha' * \beta'$. The quantale law and the complete distributivity of the lattice L yield $\bigvee_{H\in\Phi\circ\Psi} \bigwedge_{(s,t)\in H} d(s,t) \geq \alpha * \beta$, i.e. $\Phi \circ \Psi \in \Lambda^d_{\alpha*\beta}$.

The left-continuity is trivial. To show that $(X, \overline{\Lambda^d})$ is principal, let $\Phi_j \in \Lambda^d_\alpha$ for all $j \in J$. Then for all $j \in J$ and all $\epsilon \lhd \alpha$ there is $F^\epsilon_j \in \Phi_j$ such that for all $(x,y) \in F^\epsilon_j$, $d(x,y) \geq \epsilon$. Define $F = \bigcup_{j\in J} F^\epsilon_j \in \bigwedge_{j\in J} \Phi_j$. Then for all $(x,y) \in F$ we have $d(x,y) \geq \epsilon$ and hence $\bigvee_{F\in\bigwedge_{j\in J}\Phi_j} \bigwedge_{(x,y)\in F} d(x,y) \geq \epsilon$. This is true for all $\epsilon \lhd \alpha$ and by the complete distributivity of L then $\bigvee_{F\in\bigwedge_{j\in J}\Phi_j} \bigwedge_{(x,y)\in F} d(x,y) \geq \alpha$ and we obtain $\bigwedge_{j\in J} \Phi_j \in \Lambda^d_\alpha$. □

Proposition 4.3. *Let* $f : (X,d) \longrightarrow (X',d')$ *be an* $\mathsf{L\text{-}MET}$*-morphism. Then* $f : (X, \overline{\Lambda^d}) \longrightarrow (X', \overline{\Lambda^{d'}})$ *is uniformly continuous.*

Proof. We have, for $\Phi \in \Lambda^d_\alpha$, that

$$\bigvee_{G\in(f\times f)(\Phi)} \bigwedge_{(u,v)\in G} d'(u,v) \geq \bigvee_{F\in\Phi} \bigwedge_{(u,v)\in(f\times f)(F)} d'(u,v)$$
$$\geq \bigvee_{F\in\Phi} \bigwedge_{(x,y)\in F} d'(f(x),f(y)) \geq \bigvee_{F\in\Phi} \bigwedge_{(x,y)\in F} d(x,y) \ \geq \ \alpha$$

and therefore $(f\times f)(\Phi) \in \Lambda^{d'}_\alpha$. □

Hence we have a functor from $\mathsf{L\text{-}MET}$ into the category of left-continuous and principal L-uniform convergence tower spaces, $\mathsf{LCP\text{-}L\text{-}UCTS}$. This functor is injective on objects: If $d \neq d'$ then, without loss of generality, there are $x,y \in X$ such that $d(x,y) \not\leq d'(x,y)$. Then $[(x,y)] \in \Lambda^d_{d(x,y)}$ but $[(x,y)] \notin \Lambda^{d'}_{d(x,y)}$.

Let $(X, \overline{\Lambda}) \in |\mathsf{L\text{-}UCTS}|$. Define, for $x,y \in X$,

$$d^{\overline{\Lambda}}(x,y) = \bigvee_{[(x,y)]\in\Lambda_\alpha} \alpha.$$

Proposition 4.4. *Let* $(X, \overline{\Lambda}) \in |\mathsf{L\text{-}UCTS}|$. *Then* $(X, d^{\overline{\Lambda}}) \in |\mathsf{L\text{-}MET}|$.

Proof. From $[(x,x)] \in \Lambda_\top$ we obtain $d^{\overline{\Lambda}}(x,x) = \top$. The symmetry (LM2) follows from $[(x,y)]^{-1} = [(y,x)]$ and (LUC4). Transitivity, (LM3), follows from $[(x,y)] \circ [(y,z)] = [(x,z)]$ and (LUC5). □

Proposition 4.5. *Let* $f : (X, \overline{\Lambda}) \longrightarrow (X', \overline{\Lambda'})$ *be uniformly continuous. Then* $f : (X, d^{\overline{\Lambda}}) \longrightarrow (X', d^{\overline{\Lambda'}})$ *is an* $\mathsf{L\text{-}MET}$*-morphism.*

Proof. We have $d^{\overline{\Lambda'}}(f(x),f(y)) = \bigvee_{(f\times f)([(x,y)])\in\Lambda'_\alpha} \alpha \geq \bigvee_{[(x,y)]\in\Lambda_\alpha} \alpha = d^{\overline{\Lambda}}(x,y)$. □

Proposition 4.6. *Let* $(X,d) \in |\mathsf{L}\text{-}\mathsf{MET}|$. *Then* $d^{(\overline{\Lambda^d})} = d$.

Proof. We have $d^{(\overline{\Lambda^d})}(x,y) = \bigvee_{[(x,y)] \in \Lambda^d_\alpha} \alpha = \bigvee_{d(x,y) \geq \alpha} \alpha = d(x,y)$. $\square$

Proposition 4.7. *Let* $(X,\overline{\Lambda}) \in |\mathsf{LCP}\text{-}\mathsf{L}\text{-}\mathsf{UCTS}|$. *Then* $\Lambda_\alpha^{(d^{\overline{\Lambda}})} \subseteq \Lambda_\alpha$.

Proof. Let $\Phi \in \Lambda_\alpha^{(d^{\overline{\Lambda}})}$. Then $\bigvee_{F \in \Phi} \bigwedge_{(x,y) \in F} \bigvee_{[(x,y)] \in \Lambda_\beta} \beta \geq \alpha$. Let $\delta \lhd \alpha$. Then there is $F^\delta \in \Phi$ such that for all $(x,y) \in F$ there is $\beta \geq \delta$ such that $[(x,y)] \in \Lambda_\beta \subseteq \Lambda_\delta$, by (LUC3). As $(X,\overline{\Lambda})$ is principal, we get $\bigwedge_{(x,y) \in F^\delta} [(x,y)] = [F^\delta] \in \Lambda_\delta$ and hence, as $[F^\delta] \leq \Phi$ with (LUC2) we obtain $\Phi \in \Lambda_\delta$. This is true for all $\delta \lhd \alpha$ and from the left-continuity and the complete distributivity of L we finally get $\Phi \in \Lambda_\alpha$. $\square$

Theorem 4.8. $\mathsf{L}\text{-}\mathsf{MET}$ *can be coreflectively embedded into* $\mathsf{LCP}\text{-}\mathsf{L}\text{-}\mathsf{UCTS}$.

Remark 4.9 (Quasimetric case). If we leave away the symmetry requirement (LM2) in the definition of an L-metric space (X,d), then $(X,\overline{\Lambda^d})$ does not satisfy the symmetry axiom (LUC4) and conversely, if $(X,\overline{\Lambda})$ does not satisfy (LUC4), then $(X,d^{\overline{\Lambda}})$ does not satisfy (LM2). All other results of this section remain valid, so that we can say that the category of L-quasimetric spaces (where $d : X \times X \longrightarrow L$ satisfies (LM1) and (LM3)) can be coreflectively embedded into the category of L-*quasiuniform tower spaces* $(X,\overline{\Lambda})$ (where the axioms (LUC1)-(LUC7) without (LUC4) are satisfied).

Remark 4.10. In [2] for the case $\mathsf{L} = (\Delta^+, \leq, *)$ we embedded the category of probabilistic metric spaces into the category of probabilistic uniform convergence tower spaces in a different way. Following Tardiff [22], we define for an L-metric space (X,d), $\epsilon > 0$ and $\varphi \in \Delta^+$ the (φ,ϵ)-*entourage* by

$$N^{\varphi,\epsilon} = \{(x,y) \in X \times X \ : \ d(x,y)(u+\epsilon) + \epsilon \geq \varphi(u) \ \forall u \in [0, \frac{1}{\epsilon})\}.$$

and define the φ-*entourage filter*, $\mathcal{N}^\varphi$, as the filter generated by the sets $N^{\varphi,\epsilon}$, $\epsilon > 0$. If we define

$$\Phi \in \tilde{\Lambda}^d_\varphi \iff \Phi \geq \mathcal{N}^\varphi$$

then we obtain a left-continuous and principal L-uniform convergence tower space $(X,\overline{\tilde{\Lambda}^d})$. We will show that this L-unform convergence tower space coincides with the L-uniform convergence tower space $(X,\overline{\Lambda^d})$ and we need the following results from [22]. For $\varphi \in \Delta^+$ and $0 \leq \epsilon \leq 1$ we define $\varphi^\epsilon \in \Delta^+$ by

$$\varphi^\epsilon(u) = \begin{cases} 0 & \text{if} \quad u = 0 \\ (\varphi(u+\epsilon)+\epsilon) \wedge 1 & \text{if} \quad 0 < u \leq \frac{1}{\epsilon} \\ 1 & \text{if} \quad u > \frac{1}{\epsilon}. \end{cases}$$

Clearly then $\varphi \leq \varphi^\epsilon$ and Tardiff [22] shows that $(x,y) \in N^{\varphi,\epsilon}$ if and only if $d(x,y)^\epsilon \geq \varphi$ and $\varphi \geq \psi$ if and only if for all $\epsilon > 0$ we have $\varphi^\epsilon \geq \psi$. The last assertion implies that for $\varphi \in \Delta^+$ we have $\varphi = \bigwedge_{\epsilon > 0} \varphi^\epsilon$. We will need the following results [12].

- (∗) Let $\varphi_j \in \Delta^+$ for all $j \in J$ and let $0 \leq \epsilon \leq 1$. Then $(\bigvee_{j \in J} \varphi_j)^\epsilon = \bigvee_{j \in J} (\varphi_j^\epsilon)$ and $(\bigwedge_{j \in J} \varphi_j)^\epsilon = \bigwedge_{j \in J} (\varphi_j^\epsilon)$.
- (∗∗) (Cf. [16], Proposition 1.8.29) Let $\mathbb{U} \in \mathbb{U}(X)$ be an ultrafilter and $f : X \longrightarrow L$ be a mapping. Then $\bigvee_{U \in \mathbb{U}} \bigwedge_{y \in U} f(y) = \bigwedge_{U \in \mathbb{U}} \bigvee_{y \in U} f(y)$.

Let first $\Phi \geq \mathcal{N}^\varphi$. Then, for $\epsilon > 0$, we have

$$\bigvee_{F \in \Phi} \bigwedge_{(x,y) \in F} d(x,y)^\epsilon \geq \bigwedge_{(x,y) \in N^{\varphi,\epsilon}} d(x,y)^\epsilon \geq \varphi.$$

From $(*)$ we conclude

$$\varphi \leq \bigwedge_{\epsilon>0} \bigvee_{F\in\Phi} \bigwedge_{(x,y)\in F} d(x,y)^{\epsilon} = \bigwedge_{\epsilon>0} (\bigvee_{F\in\Phi} \bigwedge_{(x,y)\in F} d(x,y))^{\epsilon} = \bigvee_{F\in\Phi} \bigwedge_{(x,y)\in F} d(x,y)$$

and we have $\Phi \in \Lambda^d_\varphi$.

Let now $\Phi \in \Lambda^d_\varphi$ and $\mathbb{U} \geq \Phi$ be an ultrafilter and $\psi \lhd \varphi$. Then, by $(**)$,

$$\bigwedge_{U\in\mathbb{U}} \bigvee_{y\in U} d(x,y) = \bigvee_{U\in\mathbb{U}} \bigwedge_{y\in U} d(x,y) \geq \varphi \rhd \psi,$$

and hence, for all $U \in \mathbb{U}$ there is $(x^\psi, y^\psi) \in U$ such that $d(x^\psi, y^\psi)^\epsilon \geq d(x^\psi, y^\psi) \geq \psi$. But this means $(x^\psi, y^\psi) \in N^{\psi,\epsilon}$ for all $\epsilon > 0$. So we conclude that for all $U \in \mathbb{U}$ we have $N^{\psi,\epsilon} \cap U \neq \emptyset$, and hence, $\mathbb{U}$ being an ultrafilter, $N^{\psi,\epsilon} \in \mathbb{U}$ for all $\epsilon > 0$. Therefore $\mathbb{N}^\psi \leq \mathbb{U}$ for all ultrafilters $\mathbb{U} \geq \Phi$ and we conclude $\mathbb{N}^\psi \leq \Phi$. Therefore $\Phi \in \tilde{\Lambda}^d_\psi$ for all $\psi \lhd \varphi$ and from the left-continuity then also $\Phi \in \tilde{\Lambda}^d_\varphi$.

5. A subcategory of L-UCTS isomorphic to L-MET

We introduce the following axiom (LUM) for an L-uniform convergence tower space.

(LUM) $\forall \mathbb{U} \in \mathbb{U}(X \times X), \alpha \in L$:
$\mathbb{U} \in \Lambda_\alpha \iff \forall U \in \mathbb{U}, \beta \lhd \alpha \exists (x,y) \in U$ s.t. $[(x,y)] \in \Lambda_\beta$.

A similar axiom in the realm of probabilistic convergence spaces was first introduced in [4].

Proposition 5.1. *Let $(X,d) \in |\mathsf{L}\text{-}\mathsf{MET}|$. Then $(X, \overline{\Lambda^d})$ satisfies (LUM).*

Proof. Let first $\mathbb{U} \in \Lambda^d_\alpha$ and let $\beta \lhd \alpha$ and $U \in \mathbb{U}$. Then there is $U^\beta \in \mathbb{U}$ such that for all $(x,y) \in U^\beta$ we have $d(x,y) \geq \beta$. We choose $(x,y) \in U \cap U^\beta$. Then $\bigvee_{F\in[(x,y)]} \bigwedge_{(u,v)\in F} d(u,v) \geq \bigwedge_{(u,v)\in U\cap U^\beta} d(u,v) \geq \beta$ and this means $[(x,y)] \in \Lambda^d_\beta$.

Let now, for all $U \in \mathbb{U}, \beta \lhd \alpha$ exist $(x,y) \in U$ such that $[(x,y)] \in \Lambda^d_\beta$. We define $\mathbb{N} = \bigwedge_{\Phi\in\Lambda^d_\beta} \Phi \in \Lambda^d_\beta$ and let $N \in \mathbb{N}$. By the condition, then for $U \in \mathbb{U}$ there is $(x,y) \in U$ such that $[(x,y)] \in \Lambda^d_\beta$ and hence $\mathbb{N} \leq [(x,y)]$ and we conclude $(x,y) \in N \cap U$. Therefore $\mathbb{N} \vee \mathbb{U}$ exists and $\mathbb{U}$ being ultra, this implies $\mathbb{U} \geq \mathbb{N}$. Hence $\mathbb{U} \in \Lambda^d_\beta$. This is true for all $\beta \lhd \alpha$ and by left-continuity this implies $\mathbb{U} \in \Lambda^d_\alpha$. □

We call a space $(X, \overline{\Lambda}) \in |\mathsf{L}\text{-}\mathsf{UCTS}|$ which is left-continuous, principal and satisfies the axiom (LUM) an *L-metric uniform convergence tower space* and denote the category of these spaces by L-MUCTS.

Proposition 5.2. *Let $(X, \overline{\Lambda}) \in |\mathsf{L}\text{-}\mathsf{MUCTS}|$. Then $\Lambda^{(d^{\overline{\Lambda}})}_\alpha = \Lambda_\alpha$.*

Proof. We need to show that $\Lambda_\alpha \subseteq \Lambda^{(d^{\overline{\Lambda}})}_\alpha$. Let $\mathbb{U} \in \Lambda_\alpha$ be an ultrafilter. By the property (LUM), then for $\beta \lhd \alpha$ with $N_\beta = \{(x,y) \ : \ [(x,y)] \in \Lambda_\beta\}$ we have $U \cap N_\beta \neq \emptyset$ and hence $N_\beta \in \mathbb{U}$. Furthermore, for $[(x,y)] \in \Lambda_\beta$, we have $d^{\overline{\Lambda}}(x,y) \geq \beta$ and hence $\bigvee_{U\in\mathbb{U}} \bigwedge_{(u,v)\in U} d^{\overline{\Lambda}}(u,v) \geq \bigwedge_{(u,v)\in N_\beta} d^{\overline{\Lambda}}(u,v) \geq \beta$, i.e. $\mathbb{U} \in \Lambda^{(d^{\overline{\Lambda}})}_\beta$. This is true for all $\beta \lhd \alpha$ and by the left-continuity then also $\mathbb{U} \in \Lambda^{(d^{\overline{\Lambda}})}_\alpha$. As $\Lambda^{(d^{\overline{\Lambda}})}$ is principal and $\Phi = \bigwedge_{\mathbb{U}\geq\Phi \text{ ultra}} \mathbb{U}$, the claim follows. □

Theorem 5.3. *The categories* L-MUCTS *and* L-MET *are isomorphic.*

Remark 5.4. For $\mathsf{L} = (\Delta^+, \leq, *)$ with a continuous triangle function $*$ (see [21]), we introduced a different axiom that ensured the isomorphy of L-MUCTS and L-MET. A space $(X, \overline{\Lambda}) \in |\mathsf{L\text{-}UCTS}|$ satisfies the axiom (PUM) if for all ultrafilters $\Phi \in \mathbb{F}(X \times X)$ and all $\varphi \in \Delta^+$

$$\Phi \in \Lambda_\varphi \iff$$
$$\forall \phi \in \Phi, \epsilon > 0 \exists (x,y) \in \phi \text{ s.t. } \bigvee_{\psi : [(x,y)] \in \Lambda_\psi} \psi(s+\epsilon) + \epsilon \geq \varphi(s) \forall s \in [0, \frac{1}{\epsilon}).$$

With the notation of this paper and of Remark 4.10 then $d^{(\overline{\tilde{\Lambda}^d})} = d$ and if $(X, \overline{\Lambda})$ is left-continuous and principal, then $\tilde{\Lambda}_\varphi^{(d^{\overline{\Lambda}})}(\Phi) = \Lambda_\varphi(\Phi)$ for all $\varphi \in \Delta^+$ and all $\Phi \in \mathbb{F}(X \times X)$. It follows from this, that for an L-uniform convergence tower space $(X, \overline{\Lambda})$ that is left-continuous and principal, the axioms (PUM) and (LUM) are equivalent. In fact, if (PUM) is true, then $\tilde{\Lambda}_\varphi^{(d^{\overline{\Lambda}})} = \Lambda_\varphi$ and hence, using Remark 4.10, then also $\Lambda_\varphi^{(d^{\overline{\Lambda}})} = \Lambda_\varphi$ and as $d^{\overline{\Lambda}}$ is an L-metric on X we know that $(X, \overline{\Lambda}) = (X, \overline{\Lambda^{(d^{\overline{\Lambda}})}})$ satisfies (LUM). A similar argument shows that (LUM) implies (PUM).

6. The L-uniform tower of an L-metric space

Let (X, d) be an L-metric space and let $\epsilon \in L$, $\epsilon \prec \top$. We define

$$U_\epsilon = \{(x,y) \in X \times X \ : \ d(x,y) \succ \epsilon\}.$$

Then, for $\alpha \in L$, the collection of all U_ϵ with $\epsilon \prec \alpha$ is a filter basis. As $(x,x) \in U_\epsilon$ for all $\epsilon \prec \top$, none of the U_ϵ is empty. Moreover we have $U_{\epsilon \vee \delta} \subseteq U_\epsilon \cap U_\delta$ and for $\epsilon, \delta \prec \alpha$, also $\epsilon \vee \delta \prec \alpha$. We define $\mathcal{U}_\alpha^d = [\{U_\epsilon \ : \ \epsilon \prec \alpha\}]$ as the filter on $X \times X$ generated by this filter basis.

Lemma 6.1. *Let $(X, d) \in |\mathsf{L\text{-}MET}|$ and let $\alpha \in L$. Then $\mathcal{U}_\alpha^d = \bigwedge_{\Phi \in \Lambda_\alpha^d} \Phi$.*

Proof. We have $\bigvee_{F \in \mathcal{U}_\alpha^d} \bigwedge_{(x,y) \in F} d(x,y) \geq \bigvee_{\epsilon \prec \alpha} \bigvee_{(x,y) \in U_\epsilon} d(x,y) \geq \bigvee_{\epsilon \prec \alpha} \epsilon = \alpha$. Hence $\mathcal{U}_\alpha^d \in \Lambda_\alpha^d$ and therefore $\mathcal{U}_\alpha^d \geq \bigwedge_{\Phi \in \Lambda_\alpha^d} \Phi$. For the converse, let $U \in \mathcal{U}_\alpha^d$. Then there is $\epsilon \prec \alpha$ such that $U_\epsilon \subseteq U$. We have, for $\Phi \in \Lambda_\alpha^d$, that $\bigvee_{F \in \Phi} \bigwedge_{(x,y) \in F} d(x,y) \geq \alpha \succ \epsilon$. Noting that the set $\{\bigwedge_{(x,y) \in F} d(x,y) \ : \ F \in \Phi\}$ is directed, there is $F^\epsilon \in \Phi$ such that $F^\epsilon \subseteq U_\epsilon \subseteq U$ and hence $U \in \Phi$. This shows $\mathcal{U}_\alpha^d \leq \bigwedge_{\Phi \in \Lambda_\alpha^d} \Phi$. □

Remark 6.2. For the case $\mathsf{L} = (\Delta^+, \leq, *)$ we conclude with Remark 4.10 that the φ-entourage filter $\mathcal{N}^\varphi$ from [2] and $\mathcal{U}_\varphi^d$ coincide.

Proposition 6.3. *Let $(X, d) \in |\mathsf{L\text{-}MET}|$. The system of filters $(\mathcal{U}_\alpha^d)_{\alpha \in L}$ then has the following properties.*

(LUT1) $\mathcal{U}_\alpha^d \leq [\Delta]$ *with* $[\Delta] = \bigwedge_{x \in X} [(x,x)]$;
(LUT2) $\mathcal{U}_\alpha^d \leq (\mathcal{U}_\alpha^d)^{-1}$;
(LUT3) $\mathcal{U}_{\alpha * \beta}^d \leq \mathcal{U}_\alpha^d \circ \mathcal{U}_\beta^d$;
(LUT4) $\mathcal{U}_\alpha^d \leq \mathcal{U}_\beta^d$ *whenever* $\alpha \leq \beta$;
(LUT5) $\mathcal{U}_\bot^d = \bigwedge \mathbb{F}(X \times X)$;
(LUT6) $\mathcal{U}_{\bigvee A}^d \leq \bigvee_{\alpha \in A} \mathcal{U}_\alpha^d$ *whenever* $\emptyset \neq A \subseteq L$.

Proof. We use Lemma 6.1. (LUT1) follows as $[(x,x)] \in \Lambda_\alpha^d$ and hence $\mathcal{U}_\alpha^d \leq [(x,x)]$ for all $x \in X$. (LUT2) We have, with $\mathcal{U}_\alpha^d \in \Lambda_\alpha^d$ that also $(\mathcal{U}_\alpha^d)^{-1} \in \Lambda_\alpha^d$ and hence $\mathcal{U}_\alpha^d \leq (\mathcal{U}_\alpha^d)^{-1}$. For (LUT3) we note that $\mathcal{U}_\alpha \circ \mathcal{U}_\beta$ exists as both filters are $\leq [\Delta]$. From (LUC6) then $\mathcal{U}_\alpha^d \circ \mathcal{U}_\beta^d \in \Lambda_{\alpha * \beta}$ which implies $\mathcal{U}_{\alpha * \beta}^d \leq \mathcal{U}_\alpha^d \circ \mathcal{U}_\beta^d$. (LUT4) We have, for $\alpha \leq \beta$, that $\mathcal{U}_\beta^d \in \Lambda_\beta^d \subseteq \Lambda_\alpha^d$ and hence $\mathcal{U}_\alpha^d \leq \mathcal{U}_\beta^d$. (LUT5) follows with Lemma 6.1 from (LUC7). For (LUT6) finally, we remark that $\bigvee_{\alpha \in A} \mathcal{U}_\alpha^d$ exists by (LUT1) and clearly

$\bigvee_{\alpha\in A}\mathcal{U}^d_\alpha \in \Lambda^d_\beta$ for all $\beta\in A$. By left-continuity of Λ^d then $\bigvee_{\alpha\in A}\mathcal{U}^d_\alpha \in \Lambda^d_{\bigvee A}$ and this means $\mathcal{U}^d_{\bigvee A} \le \bigvee_{\alpha\in A}\mathcal{U}^d_\alpha$. □

We note that $\mathcal{U}^d_\top$ is a classical uniformity and that we can think of the collection of the $\mathcal{U}^d_\alpha$ as "approximations" of $\mathcal{U}^d_\top$. For $\mathsf{L}=(\{0,1\},\le,\wedge)$ we obtain classical uniformities [3], for $\mathsf{L}=([0,1],\le,*)$ with a left-continuous t-norm, we obtain probabilistic uniformities in the definition of Florescu [8] and for Lawvere's quantale, $\mathsf{L}=([0,\infty],\ge,+)$, a left-continuous L-uniform tower is an approach uniformity [15]. For $\mathsf{L}=(\Delta^+,\le,*)$, an L-uniform tower is a probabilistic uniform space in [2].

For an L-metric space (X,d) we defined in [12] the L-convergence structure $\overline{c^d}$ by

$$x\in c^d_\alpha(\mathbb{F}) \iff \bigvee_{F\in\mathbb{F}}\bigwedge_{y\in F} d(x,y)\ge\alpha.$$

Lemma 6.4. *Let $(X,d)\in|\mathsf{L}\text{-}\mathsf{MET}|$. Then $c^d_\alpha(\mathbb{F})=c^{(\overline{\Lambda^d})}_\alpha(\mathbb{F})$ for all $\mathbb{F}\in\mathbb{F}(X)$ and all $\alpha\in L$.*

Proof. Let first $x\in c^d_\alpha(\mathbb{F})$. Then we have $\alpha\le\bigvee_{F\in\mathbb{F}}\bigwedge_{y\in F}d(x,y)=\bigvee_{F\in\mathbb{F}}\bigwedge_{y\in F,z\in\{x\}}d(z,y)\le\bigvee_{G\in[x]\times\mathbb{F}}\bigwedge_{(z,y)\in G}d(z,y)$ and hence $[x]\times\mathbb{F}\in\Lambda^d_\alpha$, which means $x\in c^{(\overline{\Lambda^d})}_\alpha(\mathbb{F})$. Conversely, let $x\in c^{(\overline{\Lambda^d})}_\alpha(\mathbb{F})$. Then $[x]\times\mathbb{F}\in\Lambda^d_\alpha$ and hence $\alpha\le\bigvee_{G\in[x]\times\mathbb{F}}\bigwedge_{(u,v)\in G}d(u,v)\le\bigvee_{F\in\mathbb{F}}\bigwedge_{v\in F}d(x,v)$. Therefore, $x\in c^d_\alpha(\mathbb{F})$. □

We showed in [12] that $(X,\overline{c^d})$ is principal, i.e. that we have $x\in c^d_\alpha(\mathbb{F})$ if and only if $\mathbb{F}\ge\mathbb{U}^d_{\alpha,x}$ with the *α-neighbourhood filter of x*, $\mathbb{U}^d_{\alpha,x}=\bigwedge_{x\in c^d_\alpha(\mathbb{F})}\mathbb{F}$.

Lemma 6.5. *Let $(X,d)\in|\mathsf{L}\text{-}\mathsf{MET}|$. Then $\mathbb{U}^d_{\alpha,x}$ is generated by the filter basis $\mathbb{B}^d_{\alpha,x}=\{U_{\epsilon,x}\,:\,\epsilon\prec\alpha\}$, where $U_{\epsilon,x}=\{y\in X\,:\,d(x,y)\succ\alpha\}$. Furthermore, we have $\mathbb{U}^d_{\alpha,x}=\mathcal{U}^d_\alpha(x)$.*

Proof. Clearly $x\in U_{\epsilon,x}$ for all $\epsilon\prec\top$ and hence $U_{\epsilon,x}\ne\emptyset$. Moreover, for $y\in U_{\epsilon\vee\delta,x}$ we have $d(x,y)\succ\epsilon\vee\delta\ge\epsilon,\delta$ and hence also $y\in U_{\epsilon,x}\cap U_{\delta,x}$. As $\epsilon,\delta\succ\alpha$ implies $\epsilon\vee\delta\succ\alpha$ this shows that $\mathbb{B}^d_{\alpha,x}$ is a filter basis. The proof of $\mathbb{U}^d_{\alpha,x}=[\mathbb{B}^d_{\alpha,x}]$ is similar to the proof of Lemma 6.1 and not shown. The last part follows from the observation that $U_\epsilon(x)=\{y\in X\,:\,(x,y)\in U_\epsilon\}=\{y\in X\,:\,d(x,y)\succ\epsilon\}=U_{\epsilon,x}$. □

7. The case of L-partial metric spaces

An *L-partial metric* $p:X\times X\longrightarrow L$ satisfies the axioms, for all $x,y,z\in X$,

(LPM1) $p(x,y)\le p(x,x)$,
(LPM2) $p(x,y)=p(y,x)$ and
(LPM3) $p(x,y)*(p(y,y)\to p(y,z))\le p(x,z)$.

The pair (X,p) is then called an *L-partial metric space* [13]. We define morphisms between L-partial metric spaces in the same way as for L-metric spaces and denote the resulting category by $\mathsf{L}\text{-}\mathsf{PMET}$.

We will, in this section, discuss axioms for suitable L-uniform convergence towers. To this end, we again define, for a mapping $p:X\times X\longrightarrow L$ and $\alpha\in L$, a family $\overline{\Lambda^p}=(\Lambda^p_\alpha)_{\alpha\in L}$ with $\Lambda^p_\alpha\subseteq\mathbb{F}(X\times X)$, by

$$\Phi\in\Lambda^p_\alpha \iff \bigvee_{F\in\Phi}\bigwedge_{(x,y)\in F}p(x,y)\ge\alpha.$$

Conversely, for $\overline{\Lambda} = (\Lambda_\alpha)_{\alpha \in L}$ with $\Lambda_\alpha \subseteq \mathbb{F}(X \times X)$, we define a mapping $p^{\overline{\Lambda}} : X \times X \longrightarrow L$ by

$$p^{\overline{\Lambda}}(x,y) = \bigvee_{[(x,y)] \in \Lambda_\alpha} \alpha.$$

As in Lemma 4.1, we can show that $[(x,y)] \in \Lambda^p_\alpha$ if and only if $p(x,y) \geq \alpha$.

Lemma 7.1. (1) *Let $p : X \times X \longrightarrow L$ satisfy (LPM1). Then $[(x,x)] \in \Lambda^p_\alpha$ whenever $[(x,y)] \in \Lambda^p_\alpha$.*

(2) *Let $\Lambda_\alpha \subseteq \mathbb{F}(X \times X)$ satisfy $[(x,x)] \in \Lambda_\alpha$ whenever $[(x,y)] \in \Lambda_\alpha$. Then $p^{\overline{\Lambda}}(x,y) \leq p^{\overline{\Lambda}}(x,x)$.*

Proof. (1) If $[(x,y)] \in \Lambda^p_\alpha$, then $\alpha \leq p(x,y) \leq p(x,x)$ and hence $[(x,x)] \in \Lambda^p_\alpha$.

(2)We have $p^{\overline{\Lambda}}(x,y) = \bigvee_{[(x,y)] \in \Lambda_\alpha} \alpha \leq \bigvee_{[(x,x)] \in \Lambda_\alpha} \alpha = p^{\overline{\Lambda}}(x,x)$. □

These results show that for a suitable definition of L-uniform convergence towers for an L-partial metric space we have to replace the axiom (LUC1) by the following weaker axiom (WLUC1).

(WLUC1) $[(x,x)] \in \Lambda_\alpha$ whenever $[(x,y)] \in \Lambda_\alpha$ for all $x, y \in X$, $\alpha \in L$.

We will next consider a suitable concept of transitivity. To this end, we define, for $y \in X$, $\mathbb{E}(y) = \mathbb{E}^{\overline{\Lambda}}(y) = \bigvee_{[(y,y)] \in \Lambda_\alpha} \alpha$. We note that for $(X, \overline{\Lambda}) \in |\mathsf{L\text{-}UCTS}|$, we have $\mathbb{E}(y) = \top$ for all $y \in X$. Furthermore, for an L-partial metric space (X,p) we have $\mathbb{E}^{\overline{\Lambda^p}}(y) = p(y,y)$ for all $y \in X$.

We say that $(X, \overline{\Lambda})$ is *strongly point transitive* if $\overline{\Lambda}$ satisfies the following axiom

(SPT) $[(x,z)] \in \Lambda_{\alpha * (\mathbb{E}(y) \to \beta)}$ whenever $[(x,y)] \in \Lambda_\alpha$ and $[(y,z)] \in \Lambda_\beta$.

Proposition 7.2. *Let (X,p) be an L-partial metric space. Then $\overline{\Lambda^p}$ is strongly point transitive.*

Proof. Let $[(x,y)] \in \Lambda^p$ and $[(y,z)] \in \Lambda^p_\beta$. Then $p(x,y) \geq \alpha$ and $p(y,z) \geq \beta$. Moreover $\mathbb{E}(y) = p(y,y)$. Hence $\alpha * (\mathbb{E}(y) \to \beta) \leq p(x,y) * (p(y,y) \to p(y,z)) \leq p(x,z)$ and hence $[(x,z)] \in \Lambda^p_{\alpha * (\mathbb{E}(y) \to \beta)}$. □

In the sequel, we will need the following axiom (DM2) for the quantale L.

(DM2) $\alpha \to (\bigvee_{j \in J} \beta_j) = \bigvee_{j \in J} (\alpha \to \beta_j)$ for all $\alpha, \beta_j \in L, J \neq \emptyset$.

Typical examples for quantales satisfying (DM2) are Lawvere's quantale or complete MV-algebras.

Proposition 7.3. *Let the quantale L satisfy (DM2) and let $\overline{\Lambda}$ satisfy the axiom (SPT). Then $p^{\overline{\Lambda}}$ satisfies (LPM3).*

Proof. We have, using $p^{\overline{\Lambda}}(y,y) = \bigvee_{[(y,y)] \in \Lambda_\gamma} \gamma = \mathbb{E}(y)$,

$$\begin{aligned} p^{\overline{\Lambda}}(x,y) * (p^{\overline{\Lambda}}(y,y) \to p^{\overline{\Lambda}}(y,z)) &= \bigvee_{[(x,y)] \in \Lambda_\alpha} \alpha * (\mathbb{E}(y) \to \bigvee_{[(y,z)] \in \Lambda_\beta} \beta) \\ &\overset{(DM2)}{=} \bigvee_{[(x,y)] \in \Lambda_\alpha} \bigvee_{[(y,z)] \in \Lambda_\beta} \alpha * (\mathbb{E}(y) \to \beta) \\ &\leq \bigvee_{[(x,z)] \in \Lambda_{\alpha * (\mathbb{E}(y) \to \beta)}} \alpha * (\mathbb{E}(y) \to \beta) \ \leq \ p^{\overline{\Lambda}}(x,z). \end{aligned}$$

□

Hence, if we define the category L-PUCTS with objects the L-*partial uniform convergence tower spaces* $(X, \overline{\Lambda})$ where $\overline{\Lambda} = (\Lambda_\alpha)_{\alpha \in L}$ and $\Lambda_\alpha \subseteq \mathbb{F}(X \times X)$ satisfies the axioms (WLUC1), (LUC2) - (LUC6) and (SPT), and morphisms as defined for L-UCTS, then we deduce the following results.

Theorem 7.4. *Let the quantale* L *satisfy (DM2).*

(1) *The category* L-PMET *can be embedded into* L-PUCTS *as a coreflective subcategory.*
(2) *The category* L-PMET *is isomorphic to the category* L-PMUCTS *with objects the* L-*partial uniform convergence tower spaces that are left-continuous, principal and satisfy the axiom (LUM).*

Remark 7.5 (**Transitivity).** (1) In [12] we introduced a transivity axiom for L-convergence spaces $(X, \overline{c})$. We call $(X, \overline{c})$ *transitive* if $x \in c_{\alpha * \beta}([z])$ whenever $x \in c_\alpha([y])$ and $y \in c_\beta([z])$. Similarly, we call $(X, \overline{c})$ *strongly transitive* if $x \in c_{\alpha*(\mathbb{E}^{\overline{c}}(y) \to) \beta}([z])$ whenever $x \in c_\alpha([y])$ and $y \in c_\beta([z])$, where $\mathbb{E}^{\overline{c}}(y) = \bigvee_{y \in c_\gamma([y])}$. If we call an L-uniform convergence space *point transitive* if $[(x,z)] \in \Lambda_{\alpha * \beta}$ whenever $[(x,y)] \in \Lambda_\alpha$ and $[(y,z)] \in \Lambda_\beta$, then $(X, \overline{c^{\overline{\Lambda}}})$ is transitive if $(X, \overline{\Lambda})$ is point transitive and, noting that $\mathbb{E}^{\overline{c^{\overline{\Lambda}}}}(y) = \mathbb{E}^{\overline{\Lambda}}(y)$, we also see that $(X, \overline{c^{\overline{\Lambda}}})$ is strongly transitive if $(X, \overline{\Lambda})$ is strongly point transitive.

(2) From $[(x,y)] \circ [(y,z)] = [(x,z)]$ it is clear that the axiom (LUC6) implies the point transitivity for $(X, \overline{\Lambda}) \in |\text{L-UCTS}|$. For this reason we did not introduce such an axiom before. It is, however, not clear at present, if a suitable transitivity axiom à la (LUC6) is available for L-partial uniform convergence tower spaces.

8. Conclusion

We showed that the category of quantale-valued metric spaces can be coreflectively embedded in the category of left-continuous and principal quantale-valued uniform convergence tower spaces. We furthermore identified a subcategory, which is isomorphic to the category of quantale-valued metric spaces. These quantale-valued uniform convergence towers may lend themselves to the study of completions of quantale-valued metric spaces. We shall look into this aspect in our future work.

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