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Research Article

MS-intervals of an MS-algebra

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Abstract

The concept of MS-intervals and its properties on MS-algebras are introduced and the connection between an MS-algebra and the family of its MS-intervals is established.

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1. Introduction

In 1983, T.S. Blyth and J.C. Varlet [8] introduced the notion of an MS-algebra as a common abstraction of De Morgan algebra and Stone algebra. The class **MS** of all MS-algebras is equational. Also, in [9] they described all subvarieties of **MS**. The lattice $\Lambda(\mathbf{MS})$ of subvarieties of **MS** has 20 elements (see Figure 1 of [10]). In [11] T. S. Blyth and J. C. Varlet studied and characterized the ideal lattice of an MS-algebra. In [13] Luo Congwen introduced and characterized Stonean intervals of an MS-algebras. In 2012, A. Badawy, D. Guffova and M. Haviar [5] introduced and characterized the class of decomposable MS-algebras by means of triples. A. Badawy and R. El-Fawal [3] studied homomorphisms and subalgebras of decomposable MS-algebras. In 2014, A. Badawy and M.S. Rao [4] introduced the notion of closure ideals of MS-algebras. Recently, A. Badawy [1,2] introduced and constructed two new classes of generalized MS-algebras.

In this article, after preliminaries in Section 2, we introduce in Section 3, the concepts of central elements and MS-intervals of an MS-algebra and related properties. Also, it is proved that the set $I_{MS}(L)$, of all MS-intervals of L forms a Stone algebra on its own. The largest Stone subalgebra L_S of any MS-algebra L is obtained and characterized. In Section 4, if L_a is an MS-interval of an MS-algebra L, then the relationship between $\Gamma(L)$ and $\Gamma(L_a)$ is obtained and some related properties are investigated. Finally, we proved that the skeleton of the two MS-intervals L_c and L_d are isomorphic, whenever $(c, d) \in \Gamma(L)$.

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2. Preliminaries

In this section, we present certain definitions and results which are taken mostly from [6,8,9,12].

An MS-algebra is an algebra $(L;\vee,\wedge,^\circ,0,1)$ of type (2,2,1,0,0) where $(L;\vee,\wedge,0,1)$ is a bounded distributive lattice and $^\circ$ a unary operation of involution satisfies :

$$x\leq x^{\circ\circ},\ (x\wedge y)^\circ = x^\circ\vee y^\circ,\ 1^\circ=0.$$

The class **MS** of all MS-algebras is equational. The class **B** of Boolean algebras is a subclass of **MS** and is characterized by the identity $x\vee x^\circ=1$.

The class **M** of De Morgan algebras is a subclass of the class **MS** and is characterized by the identity, $x=x^{\circ\circ}$. The class $\mathbf{K_2}$ of K_2-algebras is a subclass of **MS** and is characterized by the additional two identities

$$x\wedge x^\circ = x^\circ\wedge x^{\circ\circ}, (x\wedge x^\circ)\vee(y\vee y^\circ)=y\vee y^\circ.$$

The class **S** of Stone algebras is a subclass of **MS** and is characterized by the identity $x\wedge x^\circ=0$. The class $\mathbf{K_3}$ is a subclass of **MS** and is characterized by the following two identities

$$(x\wedge x^\circ)\vee y^{\circ\circ}\vee y^\circ = y^{\circ\circ}\vee y^\circ,\ \ x\vee y^\circ\vee y^{\circ\circ}=x^{\circ\circ}\vee y^\circ\vee y^{\circ\circ}.$$

The class $\mathbf{M}\vee\mathbf{K_2}\vee\mathbf{K_3}$ is a subclass of **MS** and is characterized by the identity

$$(x\wedge x^\circ)\vee(y^\circ\vee y^{\circ\circ})=(x^{\circ\circ}\wedge x^\circ)\vee(y^\circ\vee y^{\circ\circ}).$$

We recall some of the basic properties of MS-algebras which were proved in [8] or [12].

Theorem 2.1. *For any two elements a,b of an MS-algebra L, we have*
(1) $0^\circ=1$
(2) $a\leq b\Rightarrow b^\circ\leq a^\circ$
(3) $a^\circ=a^{\circ\circ\circ}$
(4) $(a\vee b)^\circ=a^\circ\wedge b^\circ$
(5) $(a\vee b)^{\circ\circ}=a^{\circ\circ}\vee b^{\circ\circ}$
(6) $(a\wedge b)^{\circ\circ}=a^{\circ\circ}\wedge b^{\circ\circ}$.

It is known that the set $L^{\circ\circ}=\{x\in L: x=x^{\circ\circ}\}$ of all closed elements of an MS-algebra L is a De Morgan subalgebra of L. $L^{\circ\circ}$ is called the skeleton of L. If $L\in\mathbf{S}$, then $L^{\circ\circ}$ is a Boolean subalgebra of L.

An element d of an MS-algebra L is called a dense element if $d^\circ=0$. The set $D(L)=\{d\in L: d^\circ=0\}$ of all dense elements of L is a filter of L (see [5]).

The mapping $x\to x^{\circ\circ}$ on an MS-algebras L gives rise to the following binary relation

$$x\equiv y(\Gamma(L))\Leftrightarrow x^{\circ\circ}=y^{\circ\circ}\Leftrightarrow x^\circ=y^\circ$$

It is known that $\Gamma(L)$ is a congruence relation on L and $L/\Gamma(L)\cong L^{\circ\circ}$. The element $x^{\circ\circ}$ is the greatest element of the congruence class $[x]\Gamma(L)$. It is easy to see that $[1]\Gamma(L)=D(L)$ and $[0]\Gamma(L)=\{0\}$.

We refer the reader to [7,12,14] for the basic properties of distributive lattices and to [6,9,10,12] for the basic properties of MS-algebras.

3. MS-intervals

For any element a of an MS-algebra L, denote the interval $[0,a]$ by L_a. In this section, the answer of the following question and related results are given: Under what conditions the intervals L_a, $a\in L$ constructs MS-algebras?. We proved that the collection $I_{MS}(L)$ of all MS-intervals of L is a Stone algebra. Also, the largest Stone subalgebra L_S of an

MS-algebra L and $I_{MS}(L)$ are isomorphic.

Firstly, we mention the notion of central elements of an MS-algebra L.

Definition 3.1. An element a of an MS-algebra L is said to be a central element of L if $a \vee a^{\circ} = 1$. The set $C(L) = \{a \in L : a \vee a^{\circ} = 1\}$ of all central elements of L is called the center of L.

It is obvious that, if $L \in \mathbf{S}$, then $C(L) = L^{\circ\circ}$. Otherwise $C(L) \subset L^{\circ\circ}$, for any $L \in \mathbf{MS} - \mathbf{S}$.

The following theorem is a direct consequence of the above.

Theorem 3.2. *Let L be an MS-algebra. Then*
(1) if $a \in C(L)$, then $a^{\circ} \in C(L)$,
(2) if $a \in C(L)$, then $a \wedge a^{\circ} = 0$,
(3) $C(L)$ is a Boolean subalgebra of $L^{\circ\circ}$.

Secondly, we introduce the concept of MS-interval of an MS-algebra.

Definition 3.3. An interval L_a of an MS-algebra L is called an MS-interval if $(L_a; \vee, \wedge, ^{\circ_a}, 0, a)$ is an MS-algebra with respect to the operations $\vee, \wedge$ of L and a unary operation $^{\circ_a}$ is defined by $x^{\circ_a} = x^{\circ} \wedge a$ for every $x \in L_a$.

An MS-interval $L_a,\ a \neq 1$ is not a subalgebra of L, for it does not preserve the nullary operations.

Now, the following crucial lemma is given.

Lemma 3.4. *Let $L_a = (L_a,\ ^{\circ_a})$ be an MS-interval of an MS-algebra L. Then*
(1) In L_a, $a^{\circ_a} = 0$,
(2) $L_a^{\circ_a\circ_a} = \{x^{\circ\circ} \wedge a : x \in L_a\}$ and $D(L_a) = \{x \in L : x^{\circ} \wedge a = 0\}$,
(3) If $a \in L^{\circ\circ}$, then $L_a^{\circ_a\circ_a} = L_a \cap L^{\circ\circ}$.

Proof. The proofs of (1) and (2) are obvious.

(3) If $a \in L^{\circ\circ}$, we have to prove that $L_a^{\circ_a\circ_a} = L_a \cap L^{\circ\circ}$. Let $x \in L_a^{\circ_a\circ_a}$. Then $x = x^{\circ_a\circ_a} = x^{\circ\circ} \wedge a \in L_a \cap L^{\circ\circ}$, that is $L_a^{\circ_a\circ_a} \subseteq L_a \cap L^{\circ\circ}$. Conversely, let $x \in L_a \cap L^{\circ\circ}$. Then $x \leq a$ and $x = x^{\circ\circ}$. Now

$$x^{\circ_a\circ_a} = x^{\circ\circ} \wedge a = x \wedge a = x$$

Therefore $x \in L_a^{\circ_a\circ_a}$, and so $L_a \cap L^{\circ\circ} \subseteq L_a^{\circ_a\circ_a}$. □

A characterization of MS-intervals of an MS-algebra L in terms of the central elements of L is investigated in the following.

Theorem 3.5. *Let L_a be an interval of an MS-algebra L. Then L_a is an MS-interval of L if and only if a° is a central element of L.*

Proof. Let $L_a = (L_a, ^{\circ_a})$ be an MS-interval of L. Then $a^{\circ_a} = 0$ by Lemma 3.4(1). Hence we have

$$\begin{aligned} a^{\circ\circ} \vee a^{\circ} &= (a^{\circ} \wedge a)^{\circ} \\ &= (a^{\circ_a})^{\circ} \\ &= 0^{\circ} \\ &= 1 \end{aligned}$$

Therefore a° is a central element of L. Conversely, let $a^\circ \in C(L)$. Now we prove that $(L_a, ^{a_a})$ is an MS-interval of L, whenever $x^{\circ_a} = x^\circ \wedge a$, $\forall x \in L_a$. Let $x, y \in L_a$.

$$\begin{aligned} x^{\circ_a\circ_a} \wedge x &= ((x^{\circ_a})^\circ \wedge a) \wedge x \\ &= ((x^\circ \wedge a)^\circ \wedge a) \wedge x \\ &= ((x^{\circ\circ} \vee a^\circ) \wedge a) \wedge x \\ &= ((x^{\circ\circ} \wedge a) \vee (a^\circ \wedge a)) \wedge x \text{ (by distributivity of } L) \\ &= ((x^{\circ\circ} \wedge a) \vee 0) \wedge x \text{ (as } a^\circ \wedge a = 0 \textit{ by Lemma } 3.2(2)) \\ &= x \text{ (as } x \leq x^{\circ\circ} \wedge a). \end{aligned}$$

Therefore $x \leq x^{\circ_a\circ_a}$. Also, we get

$$\begin{aligned} (x \wedge y)^{\circ_a} &= (x \wedge y)^\circ \wedge a \\ &= (x^\circ \vee y^\circ) \wedge a \\ &= (x^\circ \wedge a) \vee (y^\circ \wedge a) \text{ (by distributivity of } L) \\ &= x^{\circ_a} \vee y^{\circ_a}, \\ a^{\circ_a} &= a^\circ \wedge a \\ &\leq a^\circ \wedge a^{\circ\circ} \\ &= 0 \text{ (by Theorem 2.3(2) with } a^\circ \in C(L)). \end{aligned}$$

Therefore $(L_a, ^{\circ_a})$ is an MS-interval of L. □

For Stone algebras, we have the following.

Corollary 3.6. *Let $L \in \mathbf{S}$. Then L_a is an MS-interval of L for every $a \in L$.*

For any MS-algebra L, we can claim that L and all its MS-intervals are belong to the same subvariety of the variety **MS**. As an example we claim this for the subclass $\mathbf{M} \vee \mathbf{K_2} \vee \mathbf{K_3}$ of **MS**.

Theorem 3.7. *Let L_a be an MS-interval of an MS-algebra L. If $L \in \mathbf{M} \vee \mathbf{K_2} \vee \mathbf{K_3}$, then $L_a \in [\mathbf{B}, \mathbf{M} \vee \mathbf{K_2} \vee \mathbf{K_3}]$.*

Proof. Let $x, y \in L_a$. Then we have

$$\begin{aligned} (x \wedge x^{\circ_a}) \vee (y^{\circ_a} \vee y^{\circ_a\circ_a}) &= (x \wedge x^\circ \wedge a) \vee ((y^\circ \wedge a) \vee (y^{\circ\circ} \wedge a) \\ &= (x \wedge x^\circ \wedge a) \vee ((y^\circ \vee y^{\circ\circ}) \wedge a) \\ &= ((x \wedge x^\circ) \vee (y^\circ \vee y^{\circ\circ})) \wedge a \\ &= ((x^\circ \wedge x^{\circ\circ}) \vee (y^\circ \vee y^{\circ\circ})) \wedge a \text{ (as } L \in \mathbf{M} \vee \mathbf{K_2} \vee \mathbf{K_3}) \\ &= ((x^\circ \wedge x^{\circ\circ} \wedge a) \vee ((y^\circ \vee y^{\circ\circ}) \wedge a) \\ &= ((x^\circ \wedge x^{\circ\circ} \wedge a) \vee ((y^\circ \wedge a) \vee (y^{\circ\circ} \wedge a)) \\ &= (x^{\circ_a} \wedge x^{\circ_a\circ_a}) \vee (y^{\circ_a} \vee y^{\circ_a\circ_a}). \end{aligned}$$

Therefore $L_a \in \mathbf{M} \vee \mathbf{K_2} \vee \mathbf{K_3}$. □

Remark 3.8. The converse of the above observation is not true. Let L be the 3-element Stone algebra, $0 < a < 1$. Clearly $L_a = [0, a]$ is an MS-interval of L and $L_a \in \mathbf{B}$, but $L \notin \mathbf{B}$.

Moreover, it is easy to observe the following remark:

Remark 3.9. Let L be an MS-algebra. Let L_a and L_b be two MS-intervals. Then we have
(1) $b \in L_a^{\circ_a\circ_a}$ if and only if $L_b^{\circ_b\circ_b} \subseteq L_a^{\circ_a\circ_a}$,
(2) $L_b^{\circ_b\circ_b} \subseteq L_a^{\circ_a\circ_a}$ implies $L_b \subseteq L_a$.

Remark 3.10. The converse of Remark 3.9(2) is not true. For example, Let $(L,^{\circ})$ be an MS-algebra, where L is the 5-element chain, $0 < a < b < d < 1$ with $a^{\circ} = a$, $b^{\circ} = d^{\circ} = 1^{\circ} = 0$ and $0^{\circ} = 1$. Clearly L_b and L_d are MS-intervals of L such that $L_b \subseteq L_d$ but $L_b^{\circ_b\circ_b} \nsubseteq L_d^{\circ_d\circ_d}$.

Use $I_{MS}(L)$ to denote the set of all MS-intervals of an MS-algebra L.

Lemma 3.11. *Let L_a, L_b be MS-intervals of an MS-algebra L. Then we have*
(1) $a \leq b \Leftrightarrow L_a \subseteq L_b$,
(2) $L_a = \{0\} \Leftrightarrow a = 0$,
(3) $L_{a\wedge b} = L_a \wedge L_b$,
(4) $L_{a\vee b} = L_a \vee L_b$.

The proof is straightforward.

The following theorem shows that the set of all MS-intervals of an MS-algebra L can constructed MS-algebra, precisely from the subvariety **S** of the variety **MS**.

Theorem 3.12. *The collection $I_{MS}(L)$ of all MS-intervals of an MS-algebra L forms a Stone algebra on its own.*

Proof. By Lemma 3.11(3),(4), $I_{MS}(L)$ is a lattice. clearly $L_0 = \{0\}$ and $L_1 = L$ are the smallest and greatest members of $I_{MS}(L)$ respectively. Then $I_{MS}(L)$ is a bounded lattice. It is easy to see that $I_{MS}(L)$ is a distributive lattice. Define * on $I_{MS}(L)$ by $L_a^{\star} = L_{a^{\circ}}$. Let $L_a, L_b \in I_{MS}(L)$. Then

$$\begin{aligned} L_a &\subseteq L_{a^{\circ\circ}} \ (as\ a \leq a^{\circ\circ}) \\ &= L_a^{\star\star}, \\ (L_a \wedge L_b)^{\star} &= L_{a\wedge b}^{\star} \\ &= L_{(a\wedge b)^{\circ}} \\ &= L_{a^{\circ}\vee b^{\circ}} \\ &= L_{a^{\circ}} \vee L_{b^{\circ}} \\ &= L_a^{\star} \vee L_b^{\star}. \end{aligned}$$

Also,

$$L_1^{\star} = L_0$$

Then $(I_{MS}(L), {}^{\star})$ is an MS-algebra. Now, $I_{MS}(L)$ is a Stone algebra, because of

$$\begin{aligned} L_a \wedge L_a^{\star} &= L_a \wedge L_{a^{\circ}} \\ &= L_{a\wedge a^{\circ}} \ (by\ Lemma\ 3.11(3)) \\ &= L_0 \text{ (by Lemma 3.2(2))}. \end{aligned}$$

for every $L_a \in I_{MS}(L)$. □

Consider the subset $L_S = \{x \in L : x^{\circ} \in C(L)\}$ of an MS-algebra L. In view of the central elements and the elements of L_S, it is observed the following.

Remark 3.13. Let L be an MS-algebra. Then
(1) For any $x \in C(L)$, $x \in L_S$,
(2) For any $x \in L_S$, $x^{\circ} \in C(L)$,

Now the concept of Stone subalgebras of an MS-algebra is given.

Definition 3.14. A subalgebra S of an MS-algebra L is called a Stone subalgebra of L if $x^{\circ} \vee x^{\circ\circ} = 1$, for all $x \in S$.

Lemma 3.15. *If S is a Stone Subalgebra of an MS-algebra L, then*
(1) For any $x \in S$, $x = x^{\circ\circ} \wedge (x \vee x^{\circ})$ with $x^{\circ\circ} \in C(S)$ and $x \vee x^{\circ} \in D(S)$,
(2) $C(S) \subseteq C(L)$ and $D(S) \subseteq D(L)$.

Proof.
(1) Let $x \in S$. Then $x^{\circ} \vee x^{\circ\circ} = 1$ implies $x^{\circ} \wedge x^{\circ\circ} = 0$. Now

$$\begin{aligned} x^{\circ\circ} \wedge (x \vee x^{\circ}) &= (x^{\circ\circ} \wedge x) \vee (x^{\circ\circ} \wedge x^{\circ}) \\ &= x \vee 0 \\ &= x. \end{aligned}$$

Clearly, $x^{\circ\circ} \in C(S)$ for every $x \in S$. Let $x \in S$. Then $x \vee x^{\circ} \in S$. Since $(x \vee x^{\circ})^{\circ} = 0$, then $x \vee x^{\circ} \in D(S)$.
(2) It is obvious. □

In the following theorem, the subset L_S of the MS-algebra L forms the greatest Stone subalgebra of L. Moreover, the centers of L and L_S are coincide and also the dense sets of L and L_S are coincide.

Theorem 3.16. *L_S is the largest Stone subalgebra of L with $C(L_S) = C(L)$ and $D(L_S) = D(L)$.*

Proof. Clearly $0, 1 \in L_S$. Let $x, y \in L_S$. Then $x^{\circ}, y^{\circ} \in C(L)$. Since $(x \vee y)^{\circ} = x^{\circ} \wedge y^{\circ} \in C(L)$ and $(x \wedge y)^{\circ} = x^{\circ} \vee y^{\circ} \in C(L)$, we get $x \vee y, x \wedge y \in L_S$. Then L_S is a sublattice of L. To prove that L_S is closed under $^{\circ}$, let $x \in L_S$. Then by Theorem 3.2(2), $x^{\circ} \in C(L)$ implies $x^{\circ\circ} \in C(L)$. Hence L_S is a subalgebra of L. Consequently, $x^{\circ\circ} \vee x^{\circ} = 1$ for all $x \in L_S$. Then L_S is a subalgebra of L. Clearly, $C(L_S) \subseteq C(L)$ and $D(L_S) \subseteq D(L)$. For the converse, let $a \in C(L)$. By Theorem 3.2(2), $a^{\circ} \in C(L)$. Then $a \in L_S$ and $C(L) \subseteq L_S$. So $C(L) \subseteq C(L_S)$. Let $x \in D(L)$. Then $x^{\circ} = 0 \in C(L)$. So $x \in L_S$. Then $D(L) \subseteq L_S$ and $D(L) \subseteq D(L_S)$. Therefore $C(L_S) = C(L)$ and $D(L_S) = D(L)$.
Assume L_{S_1} be any Stone subalgebra of L. Let $x \in L_{S_1}$. Then by Lemma 3.15(1), we have $x = x^{\circ\circ} \wedge (x \vee x^{\circ})$ with $x^{\circ\circ} \in C(L_{S_1})$ and $x \vee x^{\circ} \in D(L_{S_1})$. Now $x^{\circ\circ} \in C(L_{S_1}) \subseteq C(L_S)$ and $x \vee x^{\circ} \in D(L_{S_1}) \subseteq D(L_S)$ by Lemma 3.15(1). So, $x \in L_S$. Then $L_{S_1} \subseteq L_S$. Therefore L_S is the largest Stone subalgebra of L. □

Now, the largest Stone subalgebra L_S of L and the Stone algebra $I_{MS}(L)$ of all MS-intervals of L are isomorphic.

Theorem 3.17. *Let L be an MS-algebra. Then $L_S \cong I_{MS}(L)$.*

Proof. Define $f : L_S \to I_{MS}(L)$ by $f(a) = L_a$ for every $a \in L_S$. Since $a \in L_S$, then $a^{\circ} \in C(L)$. Hence by Theorem 3.5, L_a is an MS-interval and f is well defined. It is clear that f is an onto homomorphism. To prove the injectivity of f, let $f(a) = f(b)$. Then $L_a = L_b$ implies $a = b$. Therefore f is an isomorphism and $L_S \cong I_{MS}(L)$. □

Moreover, we have the following:

Theorem 3.18. *Let L be an MS-algebra. Then $I_{MS}(L)$ is embedded into L.*

Proof. Define a map $g : I_{MS}(L) \to L$ by $g(L_a) = a$, for all $L_a \in I_{MS}(L)$. Suppose that $L_a, L_b \in I_{MS}(L)$. One can easy show that g is a monomorphism. Then $I_{MS}(L)$ is embedded into L. □

The following corollary shows that L and $I_{MS}(L)$ are isomorphic, whenever L is a Stone algebra.

Corollary 3.19. *Let $L \in \mathbf{S}$. Then $I_{MS}(L)$ is isomorphic to L.*

Proof. Since L is an Stone algebra, then for any $a \in L$, we have $a \in C(L)$. Hence, it follows that L_a is an MS-interval. Hence the map $f : L \to I_{MS}(L)$ which is defined by $f(a) = L_a$ is an isomorphism. □

4. Γ-congruences on MS-intervals

Let L_a be an MS-interval of an MS-algebra L. In this Section, the relationship between $\Gamma(L)$ and $\Gamma(L_a)$ is investigated. Finally, we derived that the skeleton of the two MS-intervals L_c and L_d are isomorphic, whenever $(c, d) \in \Gamma(L)$.

A characterization of MS-intervals of an MS-algebra L in terms of congruence classes of $\Gamma(L)$ is given in the next theorem.

Theorem 4.1. *Let L be an MS-algebra. An interval L_x of an MS-algebra L is an MS-interval if and only if $x \in [a]\Gamma(L)$ for some $a \in C(L)$.*

Proof. Let L be an MS-algebra and let L_x be an MS-interval. By sufficiency of Theorem 3.5, $x^\circ \in C(L)$. Since $x^\circ = x^{\circ\circ\circ}$, then $x \equiv x^{\circ\circ}(\Gamma(L))$, it follows that $x \in [x^{\circ\circ}]\Gamma(L)$, where $x^{\circ\circ} \in C(L)$. Conversely, let $x \in [a]\Gamma(L)$ for some $a \in C(L)$. Then $x^\circ = a^\circ \in C(L)$. By necessity of Theorem 3.5, L_x is an MS-interval of L. □

It is derived the following crucial lemma.

Lemma 4.2. *Let L_a be an MS-interval of an MS-algebra L. Then*

$$\Gamma(L_a) = \Gamma(L) \cap (L_a \times L_a).$$

In addition, $[a]\Gamma(L) = D(L_a)$ for every $a \in C(L)$.

Proof. Denote by $\gamma = \Gamma(L) \cap (L_a \times L_a)$. Assume $(c, d) \in \gamma$. Then

$$\begin{aligned}(c,d) \in \gamma \quad &\Rightarrow \quad (c,d) \in \Gamma(L) \text{ and } (c,d) \in L_a \times L_a \\ &\Rightarrow \quad c^\circ = d^\circ \text{ and } c, d \in L_a \\ &\Rightarrow \quad c^{\circ_a} = c^\circ \wedge a = d^\circ \wedge a = d^{\circ_a} \\ &\Rightarrow \quad (c,d) \in \Gamma(L_a).\end{aligned}$$

Hence $\gamma \subseteq \Gamma(L_a)$. Conversely, let $(c, d) \in \Gamma(L_a)$ for $c, d \in L_a$. Then $c^{\circ_a} = d^{\circ_a}$ implies $c^\circ \wedge a = d^\circ \wedge a$. Then $c^\circ \wedge a^{\circ\circ} = d^\circ \wedge a^{\circ\circ}$. We claim that $c^\circ = d^\circ$. Really, as $c^\circ, d^\circ \geq a^\circ$ and $a^\circ \in C(L)$, we get

$$\begin{aligned}c^\circ &= (c^\circ \vee a^\circ) \wedge 1 \\ &= (c^\circ \vee a^\circ) \wedge (a^{\circ\circ} \vee a^\circ) \text{ (as } a^{\circ\circ} \vee a^\circ = 1) \\ &= (c^\circ \wedge a^{\circ\circ}) \vee a^\circ \text{ (by distributivity of } L) \\ &= (d^\circ \wedge a^{\circ\circ}) \vee a^\circ \\ &= (d^\circ \vee a^\circ) \wedge (a^{\circ\circ} \vee a^\circ) \text{ (by distributivity of } L) \\ &= d^\circ \vee a^\circ \\ &= d^\circ\end{aligned}$$

Therefore $(c, d) \in \Gamma(L)$ and consequently, $(c, d) \in \gamma$. Then $\Gamma(L_a) = \gamma$, as required. To show that $[a]\Gamma(L) = D(L_a)$ for any $a \in C(L)$, let $x \in [a]\Gamma(L)$. Then

$$\begin{aligned}x \in [a]\Gamma(L) \quad &\Rightarrow \quad x^\circ = a^\circ \\ &\Rightarrow \quad x^\circ \wedge a = a^\circ \wedge a \\ &\Rightarrow \quad x^\circ \wedge a = 0 \text{ (}as\ a \wedge a^\circ = 0 \text{ by Theorem 3.2(2))} \\ &\Rightarrow \quad x^{\circ_a} = 0 \\ &\Rightarrow \quad x \in D(L_a).\end{aligned}$$

Then $[a]\Gamma(L) \subseteq D(L_a)$. Conversely, let $x \in D(L_a)$. Then $x \leq a$. So we get

$$\begin{aligned} x \in D(L_a) &\Rightarrow x^{\circ_a} = 0 \\ &\Rightarrow x^{\circ} \wedge a = 0 \\ &\Rightarrow (x^{\circ} \wedge a) \vee a^{\circ} = 0 \vee a^{\circ} \\ &\Rightarrow (x^{\circ} \vee a^{\circ}) \wedge (a \vee a^{\circ}) = a^{\circ} \ (by\ distributivity\ of\ L) \\ &\Rightarrow x^{\circ} \wedge 1 = a^{\circ} \ (\text{as } x^{\circ} \geq a^{\circ} \text{ and } a \vee a^{\circ} = 1) \\ &\Rightarrow x^{\circ} = a^{\circ} \\ &\Rightarrow x \in [a]\Gamma(L). \end{aligned}$$

Therefore $D(L_a) \subseteq [a]\Gamma(L)$. Then $D(L_a) = [a]\Gamma(L)$, whenever a is a central element of L. □

In closing this paper, some important results concerning the skeleton of MS-intervals are studied.

Theorem 4.3. *Let L be an MS-algebra. Then $c, d \in [1]\Gamma(L)$ implies $L_c^{\circ_c\circ_c} \cong L_d^{\circ_d\circ_d}$.*

Proof. Let $d \in D(L)$. Then d is a Stone element of L and hence $d^{\circ} = 0 \in C(L)$. Thus L_d is an MS-interval of L, by Theorem 3.5. Consider the mapping

$$\varphi : L^{\circ\circ} \to L_d^{\circ_d\circ_d}$$

defined by

$$\varphi(x) = x \wedge d, \ \forall x \in L^{\circ\circ}$$

It is known that $x = x^{\circ\circ}$ for any $x \in L^{\circ\circ}$. Then by Lemma 3.4(2), φ is well defined. It is easy to see that φ preserves meets and joins. Also, φ preserves unary operations.

$$\begin{aligned} (\varphi(x))^{\circ_d} &= (x \wedge d)^{\circ_d} \\ &= (x \wedge d)^{\circ} \wedge d \\ &= (x^{\circ} \vee d^{\circ}) \wedge d \\ &= x^{\circ} \wedge d \ (\text{as } d^{\circ} = 0) \\ &= \varphi(x^{\circ}) \end{aligned}$$

For $x \in L_d^{\circ_d\circ_d}$, we have

$$\begin{aligned} x &= x^{\circ_d\circ_d} \\ &= x^{\circ\circ} \wedge d \ (\text{by Lemma 3.4(2)}) \\ &= \varphi(x^{\circ\circ}). \end{aligned}$$

Thus φ is an epimorphism. Let $x, y \in L^{\circ\circ}$ be such that $\varphi(x) = \varphi(y)$. Then $x \wedge d = y \wedge d$. Therefore $x = x \wedge 1 = x \wedge d^{\circ\circ} = x^{\circ\circ} \wedge d^{\circ\circ} = (x \wedge d)^{\circ\circ} = (y \wedge d)^{\circ\circ} = y^{\circ\circ} \wedge d^{\circ\circ} = y \wedge 1 = y$ as $d \in D(L)$.

Therefore φ is an isomorphism of De Morgan algebras. Then $L^{\circ\circ} \cong L_d^{\circ_d\circ_d}$, whenever $d \in [1]\Gamma(L)$. Similarly, we can get $L^{\circ\circ} \cong L_d^{\circ_d\circ_d}$. Hence, $L_c^{\circ_c\circ_c} \cong L_d^{\circ_d\circ_d}$. □

A generalization of the above theorem is given in the following.

Theorem 4.4. *Let L_c, L_d be MS-intervals of an MS-algebra L. Then $c \equiv d(\Gamma(L))$ implies $L_c^{\circ_c\circ_c} \cong L_d^{\circ_d\circ_d}$. Moreover, $L_c^{\circ_c\circ_c} = L_d^{\circ_d\circ_d}$ iff $c = d$.*

Proof. Since L_c, L_d are MS-intervals of L, then $c^{\circ\circ}$, $d^{\circ\circ} \in C(L)$. Assume that $c \equiv d(\Gamma(L))$ with $c^{\circ\circ} = d^{\circ\circ} = w < 1$. Consider the new MS-algebra L_w (see Theorem 3.5), with the congruence $\Gamma(L_w)$ on it. Evidently, $c, d \in [w]\Gamma(L) = D(L_w)$, by Lemma 4.2. Therefore by the above theorem, $L_c^{\circ_c\circ_c} \cong L_d^{\circ_d\circ_d}$. The last part of the proof is obvious and the proof is finished. □

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