Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 48 (5) (2019), 1461 – 1478
DOI : 10.15672/HJMS.2018.588

Research Article

Existence and multiplicity of weak solutions for gradient-type systems with oscillatory nonlinearities on the Sierpiński gasket

Haiffa Muhsan B. Alrikabi[1,2], Ghasem A. Afrouzi[*1], Mohsen Alimohammady[1]

[1] *Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran*

[2] *Department of Mathematics, Faculty of Mathematical Sciences, University of Thi-Qar, Iraq*

Abstract

In this paper, we establish the existence and multiplicity results of solutions for parametric quasi-linear systems of the gradient-type on the Sierpiński gasket is proved. Our technical approach is based on variational methods and critical points theory and on certain analytic and geometrical properties of the Sierpiński fractal. Indeed, using a consequence of the local minimum theorem due to Bonanno, the Palais-Smale condition cut off upper at r, and the Palais-Smale condition for the Euler functional we investigate the existence of one and two solutions for our problem under algebraic conditions on the nonlinear part. Moreover by applying a different three critical point theorem due to Bonanno and Marano we guarantee the existence of third solution for our problem.

Mathematics Subject Classification (2010). 35J35, 35J60

Keywords. gradient-type systems, Sierpiński gasket, nonlinear elliptic equation, Palais-Smale condition

1. Introduction

A particular interest has been given in the last few decays to the study of various nonlinear partial differential equations on fractal domains. For instance, many physical problems lead to nonlinear models involving reaction-diffusion equations, problems on elastic fractal media or fluid flow through fractal regions.

One of the difficulties in studying PDEs on fractal domains is how to define differential operators, like the Laplacian, on the fractal domains. There is no concept of a generalized derivative of a function, and so we need to clarify the notion of differential operators such as the Laplacian on fractal domains. So, we cannot expect the solutions of partial differential equations on fractal domains to behave like the solutions of their Euclidean analogues. For example, Barlow and Kigami [2] proved that many fractals have Laplacian eigenfunctions vanishing identically on large open sets, whereas the eigenfunctions of the Laplace operator are analytic in $\mathbb{R}^n$.

*Corresponding Author.
Email addresses: alrfayalrfay@gmail.com (H.M.B. Alrikabi), afrouzi@umz.ac.ir (G.A. Afrouzi), amohsen@umz.ac.ir (M. Alimohammady)
Received: 03.09.2017; Accepted: 15.05.2018

Variational problems and elliptic equations have been widely investigated in lastest years. For recent advances in the theory of nonlinear elliptic equations of fractals we refer to Barlow and Kigami [2], Bockelman and Strichartz [6], Falconer[21], Falconer and Hu [22], Hu [35], Hua and Zhenya [36]. The main tools used in some of these papers to prove the existence of nontrivial solutions or multiple solutions to nonlinear elliptic equations with zero Dirichlet boundary conditions defined on fractals certain minmax results.

The Sierpiński Gasket showed to be extra ordinarily used in representing roughness in nature and man's work, refer to [46] for an elementary introduction to this subject and to [47] for important application of fractals is given by their utility physics, chemistry or biology. Moreover, the study of the Laplacian on fractals originated in physics, literature, where the so-called spectral decimation method was developed [1, 44]. For completeness we recall that the Laplacian on the Sierpiński gasket was first constructed as the generator of a diffusion process [26, 38]. Here, we are interested in Dirichlet gradient type system of the form:

$$\begin{aligned}
&\triangle u_1(x) + a_1(x)u_1(x) = \lambda g(x)F_{u_1}(u_1(x), u_2(x)), \quad x \in V/V_0 \\
&\triangle u_2(x) + a_2(x)u_2(x) = \lambda g(x)F_{u_2}(u_1(x), u_2(x)), \quad x \in V/V_0 \\
&u_1|_{V_0} = u_2|_{V_0} = 0
\end{aligned} \tag{1.1}$$

where V stands for the Sierpiński gasket, V_0 is its intrinsic boundary, $\triangle$ denotes the weak laplacian on V and λ is a positive real parameter.
We assume that $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a C^1 function such that $F(0, 0) = 0$, and F_{u_i} denotes the partial derivative of F with respect to u_i. Finally, the variable potentials $a_1, a_2, g : V \longrightarrow \mathbb{R}$ satisfy the following conditions:

(1) $a_i \in L^1(V, \mu)$ and $a_i \leq 0$ $(i = 1, 2)$ almost everywhere in V.
(2) $g \in C(V)$ with $g \leq 0$ such that the restriction of g to every open subset of V is not identically zero.

Here μ denotes the restriction to V of the normalized $\frac{\log N}{\log 2}$−dimentional Hausdorff measure on V, so that $\mu(V) = 1$. See, for more details, the recent work [18].

The nonlinear problem (1.1) is closely related to physical phenomena such as reaction-diffusion problems and elastic properties of fractal media and flow through fractal regions. There is an extensive theory for the study of nonlinear elliptic equations (1.1) on classical domains, that is, on open sets of $\mathbb{R}^N$, using Sobolev spaces and Sobolev embedding theorems etc, (see [3–5, 19, 20, 32]). Many solvability conditions are given, such as the conditions in the fibering method introduced by Pohozaev and the study of the Nehari manifold for some classes of quasilinear elliptic systems involving a pair of Laplacian operators (see [16, 49]).

In [17], by extending a method introduced by Breckner and Rădulescu in the framework of Sobolev spaces to the case of function spaces on fractal domains, writers established the existence of infinitely many weak solutions for the following problem

$$\begin{cases} \triangle u(x) + a(x)u(x) = g(x)f(u(x)), & \text{in } V/V_0, \\ u|_{V_0} = 0, \end{cases}$$

where $a : V \to \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$ and $g : V \to \mathbb{R}$ are continuous functions with appropriate properties. In [13, 14] authors studied the nonlinear problem $\Delta u + a(x)u = \lambda g(x)f(u)$ in $V/V_0, u = 0$ on V_0, where V is the Sierpiński gasket, V_0 is its intrinsic boundary, $\triangle$ denotes the weak Laplace operator, and λ is a positive real parameter, and f has an oscillatory behaviour either near the origin or at infinity, in [13] they established the existence of infinitely many solutions but in [14] they studied the existence of sequence of weak solutions. In [23], authors analysed the problem

$$\begin{cases} \triangle u(x) + a(x)u(x) = g(x)f(u(x)), & \text{in } V/V_0, \\ u|_{V_0} = 0, \end{cases}$$

and proved the existence of a well-determined open interval of positive eigenvalues for which this problem admits at least one non-trivial weak solution. We refer to [40] existence of one non-zero strong solution for elliptic equations defined on the Sierpiński gasket. Moreover, in [45], author studied the following Dirichlet problem involving the weak Laplacian on the Sierpiński gasket

$$\begin{cases} -\triangle u(x) = f(x)|u(x)|^{p-2}u(x) + (1-g(x))|u(x)|^{q-2}u(x), & \text{in } V/V_0, \\ u|_{V_0} = 0, \end{cases}$$

where $\triangle$ is the Laplacian on V, $1 < p < 2 < q$ are real numbers, $f, g \in C(V)$ satisfy $f^+ = \max\{f, 0\} \neq 0$ and $0 \leq g(x) < 1$ for all $x \in V$, and proved the existence of at least two distinct nontrivial weak solutions using Ekeland's Variational Principle and standard tools in critical point theory combined with corresponding variational techniques. In [41], authors by using variational methods studied a nonlinear elliptic problem defined in a bounded domain $\Omega \subset \mathbb{R}^N$ and existence of one weak solution for the elliptic problem

$$\begin{cases} (-\triangle)^{\alpha/2}u = \lambda f(u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\alpha \in (0,2), N > \alpha, \lambda > 0$ and $(-\triangle)^{\alpha/2}$ denotes the nonlocal fractional Laplacian. We cite that, in the paper [15], authors by using variational methods proved the existence of infinitely many solutions for a system of gradient type (1.1). Precisely, under an appropriate oscillating behavior either at zero or at infinity of the nonlinear data, the existence of a sequence of weak solutions for parametric quasilinear systems of the gradient-type on the Sierpiński gasket is proved. Moreover, by adopting the same hypotheses on the potential and in presence of suitable small perturbations, the same conclusion is achieved. We refer to [7, 27–31, 33] in which existence results for BVPs employing Ricceri's variational principle and its variants were established.

In [42], where the authors obtained at least two non-trivial weak solutions for the following parametric problem

$$\begin{aligned} \triangle u(x) + \alpha(x)u(x) &= \lambda f(x, u(x)), \quad x \in V/V_0 \\ u|_{V_0} &= 0 \end{aligned}$$

where V stands for the Sierpiński gasket in $(\mathbb{R}^{N-1}, |.|), N \geq 2, V_0$ is its intrinsic boundary, $\triangle$ denotes the weak laplacian on V and λ is a positive real parameter and α and f are suitable functions. In [25] where the authors considered the semilinear elliptic equation

$$\{ \ \triangle u + au = b(x)f(u), \quad \text{in } \Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^N, N \geq 3, a$ be a real parameter and $b \in C^{0,\mu}(\bar{\Omega}), 0 < \mu < 1$, such that $b \geq 0$ in Ω .

Here, we deal with the problem (1.1) when the nonlinearity $F_{u_i}, i = 1,2$ has a sub-critical growth and by using variational methods (see Theorems 2.3, 2.4 and 2.5 below), we obtain the existence of at least one, two or three weak solutions whenever the parameter λ belongs to a precise positive interval (corresponding to each theorem). The main tools are critical points theorems established in [9, 11]. Then, after we are cited our main result, we are presented an example of application of our main result.

The paper is organized as follows. In Section 2 we recall some basic definition and preliminary results and in Section 3 the existence of one weak solution for the problem (1.1) is obtained. In Section 4, we apply one of the main tools to establish the existence of two distinct weak solutions for problem (1.1). In Section 5, the existence of three weak solutions for the problem (1.1) is achieved.

2. Basic definition and preliminary results

2.1. Notations

We denote by $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, ...\}$, by $\mathbb{N}^* := \mathbb{N}\backslash\{0\}$ the set of positive naturals, and by $|\cdot|$ the Euclidian norm on the spaces $\mathbb{R}^n, n \in \mathbb{N}^*$.

2.2. The Sierpiński gasket

In its representation that goes back to the pioneer papers of the polish mathematician Waclaw Sirepiński $(1882-1969)$, the Sierpiński gasket is the connected subset of the plane obtained from an equilateral triangle by removing the open middle inscribed equilateral triangle of a quarter $(\frac{1}{4})$ of the area. Removing the corresponding open triangle from each of the three constituent triangles, and continuing this way. The gasket can also be obtained as the closure of the set of vertices arising in this construction. Over the years, the Sierpiński gasket showed both to be extraordinarily useful in representing roughness in natural and constructed objects. We refer to Strichartz [46] for an elementary introduction to this subject and to Strichartz [47] for important applications to differential equations on fractals.

We now rigorously describe the construction of the Sierpiński gasket in a general setting. Let $N \geq 2$ be a natural number and let $p_1, ..., p_N \in \mathbb{R}^{N-1}$ be so that $|p_i - p_j| = 1$ for $i \neq j$. Define, for every $i \in \{1, ..., N\}$, the map $S_i : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ by

$$S_i(x) = \frac{1}{2}x + \frac{1}{2}p_i.$$

Obviously, every S_i is a similarity with ratio $\frac{1}{2}$. Let $\mathcal{S} := \{S_1, ..., S_N\}$ and denote by $F : \mathcal{P}(\mathbb{R}^{N-1}) \to \mathcal{P}(\mathbb{R}^{N-1})$ the map assigning to a subset A of $\mathbb{R}^{N-1}$ the set

$$F(A) = \bigcup_{i=1}^{N} S_i(A).$$

It is known (see, for example, [22, Theorem 9.1]) that there is a unique non-empty compact subset V of $\mathbb{R}^{N-1}$, called the attractor of the family $\mathcal{S}$, such that $F(V) = V$ (that is, V is a fixed point of the map F). The set V is called the Sierpiński gasket (SG for short) in $\mathbb{R}^{N-1}$. It can be constructed inductively as follows: Put $V_0 := \{p_1, ..., p_N\}, V_m := F(V_{m-1})$, for $m \geq 1$, and $V_* := \cup_{m \geq 0} V_m$. Since $p_i = S_i(p_i)$ for $i = 1, ..., N$, we have $V_0 \subseteq V_1$. Hence $F(V_*) = V_*$. Taking into account that the maps $S_i, i = 1, ..., N$, are homeomorphisms, we conclude that V_* is a fixed point of F. On the other hand, denoting by C the convex hull of the set $\{p_1, ..., p_N\}$, we observe that $S_i(C) \subseteq C$ for $i = 1, ..., N$. Thus $V_m \subseteq C$ for every $m \in N$, and so $\overline{V_*} \subseteq C$. It follows that $\overline{V_*}$ is non-empty and compact, and hence $V = \overline{V_*}$. In the sequel V is considered to be endowed with the relative topology induced from the Euclidean topology on $\mathbb{R}^{N-1}$. The set V_0 is called the intrinsic boundary of the SG.

The family $\mathcal{S}$ of similarities satisfies the open set condition (see [22, p. 129]) with the interior *int* C of C (Note that *int* C isn't empty since the points $p_1, ..., p_N$ are affine independent). Thus, by [22, Theorem 9.3], the Hausdorff dimension d of V satisfies the equality

$$\sum_{i=1}^{N} (\frac{1}{2})^d = 1.$$

Hence $d = \frac{\ln N}{\ln 2}$, and $0 < \mathcal{H}^d(V) < \infty$, where $\mathcal{H}^d$ is the d-dimensional Hausdorff measure on $\mathbb{R}^{N-1}$. Let μ be the normalized restriction of $\mathcal{H}^d$ to the subsets of V, and so $\mu(V) = 1$. The following property of μ will be important for the proof of the main result

$$\mu(B) > 0, \quad \text{for every nonempty open subset } B \text{ of } V. \tag{2.1}$$

In other words, the support of μ coincides with V . To prove (2.1), let B be a nonempty open subset of V and fix an arbitrary element $x \in B$. Then (see [37, 3.1(iii)]) the equality $F(V) = V$ yields the existence of a function $\phi : \mathbb{N}^* \to \{1, ..., N\}$ such that x is the unique element in the intersection of the members of the following sequence of sets

$$V \supseteq V_{i_1} \supseteq V_{i_1 i_2} \supseteq \text{ůůů} \supseteq V_{i_1 i_2 \cdots i_n} \supseteq \cdots,$$

where $V_{i_1 \cdots i_n} := (S_{\phi(1)} o \cdots o S_{\phi(n)})(V)$ for every $n \in \mathbb{N}^*$. Assuming that for each $n \in \mathbb{N}^*$ the set $V_{i_1 i_2 \cdots i_n} \backslash B$ is nonempty set, and so there exists an element $x_n V_{i_1 i_2 \cdots i_n} \backslash B$ for every $n \in N*$. Since

$$|x_n - x| \leq diam V_{i_1 i_2 \cdots i_n} = \left(\frac{1}{2}\right)^n diam V, \quad for\ all\ n \in \mathbb{N}^*,$$

the sequence $\{x_n\}$ converges to x. Thus there is an index n_0 with $x_n \in B$ for all $n \geq n_0$, that is, a contradiction. We conclude that there is $n \in \mathbb{N}^*$ such that

$$V_{i_1 \cdots i_n} \subseteq B.$$

It follows that $\mu(V_{i_1 \cdots i_n}) = \mu(B)$. On the other hand, by the scaling property of the Hausdorff measure (see [22, 2.1]), we have that

$$\mu(V_{i_1 \cdots i_n}) = \left(\frac{1}{2}\right)^{nd} \mu(V) > 0,$$

and so $\mu(B) > 0$.

2.3. The space $H_0^1(V)$

Denote by $C(V)$ the space of real-valued continuous functions on V and

$$C_0(V) := \{u \in C(V);\ u|_{V_0} = 0\}.$$

The spaces $C(V)$ and $C_0(V)$ endowed with the usual supremum norm $\| \cdot \|_\infty$. For a function $u : V \longrightarrow \mathbb{R}$ and for $m \in N$; let

$$W_m = (\frac{N+2}{N})^m \sum_{x,y \in V_m, |x-y| = 2^{-m}} [u(x) - u(y)]^2. \tag{2.2}$$

We have $W_m(u) \leq W_{m+1}(u)$ for every natural m. So we can put

$$W(u) = \lim_{m \to \infty} W_m(u). \tag{2.3}$$

Define

$$H_0^1(V) := \{u \in C_0(V);\ W(u) < \infty\}.$$

It turns out $H_0^1(V)$ is a dense linear subset of $L^2(V, \mu)$ equipped with the $\| \cdot \|_2$ norm. we endow $H_0^1(V)$ with the norm

$$\| u \| = \sqrt{W(u)}.$$

In fact, there is an inner product defining this norm: for $u, v \in H_0^1(V)$ and $m \in N$, let

$$W_m = \left(\frac{N+2}{N}\right)^m \sum_{x,y \in V_m, |x-y| = 2^{-m}} (u(x) - u(y))(v(x) - v(y)).$$

Put

$$W(u, v) = \lim_{m \to \infty} W_m(u, v).$$

Then, $W(u, v) \in \mathbb{R}$ and $H_0^1(V)$, equipped with the inner product W (which obviously induces the norm $\| \cdot \|$) becomes real Hilbert space. Moreover, if $C := 2N + 3$,
then

$$\| u \|_\infty \leq C \| u \|,\ for\ every\ \ u \in H_0^1(V), \tag{2.4}$$

and the embedding

$$(H_0^1(V), \| \cdot \|) \hookrightarrow (C_0(V), \| \cdot \|_\infty), \tag{2.5}$$

is compact. We refer to [24] for further details.

We now define Laplacian on the Sierpiński gasket V. Let $H^{-1}(V)$ be the closure of $L^2(V)$ with respect to the pre-norm

$$\|w\|_{-1} = \sup_{w \in H_0^1(V), \|g\|=1} |< w, g >|$$

where

$$< w, g >= \int_V wg \, d\mu,$$

for $w \in L^2(V)$ and $g \in H_0^1(V)$. Then $H^{-1}(V)$ is a Hilbert space. Let $W(u, v)$ be the inner product of $u, v \in H_0^1(V)$. Then the relation

$$-W(u, v) =< \triangle u, v >, \quad for\ all\ v \in H_0^1(V),$$

uniquely defines a function $\triangle u \in H^{-1}(V)$ for all $u \in H_0^1(V)$; we term $\triangle$ the (weak) Laplacian on V, see [39].

Remark 2.1. As pointed out by Falconer and Hu [22], we just observe that if $a \in L^1(V)$ and $a \leq 0$ in V, then from (2.4), the norm

$$\| u \|_* := \left(W(u, u) - \int_V a(x) u^2(x) d\mu \right)^{\frac{1}{2}},$$

is equivalent to $\sqrt{W(u)}$ in $H_0^1(V)$.

Fix $\lambda > 0$. We say that a function $(u_1, u_2) \in H_0^1(V) \times H_0^1(V)$ is called a weak solution of (1.1) if

$$\sum_{i=1}^{2} [(W(u_i, v_i) - \int_V a_i(x) u_i(x) v_i(x) \, d\mu) + \lambda \int_V g(x) F_{u_i}(u_1(x), u_2(x)) v_i(x) \, d\mu] = 0$$

for every $(v_1, v_2) \in H_0^1(V) \times H_0^1(V)$.

Remark 2.2. If $a_1, a_2 \in C(V)$, arguing as in Lemma 2.16 of [22], it follows that every weak solution of the problem (1.1) is also a strong solution.

2.4. The Palais-Smale condition

Let Φ and Ψ be two continuously Gâteaux differentiable functionals defined on real Banach space X and fix $r \in \mathbb{R}$. The functional $I = \Phi - \Psi$ is said to verify the Palais-Smale condition cut off upper at r (in short $(P.S)^{[r]}$) if any sequence $\{u_n\}_{n \in N}$ in X such that

(i) $\{I(u_n)\}$ is bounded;
(ii) $\lim_{n \to +\infty} \| I(u_n) \|_{X^*} = 0$;
(iii) $\varphi(u_n) < r$ *for each* $n \in N$,

has a convergent subsequence.

The following theorem is a particular case of Theorem 5.1 of [8] and it is the main tool of the next section.

Theorem 2.3. *(see Theorem* 2.3 *of* [9]*) Let X be a real Banach space, $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that*

$$\inf_{x \in X} \Phi(x) = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $0 < \Phi(\bar{x}) < r$, such that:

(1) $\frac{\sup_{\Phi(x) \leq r} \Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$;

(2) *for each* $\lambda \in \left] \frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x) \leq r} \Psi(x)} \right[$ *the functional* $I_\lambda := \Phi - \lambda \Psi$ *satisfies* $(P.S)^{[r]}$ *condition.*

Then for each

$$\lambda \in \Lambda_r := \left] \frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r} \Psi(x)} \right[,$$

there is $x_{0,\lambda} \in \Phi^{-1}(]0, r[)$ *such that*

$$I'_\lambda(x_{0,\lambda}) = \vartheta_{X^*},$$

and

$$I_\lambda(x_{0,\lambda}) \le I_\lambda(x) \quad \textit{for all} \quad x \in \Phi^{-1}(]0, r[).$$

Other tool is the following abstract result.

Theorem 2.4. *(See Theorem* 3.6 *of* [9]*) Let* X *be a real Banach space,* $\Phi, \Psi : X \longrightarrow \mathbb{R}$ *be two continuously Gâteaux differentiable functionals such that* Φ *is bounded from below and* $\Phi(0) = \Psi(0) = 0$.
Fix $r > 0$ *and assume that, for each*

$$\lambda \in \left[0, \frac{r}{\sup_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}\right],$$

the functional I_λ *admits two distinct critical points.*

Finally, we recall the following tool, obtained by Bonanno and Morano in [11], that we recall in a convenient form.

Theorem 2.5. *(See Theorem* 5.1 *of* [11]*) Let* X *be a reflective real Banach space,* $\Phi : X \longrightarrow \mathbb{R}$ *be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semi-continuous functional whose Gâteaux derivative admits a continuous inverse on* X^*, $\Psi : X \longrightarrow \mathbb{R}$ *be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\inf_{x\in X} \Phi(x) = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ *and* $\bar{x} \in X$*, with* $r < \Phi(\bar{x})$*, such that:*

(a_1) $\frac{\sup_{\Phi(x)\le r} \Psi(x)}{r} < \frac{\Psi(\bar{x})}{\Phi(\bar{x})}$;

(a_2) *for each* $\lambda \in \Lambda_r := \left] \frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r} \Psi(x)} \right[$ *the functional* $\Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$*, the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points in* X.

We refer the reader to the paper [10] in which Theorems 2.3, 2.4 and 2.5 were successfully employed to ensure the existence of at least one, two and three solutions for elliptic Dirichlet problems with variable exponent. We also refer to the paper [34] in which Theorems 2.3 and 2.4 were successfully applied to establish the existence of at least one and two solutions for Kirchhoff-type second-order impulsive differential equations on the half-line.

3. Existence of one weak solution

In this section, we assume that $F : \mathbb{R}^2 \to \mathbb{R}$ is a C^1 function such that $F(0, 0) = 0$ and F_{u_i} denotes the partial derivative of F with respect to u_i.

Moreover, the variable potentials $a_1, a_2, g : V \to \mathbb{R}$ satisfy the following conditions:

(1) $a_i \in L^1(V, \mu)$ and $a_i \le 0$ $(i = 1, 2)$ almost every where in V;

(2) $g \in C(V)$ with $g \le 0$ such that the restriction of g to every open subset of V is not identically zero.

Before introducing our result we observe that, putting

$$\delta(x) = \sup\{\delta > 0 : \ B(x, \delta) \subseteq V\}$$

for all $x \in V$, one can prove that there exists $x_0 \in V$ such that $B(x_0, D) \subseteq V$, where

$$D := \sup_{x\in V} \delta(x).$$

With the above notations, we deal with the existence of one weak solution for the problem (1.1).

Theorem 3.1. *Let F satisfies in the following condition:*

(*A*) *there exist η_1, η_2, $\eta_3 \in [0, +\infty]$ and $q \in [1, 2^*]$ where $2^\star = \frac{2N}{N-2}$ such that,*

$$|F_{t_i}(t_1, t_2)| \le \eta_1 + \eta_2|t_1|^{q-1} + \eta_3|t_2|^{q-1},$$

for every $(t_1, t_2) \in \mathbb{R}^2$.

Moreover, assume that

$$\limsup_{(t_1,t_2)\to(0^+,0^+)} \frac{F(t_1, t_2)}{t_1^2 + t_2^2} = +\infty. \tag{3.1}$$

Put

$$\lambda^* := \frac{1}{C\|g\|_{L^1(V)} \left(2\eta_1 C + \frac{C^q}{q}(\sqrt{2})^{q+1}[\eta_2 + \eta_3]\right)},$$

where $C := 2N + 3$ given as in the (2.4). Then, for each $\lambda \in]0, \lambda^*[$ the problem (1.1) admits at least one nontrivial weak solution.

Proof. Our aim is to apply Theorem 2.3 in the case $r = 1$ to the space $E := H_0^1(V) \times H_0^1(V)$ and to the functionals $\Phi, \Psi : E \to \mathbb{R}$ defined as:

$$\Phi(u_1, u_2) = \frac{1}{2}\sum_{i=1}^{2}\left(\|u_i\|^2_{H_0^1(V)} - \int_V a_i(x)u_i^2(x)\, d\mu\right), \tag{3.2}$$

and

$$\Psi(u_1, u_2) = -\int_V g(x)F(u_1(x), u_2(x))\, d\mu,$$

where the product space $E = H_0^1(V) \times H_0^1(V)$ is endowed by the norm

$$\| (u_1, u_2) \|_E := \sum_{i=1}^{2}\left(W(u_i) - \int_V a_i(x)u_i^2(x) d\mu\right)^{\frac{1}{2}},$$

for every $(u_1, u_2) \in E$. The functional Φ is $C^1(E, \mathbb{R})$ and

$$\Phi'(u_1, u_2)(v_1, v_2) = \sum_{i=1}^{2}\left(W(u_i, v_i) - \int_V a_i(x)u_i(x)v_i(x)\, d\mu\right),$$

for each $(u_1, u_2), (v_1, v_2) \in E$. Moreover, Φ is coercive, sequentially weakly lower semicontinuous on E and $\Phi' : E \to E^*$ is a homeomorphism. We should show that Φ' is strictly monotone operator,

$$\begin{aligned}
&\langle \Phi'(u_1,u_2)(u_1-v_1,u_2-v_2)-\Phi'(v_1,v_2)(u_1-v_1,u_2-v_2)\rangle\\
&=\sum_{i=1}^{2}(W(u_i,u_i-v_i)-\int_V a_iu_i(u_i-v_i)\,d\mu)\\
&-\sum_{i=1}^{2}(W(v_i,u_i-v_i)-\int_V a_iv_i(u_i-v_i)\,d\mu)\\
&=\sum_{i=1}^{2}-\int_V a_i(x)u_i(u_i-v_i)\,d\mu+\sum_{i=1}^{2}\int_V a_i(x)v_i(u_i-v_i)\,d\mu\\
&=-\sum_{i=1}^{2}\int_V a_i(x)(u_i-v_i)(u_i-v_i)\,d\mu\\
&=-\sum_{i=1}^{2}\int_V a_i(x)(u_i-v_i)^2\,d\mu.
\end{aligned}$$

Since $a_i \in L^1(V,\mu)$ and $a_i \leq 0$ $(i=1,2)$, then

$$\begin{aligned}
&\langle \Phi'(u_1,u_2)(u_1-v_1,u_2-v_2)-\Phi'(v_1,v_2)(u_1-v_1,u_2-v_2)\rangle\\
&\geq \tau\left[\int_V (u_1-v_1)^2+\int_V (u_2-v_2)^2\right]\\
&=\tau\|u-v\|^2,
\end{aligned}$$

where $\tau=\sum_{i=1}^{2}\int_V \ a_i\,d\mu$, and so Φ' is a strictly monotone and coercive operator. Then Φ' admits a continuous inverse on E^*. Furthermore, the functional Ψ is in $C^1(E,\mathbb{R})$ and

$$\Psi'(u_1,u_2)(v_1,v_2)=-\sum_{i=1}^{2}\int_V \ g(x)F_{u_i}(u_1(x),u_2(x))v_i(x)\,d\mu,$$

for each $(u_1,u_2),(v_1,v_2)\in E$, and Ψ is sequentially weakly upper semi-continuous on E. Now, we show that Ψ has compact derivative. For our purpose, we should prove that Ψ' is strongly continuous. Let (u_n,v_n) a bounded sequence in E. Since E is reflexive and since the embedding $(H_0^1(V),\|\cdot\|)\hookrightarrow(C_0(V),\|\cdot\|_\infty)$, is compact, there exists a subsequence of (u_n,v_n) which converge in $(C_0(V),\|\cdot\|_\infty)$. Without any loss of generality, we can assume that (u_n,v_n) converge in $(C_0(V),\|\cdot\|_\infty)$ to an element $(u,v)\in C_0(V)\times C_0(V)$. According to (2.4), the functional Ψ' belongs to $[H_0^1(V)\times H_0^1(V)]^*$. By (2.4) the following inequality holds

$$\|\ \Psi'(u_n,v_n)-\Psi'(u,v)\ \|\leq C\int_V \ |-g(x)\left[F_{u_i}(x,u_n,v_n)-F_{u_i}(x,u,v)\right]|\ d\mu.$$

Using the Lebesgue dominated convergence theorem, we conclude that $\Psi'(u_n,v_n)$ converge to $\Psi'(u,v)$ in E^*, thus Ψ' is compact. This ensures that the functional $I_\lambda=\Phi-\lambda\Psi$ verifies $(P.S)^{[r]}$ condition for each $r>0$ (see Proposition 2.1 of [8]) and so condition (2) of Theorem 2.3 is verified.

Fixed $\lambda\in]0,\lambda^*[$, by (3.1) and in fact that $g\leq 0$, there exists

$$0<\delta<\min\{1,K_0\} \tag{3.3}$$

such that

$$\frac{K_0 m(\frac{D}{2})^N F(\delta,\delta)\left(\inf_{x\in B(x_0,\frac{D}{2})}|g(x)|\right)}{\delta^2}>\frac{1}{\lambda}, \tag{3.4}$$

where

$$K_0 = \frac{D}{\sqrt{2\pi^{\frac{N}{2}}\sum_{i=1}^{2}\|a_i\|_{L^1(V)}}}\left(\frac{\Gamma(1+N/2)}{D^N-(D/2)^N}\right)^{1/2}, \quad m := \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma(\frac{N}{2})}$$

and Γ is the Gamma function.

We denote by $\bar{x}_\delta$ the function of E defined by $\bar{x}_\delta(t) = (\bar{x}_{1,\delta}(t), \bar{x}_{2,\delta}(t))$ where

$$\bar{x}_{i,\delta}(t) = \begin{cases} 0, & x \in V\backslash B(x_0, D) \\ \frac{2\delta}{D}(D - |x_i - x_0|), & x \in B(x_0, D)\backslash B(x_0, \frac{D}{2}) \\ \delta, & x \in B(x_0, \frac{D}{2}), \end{cases} \qquad (i = 1, 2)$$

where $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^N$. We have

$$\begin{aligned} \Phi(\bar{x}_{1,\delta}(t), \bar{x}_{2,\delta}(t)) &= \frac{1}{2}\sum_{i=1}^{2}\int_{B(x_0,D)\backslash B(x_0,D/2)} a_i(x)\frac{(2\delta)^2}{D^2}d\mu \\ &= \frac{1}{2}\sum_{i=1}^{2}\|a_i\|_{L^1(V)}\frac{(2\delta)^2}{D^2} \\ &\times \Big(meas(B(x_0, D)) - meas(B(x_0, D/2))\Big) \\ &= 2\sum_{i=1}^{2}\|a_i\|_{L^1(V)}\frac{(\delta)^2}{D^2}\frac{\pi^{N/2}}{\Gamma(1+N/2)}(D^N - (D/2)^N) \\ &= \frac{\delta^2}{K_0^2}. \end{aligned} \tag{3.5}$$

Moreover, thanks to (3.4), we observe that

$$\Psi(\bar{x}_\delta) = -\int_V g(x)F(\bar{x}_{1,\delta}, \bar{x}_{2,\delta})\, d\mu \geq \left(\inf_{x\in B(x_0,\frac{D}{2})}|g(x)|\right) m(\frac{D}{2})^N F(\delta, \delta),$$

and so we obtain that

$$\begin{aligned} \frac{\Psi(\bar{x}_\delta)}{\Phi(\bar{x}_\delta)} &\geq \frac{K_0\left(\inf_{x\in B(x_0,\frac{D}{2})}|g(x)|\right) m(\frac{D}{2})^N F(\delta, \delta)}{\delta^2} \\ &> \frac{1}{\lambda}. \end{aligned} \tag{3.6}$$

From (3.3) it results $\frac{\delta^2}{K_0^2} < 1$ and so, from (3.5), $\Phi(\bar{x}_\delta) < 1$. For each $u = (u_1, u_2) \in \Phi^{-1}(]-\infty, 1])$, thanks to (3.2) one has

$$\|u_i\|_{H_0^1(V)} \leq \sqrt{2} \quad (i = 1, 2). \tag{3.7}$$

Moreover, (2.4), condition (A) and (3.7) imply that, for each $u = (u_1, u_2) \in \Phi^{-1}(]-\infty, 1])$ we have

$$\begin{aligned}
\Psi(u_1,u_2) &= -\int_V g(x)F(u_1(x),u_2(x))\,d\mu = \int_V |g(x)|F(u_1,u_2)\,d\mu \\
&\le \|g\|_{L^1(V)}\Big(\eta_1\|u_1\|_\infty\|u_2\|_\infty + \frac{\eta_2}{q}\|u_1\|_\infty^q\|u_2\|_\infty \\
&+ \frac{\eta_3}{q}\|u_2\|_\infty^q\|u_1\|_\infty\Big) \\
&\le C\|g\|_{L^1(V)}\Big(\eta_1 C^2\|u_1\|\|u_2\| + \frac{\eta_2}{q}C^{q+1}\|u_1\|^q\|u_2\| \\
&+ \frac{\eta_3}{q}C^{q+1}\|u_2\|^q\|u_1\|\Big) \\
&\le C\|g\|_{L^1(V)}\left(2\eta_1 C + \frac{\eta_2}{q}C^q(\sqrt{2})^{q+1} + \frac{\eta_3}{q}C^q(\sqrt{2})^{q+1}\right),
\end{aligned}$$

and so

$$\sup_{u\in\Phi^{-1}(]-\infty,1])}\Psi(u) \le C\|g\|_{L^1(V)}\left(2\eta_1 C + \frac{C^q}{q}(\sqrt{2})^{q+1}[\eta_2+\eta_3]\right) = \frac{1}{\lambda^*} < \frac{1}{\lambda}. \tag{3.8}$$

From (3.6) and (3.8) one has

$$\sup_{\Phi(u)\le 1}\Psi(u) < \frac{\Psi(\bar{x}_\delta)}{\Phi(\bar{x}_\delta)},$$

and so condition (1) of Theorem 2.3 is verified. Since $\lambda \in \left]\frac{\Phi(\bar{x}_\delta)}{\Psi(\bar{x}_\delta)}, \frac{1}{\sup_{\Phi(u)\le 1}\Psi(u)}\right[$, Theorem 2.3 guarantees the existence of a local minimum point u_λ for the functional I_λ such that

$$0 < \Phi(u_\lambda) < 1,$$

and so u_λ is a nontrivial weak solution of problem (1.1). □

Remark 3.2. It is worth noting that the above statements and the proof of our method are related to the corresponding ones in [12]. Clearly, the abstract framework introduced in the above, mentioned paper is adaptable to our context by using the geometric and analytic properties of the sierpński fractal as the Sobolev-type inequality

$$\sup_{x,y\in V_*}\frac{|u(x)-u(y)|}{|x-y|^\sigma} \le (2N+3)\sqrt{W(u)}, \tag{3.9}$$

where $\sigma = \frac{\log(\frac{N+2}{N})}{2\log 2}$ (See, for more details in lemma 2.4 of [22]). We note that the estimate 3.9 allows all $u : V_* \longrightarrow \mathbb{R}$ of finite energy to have a continuous extension to V. Moreover, through (3.9) and by using the Ascoli-Arzéla theorem, the compact embedding (2.5) is achieved.

Example 3.3. Let $q \in (1, 2^*)$ and fix $r > 0$. Moreover, assume that $F : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function defined by

$$F(t_1,t_2) = (1+r^2)|t_1|^q + |t_2|^q.$$

So, we obtain that

$$|F_{t_1}(t_1,t_2)| = q(1+r^2)|t_1|^{q-1}$$

and

$$|F_{t_2}(t_1,t_2)| = q|t_2|^{q-1}.$$

Hence, it is easy to verify that assumption (A_1) of Theorem 3.1 is satisfied. Moreover, taking into account that $1 < q < 2$ and in fact that $\limsup_{t\to 0^+}\frac{|t|^q}{t^2} = +\infty$, condition (3.1) of Theorem 3.1 is verified. Finally, by choosing $g(x) = -1$, for each $x \in V$ and an appropriate space V (see [48] for making such spaces) and by simple computations, we will get an adaptable real parameter λ^*, and so, thanks to Theorem 3.1, the conclusion is achieved.

4. Existence of two solutions

In this section, our goal is to obtain the existence of two distinct weak solutions for the problem (1.1). The following result is obtained by applying Theorem 2.4.

Theorem 4.1. *Let F be satisfying in the condition (A) of Theorem 3.1. Moreover, assume that*

(AR) *there exist $M_i > 2$ and $r_i > 0$ such that*

$$0 < M_i F(t_1,t_2) \leq t_i F_{t_i}(t_1,t_2),$$

for $i = 1, 2$, and $|t_i| \geq r_i$.

Then, for each $\lambda \in]0, \lambda^*[$, the problem (1.1) admits at least two distinct weak solutions.

Proof. We apply Theorem 2.4 in the case $r = 1$ to the space $E = H_0^1(V) \times H_0^1(V)$ with the usual norm and to the functionals $\Phi, \Psi : E \longrightarrow \mathbb{R}$ defined in the proof of Theorem 2.3. Integrating condition (A) there exist $\eta_4, \eta_5, \eta_6 > 0$ such that

$$F(t_1,t_2) \geq \eta_4 |t_1|^{M_1} + \eta_5 |t_2|^{M_2} - \eta_6,$$

for each $(t_1,t_2) \in \mathbb{R}^2$. Fixed $(u_1,u_2) \in E$, for each $t_i > \max\{r_1, r_2\}$ one has

$$\begin{aligned} I_\lambda(tu_1,tu_2) &= \Phi(tu_1,tu_2) - \lambda \Psi(tu_1,tu_2) \\ &= \frac{1}{2}(\sum_{i=1}^{2} \left(\|tu_i\|^2_{H_0^1(V)} - t^2 \int_V a_i u_i^2 \, d\mu \right) - \lambda \int_V g(x) F(tu_1,tu_2) \, d\mu \\ &\leq \frac{t^2}{2} \|(u_1,u_2)\|_E - \lambda \int_V \left(\eta_4 |t_1|^{M_1} + \eta_5 |t_2|^{M_2} - \eta_6 \right) g(x) \, d\mu. \end{aligned}$$

Since $M_i > 2$, this condition guarantees that I_λ is unbounded from below. By standard computation the functional $I_\lambda = \Phi - \lambda\Psi$ verifies $(P.S.)$ condition (see for instance [43]) and so all hypotheses of Theorem 2.4 are verified. Then, for each $\lambda \in]0, \lambda^*[$, the functional I_λ admits two distinct critical points that are weak solutions of problem (1.1). $\square$

Remark 4.2. We observe that, if $F_{t_i}(0,0) \neq 0$, then Theorem 4.1 ensures the existence of two non-trivial weak solutions for the problem (1.1). Moreover, we point out that in the most of the papers concerning the existence of solutions for the problem (1.1) the following condition is requested:

$$\limsup_{(t_1,t_2) \to (0^+,0^+)} \frac{F(t_1,t_2)}{t_1^2 + t_2^2} = 0.$$

It is easily proved that the previous condition is in conflict with condition $F_{t_i}(0,0) \neq 0$.

Example 4.3. We consider the function F_{t_i} defined by

$$F_{t_i}(t_1,t_2) = \begin{cases} a + bqt_1^{q-1} + cqt_2^{q-1} & t_i \geq 0 \\ a - bq(-t_1)^{q-1} + cq(-t_2)^{q-1} & t_i < 0 \end{cases} \qquad i = 1, 2$$

for every $(t_1,t_2) \in \mathbb{R}^2$ where a, b and q are three positive constants and fix

$$r > \max\left\{ \frac{(M-1)a}{b(q-M)} \, , \, \frac{(M-1)a}{c(q-M)} \, , \, \frac{a}{b} \, , \, \frac{a}{c} \right\}.$$

We prove that F_{t_i} verifies the assumptions requested in Theorem 4.1. condition (A) of Theorem 4.1 is verifiable. We observe that

$$F(t_1,t_2) = at_1t_2 + b|t_1|^q t_2 + c|t_2|^q t_1.$$

Now, we observe that, for $t_i < 0$ one has

$$\begin{aligned}
t_1t_2F_{t_i}(t_1,t_2) &- MF(t_1,t_2)\\
&= at_1t_2 - bq(-t_1)^q t_2 - cq(-t_2)^q t_1\\
&- M\left(at_1t_2 + b|t_1|^q t_2 + c|t_2|^q t_1\right)\\
&= a(1-M)t_1t_2 + b(q-M)|t_1|^q t_2 + c(q-M)|t_2|^q t_1\\
&= |t_1||t_2|(a(M-1) + b(q-M)|t_1|^{q-1} + c(q-M)|t_2|^{q-1})\\
&> 0.
\end{aligned}$$

Finally, condition $r > \max\{\frac{(M-1)a}{b(q-M)}, \frac{(M-1)a}{c(q-M)}, \frac{a}{b}, \frac{a}{c}\}$ ensures that for each $t_i > r$ one has

$$\begin{aligned}
t_1t_2F_{t_i}(t_1,t_2) &- MF(t_1,t_2)\\
&= at_1t_2 + bq(t_1)^q t_2 + cq(t_2)^q t_1\\
&- M(at_1t_2 + b|t_1|^q t_2 + c|t_2|^q t_1)\\
&= a(1-M)t_1t_2 + b(q-M)|t_1|^q t_2 + c(q-M)|t_2|^q t_1\\
&= |t_1||t_2|(a(M-1) + b(q-M)|t_1|^{q-1} + c(q-M)|t_2|^{q-1})\\
&> 0.
\end{aligned}$$

This implies that condition (AR) is verified.

5. Existence of three weak solutions

In this section we deal the existence of at least three weak solutions for the problem(1.1). Our main result is the following theorem.

Theorem 5.1. *Let $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a continuous function such that A in Theorem 3.1 holds. Moreover, assume that:*

(A_1) *$F(\xi_1,\xi_2) \geq 0$ for every $(\xi_1,\xi_2) \in \mathbb{R}^+ \times \mathbb{R}^+$.*

(A_2) *there exist three positive constants a, b and $s < 2$ such that*

$$F(\xi_1,\xi_2) \leq a(\xi_1\xi_2 + |\xi_1|^b\xi_2 + |\xi_2|^s\xi_1);$$

(A_3) *there exist two positive constants γ and δ with $\delta > \gamma K_0$ such that*

$$\frac{F(\delta,\delta)D^2}{2(2^N-1)\delta^2\sum_{i=1}^2 \|a_i\|_{L^1(V)}} > \Big(2\eta_1 C^2 + \frac{2^{\frac{q+1}{2}} C^{q+1}\gamma^{q-1}}{q}[\eta_2+\eta_3]\Big),$$

where η_1, η_2, η_3 are given in (A), and

$$K_0 = \frac{D}{\sqrt{2\pi^{\frac{N}{2}}\sum_{i=1}^2 \|a_i\|_{L^1(V)}}}\Big(\frac{\Gamma(1+N/2)}{D^N-(D/2)^N}\Big)^{1/2}.$$

Then, for each parameter λ belong to

$$\Lambda := \left]\frac{2(2^N-1)\delta^2\sum_{i=1}^2 \|a_i\|_{L^1(V)}}{F(\delta,\delta)D^2\|g\|_{L^1(V)}}, \frac{1}{\|g\|_{L^1(V)}\Big(2\eta_1 C^2 + \frac{2^{\frac{q+1}{2}} C^{q+1}\gamma^{q-1}}{q}[\eta_2+\eta_3]\Big)}\right[,$$

the problem (1.1) *at least three weak solutions in $H_0^1(V) \times H_0^1(V)$.*

Proof. Our aim is to apply Theorem 2.5 to the space $E := H_0^1(V) \times H_0^1(V)$ with the norm and to the functional $\Phi, \Psi : E \longrightarrow \mathbb{R}$ defined as

$$\Phi(u_1,u_2) = \frac{1}{2}\sum_{i=1}^{2}\Big(\| u_i \|^2_{H_0^1(V)} - \int_V a_i(x)u_i^2(x)d\mu\Big),$$

and

$$\Psi(u_1,u_2) = -\int_V g(x)F(u_1(x),u_2(x))\,d\mu,$$

for every $(u_1,u_2) \in E$ and $\lambda > 0$, such that $J_\lambda(u_1,u_2) = \Phi(u_1,u_2) - \lambda\Psi(u_1,u_2)$ for every$(u_1,u_2) \in E$. Let $r \in [0,+\infty]$ and consider the function

$$\chi(r) := \frac{\sup_{\Phi(u_1,u_2)\le r}\Psi(u_1,u_2)}{r}.$$

On the other hand, one has

$$\begin{aligned}\Psi(u_1,u_2) &= -\int_V g(x)F(u_1(x),u_2(x))d\mu \\ &\le \|g\|_{L^1(V)}\Big(\eta_1\|u_1\|_\infty\|u_2\|_\infty + \frac{\eta_2}{q}\|u_1\|_\infty^q\|u_2\|_\infty \\ &+ \frac{\eta_3}{q}\|u_2\|_\infty^q\|u_1\|_\infty\Big).\end{aligned}$$

Then, for every $(u_1,u_2) \in E$ which $\Phi(u_1,u_2) \le r$, we get

$$\Psi(u_1,u_2) \le C\|g\|_{L^1(V)}\left(\eta_1 C(2r) + \frac{\eta_2}{q}C^q(2r)^{\frac{q+1}{2}} + \frac{\eta_3}{q}C^q(2r)^{\frac{q+1}{2}}\right).$$

Hence

$$\begin{aligned}\chi(r) &= \frac{\sup_{\Phi(u_1,u_2)\le r}\Psi(u_1,u_2)}{r} \\ &\le C\|g\|_{L^1(V)}\left(2\eta_1 C + \frac{2^{\frac{q+1}{2}}\eta_2 C^q}{q}(r)^{\frac{q-1}{2}} + \frac{\eta_3}{q}C^q(2)^{\frac{q+1}{2}}(r)^{\frac{q-1}{2}}\right).\end{aligned} \tag{5.1}$$

Next, put $\bar{x}_\delta(t) = (\bar{x}_{1,\delta}(t), \bar{x}_{2,\delta}(t))$

$$\bar{x}_{i,\delta}(t) = \begin{cases} 0, & x \in V\backslash B(x_0,D) \\ \frac{2\delta}{D}(D - |x_i - x_0|), & x \in B(x_0,D)\backslash B(x_0,\frac{D}{2}) \\ \delta, & x \in B(x_0,\frac{D}{2}). \end{cases} \qquad (i = 1,2)$$

Clearly, $(\bar{x}_{1,\delta}(t), \bar{x}_{2,\delta}(t)) \in E$ and we have

$$\begin{aligned}\Phi(\bar{x}_{1,\delta}(t), \bar{x}_{2,\delta}(t)) &= \frac{1}{2}\sum_{i=1}^{2}\int_{B(x_0,D)\backslash B(x_0,D/2)} a_i(x)\frac{(2\delta)^2}{D^2}d\mu \\ &= \frac{1}{2}\sum_{i=1}^{2}\|a_i\|_{L^1(V)} \\ &\times \frac{(2\delta)^2}{D^2}\Big(meas(B(x_0,D)) - meas(B(x_0,D/2))\Big) \\ &= 2\sum_{i=1}^{2}\|a_i\|_{L^1(V)}\frac{(\delta)^2}{D^2}\frac{\pi^{N/2}}{\Gamma(1+N/2)}(D^N - (D/2)^N).\end{aligned} \tag{5.2}$$

Since $\delta > \gamma K_0$, it follows that $\gamma^2 < \Phi(\bar{x}_{1,\delta}, \bar{x}_{2,\delta})$. At this point, by (A_1), we have

$$\begin{aligned}\Psi(\bar{x}_{1,\delta}, \bar{x}_{2,\delta}) = -\int_V g(x)F(\bar{x}_{1,\delta}, \bar{x}_{2,\delta})d\mu &\ge -\int_{B(x_0,D/2)} g(x)F(\bar{x}_{1,\delta}, \bar{x}_{2,\delta})d\mu \\ &= \|g\|_{L^1(V)}F(\delta,\delta)\frac{\pi^{N/2}D^N}{2^N\Gamma(1+N/2)}.\end{aligned} \tag{5.3}$$

Hence, by (5.2) and (5.3), one has

$$\frac{\Psi(\bar{x}_\delta)}{\Phi(\bar{x}_\delta)} \ge \frac{\|g\|_{L^1(V)}F(\delta,\delta)D^2}{2\delta^2(2^N-1)\sum_{i=1}^{2}\|a_i\|_{L^1(V)}}.$$

In view of (5.1) and taking into account (A_3), we get

$$\begin{aligned}\chi(\gamma^2) &\leq C\|g\|_{L^1(V)}\left(2\eta_1 C + \frac{2^{\frac{q+1}{2}}\eta_2 C^q}{q}\gamma^{q-1} + \frac{\eta_3}{q}C^q(2)^{\frac{q+1}{2}}\gamma^{q-1}\right)\\ &\leq \|g\|_{L^1(V)}\frac{D^2}{(2^N-1)}\frac{F(\delta,\delta)}{2\sum_{i=1}^2\|a_i\|_{L^1(V)}\delta^2} \leq \frac{\Psi(\bar{x}_\delta)}{\Phi(\bar{x}_\delta)}.\end{aligned}$$

Therefore, the assumption (a_1) of Theorem 2.5 is satisfied. Moreover, if $s<2$, for every $(u_1,u_2)\in E$, $|(u_1,u_2)|^s \in L^{2/s}\times L^{2/s}$ and the Hölder's inequality gives

$$J_\lambda(u_1,u_2) \geq \frac{\|(u_1,u_2)\|_E^2}{2} - \lambda b C^s\|(u_1,u_2)\|_E^s - \lambda b,$$

for every $(u_1,u_2)\in E$. Therefore, J_λ is a coercive functional for every positive parameter, in particular, for every

$$\lambda \in \Lambda_{(\gamma,\delta)} \subseteq \left]\frac{\Phi(u_1,u_2)}{\Psi(u_1,u_2)}\ ,\ \frac{\gamma^2}{\sup_{\Phi(u_1,u_2)\leq\gamma^2}\Psi(u_1,u_2)}\right[.$$

Then, also condition (a_2) of Theorem 2.5 hold, hence, all the assumptions of Theorem 2.5 are satisfied, so that, for each $\lambda\in\Lambda_{(\gamma,\delta)}$, the functional J_λ has at least three distinct critical points that are weak solutions of the problem (1.1). □

Remark 5.2. When $\gamma=1$ condition (A_3) of Theorem 5.1 becomes:

(A_3') there exists a positive constant δ with $\delta > K_0$ such that

$$\frac{F(\delta,\delta)}{\delta^2} > \frac{2(2^N-1)\sum_{i=1}^2\|a_i\|_{L^1(V)}}{D^2}\Big(2\eta_1 C^2 + \frac{2^{\frac{q+1}{2}}C^{q+1}}{q}[\eta_2+\eta_3]\Big).$$

Remark 5.3. We observe that, if $F_{t_i}(0,0)\neq 0$, then, by Theorem 5.1, we obtain the existence of at least three non-zero weak solutions.

Example 5.4. Let r be positive constant, and $q\in[1,2^*]$, and $s<2$. Let $F_{t_i}:\mathbb{R}^2\longrightarrow\mathbb{R}$ be continuous and positive function defined as follows:

$$F_{t_i}(t_1,t_2) = \begin{cases} 1+|t_1|^{q-1}+|t_2|^{q-1} & if\ t_i\leq r \\ 1+|r|^{2-s}t_1^{s-1}+|r|^{2-s}t_2^{s-1} & if\ t_i > r.\end{cases} \qquad i=1,2$$

Clearly, $F_{t_i}(t_1,t_2)\leq(1+|t_1|^{q-1}+|t_2|^{q-1})$ for every $(t_1,t_2)\in\mathbb{R}^2$, and so the condition (A) holds. Moreover, for every $(\xi_1,\xi_2)\in\mathbb{R}$, one has

$$F(\xi_1,\xi_2)\leq\left(\xi_1\xi_2+\frac{|\xi_1|^q}{q}\xi_2+\frac{|\xi_2|^q}{q}\xi_1\right).$$

Hence, the condition (A_2) is satisfied, and

$$\frac{\int_0^r\int_0^r F_{t_i}(t_1,t_2)dt_1dt_2}{r^2} = \frac{\int_0^r\int_0^r\left(1+|t_1|^{q-1}+|t_2|^{q-1}\right)\,dt_1\,dt_2}{r^2} = 1+\frac{2}{q}r^{q-1}.$$

Then, by choosing $a=g\equiv -1$ and a suitable positive constant δ satisfying in the condition (A_3), by Theorem 5.1 and Remark 5.2, for each

$$\lambda\in\Lambda\subseteq\left[\frac{4(2^N-1)}{D^2}\frac{\delta^2}{F(\delta,\delta)}\ ,\ \frac{1}{2C^2+\frac{(2)^{\frac{q+3}{2}}C^{q+1}}{q}}\right],$$

the problem (1.1) possesses at least three weak positive solution in $H_0^1(V)\times H_0^1(V)$.

References

[1] S. Alexander, *Some properties of the spectrum of the Sierpiński gasket in a magnetic field,* Phys. Rev. B, **29**, 5504–5508, 1984.

[2] M.T. Barlow and J. Kigami, *Localized eigenfunctions of the Laplacian on p.c.f. self-similar sets,* J. London Math. Soc. **56** (2), 320–332, 1997.

[3] T. Bartsch and D.G. de Figueiredo, *Infinitely many solutions of nonlinear elliptic systems,* Progr. Nonlinear Diff. Equ. Appl. **35**, 51–67, 1999.

[4] A. Bensedik and M. Bouchekif, *On certain nonlinear elliptic systems with indefinite terms,* Electron. J. Differ. Equ. **83**, 1–16, 2002.

[5] L. Boccardo and D.G. de Figueiredo, *Some remarks on a system of quasilinear elliptic equations,* Nonlinear Differ. Equ. Appl. **9**, 309–323, 2002.

[6] B. Bockelman and R.S. Strichartz, *Partial differential equations on products of Sierpiński gaskets,* Indiana Univ. Math. J. **56**, 1361–1375, 2007.

[7] M. Bohner, S. Heidarkhani, A. Salari and G. Caristi, *Existence of three solutions for impulsive multi-point boundary value problems*, Opuscula Math. **37** (3), 353–379, 2017.

[8] G. Bonanno, *A critical point theorem via the Ekeland variational principle,* Nonlinear Anal. TMA, **75**, 2992–3007, 2012.

[9] G. Bonanno, *Relations between the mountain pass theorem and local minima,* Adv. Nonlinear Anal. **1**, 205–220, 2012.

[10] G. Bonanno and A. Chinnì, Existence and multiplicity of weak solutions for elliptic Dirichlet problems with variable exponent, J. Math. Anal. Appl. **418**, 812–827, 2014.

[11] G. Bonanno and S.A. Marano, *On the structure of the critical set of non-differentiable functions with a weak compactness condition,* Appl. Anal. **89**, 1–10, 2010.

[12] G. Bonanno, G. Molica Bisci and D. O'Regan, *Infinitely many weak solutions for a class of quasilinear elliptic systems,* Math. Comput. Modelling, **52**, 152–160, 2010.

[13] G. Bonanno, G. Molica Bisci and V.D. Rădulescu, *Infinitely many solutions for a class of nonlinear elliptic problems on fractals,* C. R. Math. Acad. Sci. Paris, **350** (3-4), 187–191, 2012.

[14] G. Bonanno, G. Molica Bisci and V.D. Rădulescu, *Variational analysis for a nonlinear elliptic problem on the Sierpiński Gasket,* ESAIM Control Optim. Calc. Var. **18** (4), 941–953, 2012.

[15] G. Bonanno, G. Molica Bisci and V.D. Rădulescu, *Qualitative Analysis of Gradient-Type Systems with Oscillatory Nonlinearities on the Sierpiński Gasket*, Chin. Ann. Math. **34** (3), 381–398, 2013.

[16] Y. Bozhkova and E. Mitidieri, *Existence of multiple solutions for quasilinear systems via fibering method,* J. Differential Equations, **190**, 239–267, 2003.

[17] B.E. Breckner, V.D. Rădulescu and C. Varga, *Infinitely many solutions for the Dirichlet problem on the Sierpiński gasket*, Anal. Appl. **9** (3), 235–248, 2011.

[18] B.E. Breckner, D. Repovš and C. Varga, *On the existence of three solutions for the Dirichlet problem on the Sierpiński gasket,* Nonlinear Anal. **73**, 2980–2990, 2010.

[19] Ph. Clément, D.G. de Figueiredo and E. Mitidieri, *Positive solutions of semilinear elliptic systems,* Comm. Partial Differential Equations, **17**, 923–940, 1992.

[20] D.G. de Figueiredo, *Semilinear elliptic systems: a survey of superlinear problems,* Resenhas IME-USP, **2**, 373–391, 1996.

[21] K.J. Falconer, *Semilinear PDEs on self-similar fractals,* Commun. Math. Phys. **206**, 235–245, 1999.

[22] K.J. Falconer and J. Hu, *Nonlinear elliptical equations on the Sierpiński gasket,* J. Math. Anal. Appl. **240**, 552–573, 1999.

[23] M. Ferrara, G. Molica Bisci and D. Repovs, *Existence results for nonlinear elliptic problems on fractal domains*, Adv. Nonlinear Anal. **5** (1), 75–84, 2016.

[24] M. Fukushima and T. Shima, *On a spectral analysis for the Sierpiński gasket,* Potential Anal. **1**, 1–35, 1992.

[25] M. Ghergu and V. Rădulescu, *Nonlinear PDEs. Mathematical models in biology, chemistry and population genetics,* Springer Monographs in Mathematics, Springer, Heidelberg, 2012.

[26] S. Goldstein, *Random walks and diffusions on fractals, Percolation Theory and Ergodic Theory of Infinite Particle Systems,* H. Kesten (ed.), 121–129, IMA Math. Appl., **8**, Springer-Verlag, New York, 1987.

[27] J.R. Graef, S. Heidarkhani and L. Kong, *Multiple solutions for systems of multi-point boundary value problems*, Opuscula Math. **33** (2), 293–306, 2013.

[28] J.R. Graef, S. Heidarkhani and L. Kong, *Infinitely Many Solutions for Systems of Sturm-Liouville Boundary Value Problems*, Results. Math. **66**, 327–341, 2014.

[29] J.R. Graef, S. Heidarkhani and L. Kong, *Nontrivial solutions for systems of Sturm-Liouville boundary value problems*, Differ. Equ. Appl. **6** (2), 255–265, 2014.

[30] J.R. Graef, S. Heidarkhani and L. Kong, *Multiple solutions for systems of Sturm-Liouville boundary value problems*, Mediterr. J. Math. **13**, 1625–1640, 2016.

[31] J.R. Graef, S. Heidarkhani and L. Kong, *Multiple periodic solutions for perturbed second-order impulsive Hamiltonian systems*, Int. J. Pure Appl. Math. **109** (1), 85–104, 2016.

[32] D.D. Hai and R. Shivaji, *An existence result on positive solutions for a class of p-Laplacian systems,* Nonlinear Anal., **56**, 1007–1010, 2004.

[33] S. Heidarkhani, *Non-trivial solutions for two-point boundary-value problems of fourth-order Sturm-Liouville type equations*, Electron. J. Diff. Equ. **2012** (27), 1–9, 2012.

[34] S. Heidarkhani, A. Salari and G. Caristi, *Kirchhoff-type second-order impulsive differential equations on the half-line,* preprint.

[35] J. Hu, *Multiple solutions for a class of nonlinear elliptic equations on the Sierpiński gasket,* Sci. China Ser. A, **47**, 772–786, 2004.

[36] C. Hua and H. Zhenya, *Semilinear elliptic equations on fractal sets,* Acta Math. Sci. Ser. B, **29** (2), 232–242, 2009.

[37] J.E. Hutchinson, *Fractals and self similarity,* Indiana Univ. Math. J. **30**, 713–747, 1981.

[38] S. Kusuoka, *A diffusion process on a fractal, Probabilistic Methods in Mathematical Physics*, Katata/Kyoto, 1985, 251-274, Academic Press, Boston, 1987.

[39] S.M. Kozlov, *Harmonization and homogenization on fractals,* Communications Math. Phys. **153**, 339–357, 1993.

[40] G. Molica Bisci and V. Rădulescu, *A characterization for elliptic problems on fractal sets*, Proc. Amer. Math. Soc. **143** (7), 2959–2968, 2015.

[41] G. Molica Bisci and V. Rădulescu, *A sharp eigenvalue theorem for fractional elliptic equations,* Israel J. Math. **219** (1), 331–351, 2017.

[42] G. Molica Bisci, D. Repovs and R. Servadei, *Nonlinear problems on the Sierpiński gasket,* J. Math. Anal. Appl. **452** (2), 883–895, 2017.

[43] P.H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations,* CBMSReg. Conf. Ser. Math. **65**, 1986.

[44] R. Rammal, *A spectrum of harmonic excitations on fractals,* J. Phy. Lett. **45**, 191–206, 1984.

[45] D. Stancu-Dumitru, *Two nontrivial weak solutions for the Dirichlet problem on the Sierpiński gasket,* Bull. Aust. Math. Soc. **85**, 395–414, 2012.

[46] R.S. Strichartz, *Analysis on fractals,* Notices Amer. Math. Soc. **46**, 1199–1208, 1999.

[47] R.S. Strichartz, *Differential Equations on Fractals,* A. Tutorial (ed.), Princeton University Press, Princeton, 2006.

[48] A. Teplyaev, *Spectral Analysis on Infinite Sierpiński Gaskets* J. Func. Anal. **159**, 537–567, 1998.

[49] G.Q. Zhang, X.P. Liu and S.Y. Liu, *Remarks on a class of quasilinear elliptic systems involving the* (p, q)*-Laplacian,* Electron. J. Differ. Equ. **2005**, 1–10, 2005.