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Free Vibration Analysis of Multi-span Timoshenko Beams on Elastic Foundation Using Dynamic Stiffness Method

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Abstract

In this study, the exact first five natural frequencies of three-span Timoshenko beams on Winkler foundation are calculated using dynamic stiffness formulation. Different elastic foundation spring constants and different beam cross-sections are used to reflect their effects on natural frequencies. Moreover, the natural frequencies are also calculated via structural analysis software SAP2000 and tabulated with exact results. It is seen that the influence of elastic foundation spring stiffness in inner span is high in comparison with outer spans. The cross-section of the beam plays an important role on natural frequencies of multi-span Timoshenko beams on Winkler foundation.

Key words

Dynamic stiffness, free vibration, multi-span Timoshenko beam, Winkler foundation

1. INTRODUCTION

The calculation of exact natural frequencies of beams is of great interest to researchers for a long time. In the recent years, free vibration analyses of different types of beams with various loading conditions are performed using different methods [1-5]. The vibration problem of beams on elastic foundations can be encountered especially in civil engineering and mechanical engineering applications. The elastic foundations are modeled using elastic springs. The effects of elastic foundation can be important for beams or beam assembly structures. Thus, there are numerous studies about different foundation models carrying various types of beams and structures [6-10]. The dynamic stiffness method (DSM) is an effective method for calculating exact natural frequencies of beams or beam like structures as the method uses the exact shape functions. Vibration analyses of many types of beams and plates are performed by using DSM in recent years [11-17].

In this study, free vibration analysis of three-span simply supported beams on Winkler foundation is performed using DSM. Timoshenko Beam Theory (TBT) which considers shear deformation and rotational inertia is used. In the numerical analysis, different spring stiffness values for spans are selected to reveal the effect of elastic foundation on natural frequencies. To reflect the importance of beam geometry, several analyses are completed for different beam geometries. SAP2000 is a well known and widely used structural analysis software worldwide.

Thus, SAP2000 is used to obtain natural frequencies of beams on Winkler foundation and the results are compared with exact values.

2. MODEL AND FORMULATION

The mathematical model of three-span Timoshenko beam on Winkler foundation can be seen in Figure 1. Here, x and y represents the axes, k_{s1}, k_{s2} and k_{s3} are Winkler foundation spring stiffnesses, L is the span length, b is width of the beam and h is the height of the beam.

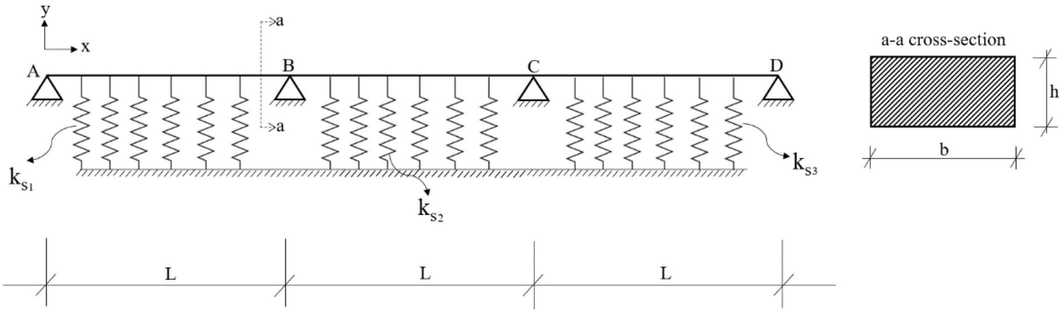

Figure 1. Three-span Timoshenko beam on Winkler foundation

Let the denomination of span AB as 1, span BC as 2 and span CD as 3.

The assumptions listed below are considered to clarify and simplificate the analysis procedure:

1) The beam is constructed by using an isotropic and homogenous material.
2) The cross-section of the beam is uniform.
3) The beam behaves linear and elastic.
4) The damping is neglected.
5) The foundation springs are linear and distributed along the beam length.

The governing equations of motion of vibrating Timoshenko beam resting on Winkler foundation are given as follows:

$$\frac{AG}{\bar{k}}\left(\frac{\partial^2 y_n(x,t)}{\partial x^2}-\frac{\partial \theta_n(x,t)}{\partial x}\right)-\bar{m}\frac{\partial^2 y_n(x,t)}{\partial t^2}-k_{sn}y_n(x,t)=0$$
$$EI\frac{\partial^2 \theta_n(x,t)}{\partial x^2}-\bar{m}\frac{I}{A}\frac{\partial^2 \theta_n(x,t)}{\partial t^2}+\frac{AG}{\bar{k}}\left(\frac{\partial y_n(x,t)}{\partial x}-\theta_n(x,t)\right)=0 \tag{1}$$

In Eq. (1), *A* is cross-sectional area, *I* is area moment of inertia, *G* is shear modulus, *E* is Young's modulus, $\bar{k}$ is shear coefficient, $\bar{m}$ is mass per unit length. $y_n(x,t)$ and $\theta_n(x,t)$ are n th beam span's deflection function and rotation function, respectively (n=1, 2, 3).

If the motion of the beam is harmonic and separation of variables method is applied, the following equation is obtained:

$$\frac{AG}{\bar{k}L^2}\frac{d^2 y_n(z)}{dz^2}-\frac{AG}{\bar{k}L}\frac{d\theta_n(z)}{dz}+\bar{m}\omega^2 y_n(z)-k_{sn}y_n(z)=0$$
$$\frac{EI}{L^2}\frac{d^2\theta_n(z)}{dz^2}+\frac{AG}{\bar{k}L}\frac{dy_n(z)}{dz}+\left(\frac{\bar{m}I\omega^2}{A}-\frac{AG}{\bar{k}}\right)\theta_n(z)=0 \tag{2}$$

where z=x/L and ω is natural frequency.

The solution is assumed as:

$$y_n(z)=\left\{\bar{C}\right\}_n e^{isz}$$
$$\theta_n(z)=\left\{\bar{D}\right\}_n e^{isz} \tag{3}$$

Substituting Eq.(3) into Eq.(2), yn(z) and θn(z) functions are given in Eq.(4) and Eq. (5), respectively.

$$y_n(z)=(\bar{C}_{n1}e^{is_{n1}z}+\bar{C}_{n2}e^{is_{n2}z}+\bar{C}_{n3}e^{is_{n3}z}+\bar{C}_{n4}e^{is_{n4}z}) \tag{4}$$

$$\theta_n(z)=(K_{n1}\bar{C}_{n1}e^{is_{n1}z}+K_{n2}\bar{C}_{n2}e^{is_{n2}z}+K_{n3}\bar{C}_{n3}e^{is_{n3}z}+K_{n4}\bar{C}_{n4}e^{is_{n4}z}) \tag{5}$$

where $K_{nm}=\dfrac{-\left(\dfrac{AG}{\bar{k}L^2}\right)+\left(m\omega^2\right)-k_{sn}}{\left(\dfrac{AG}{\bar{k}L^2}\right)is_j}$; m=1,2,3,4; j=1,2,3,4

The bending moment function and shear force function are defined in Eq.(6) and Eq.(7), respectively.

$$M_n(z)=\frac{EI}{L}\frac{d\theta_n(z)}{dz} \tag{6}$$

$$Q_n(z)=\frac{AG}{\bar{k}L}\frac{dy_n(z)}{dz}-\frac{AG}{\bar{k}}\theta_n(z) \tag{7}$$

3. DYNAMIC STIFFNESS METHOD (DSM) FOR CALCULATING NATURAL FREQUENCIES

DSM is a technique that can be used for calculating exact natural frequencies using exact mode shapes. First of all, the dynamic stiffness matrix should be obtained. The dynamic stiffness matrix can be constructed by using end displacements and end forces of beam. The vector of end displacements of beam and the vector of coefficients are given in Eqs. (8) and (9), respectively.

$$\delta_n=[y_{n0}\quad \theta_{n0}\quad y_{n1}\quad \theta_{n1}]^T \tag{8}$$

$$\bar{C}_n=[\bar{C}_{n1}\quad \bar{C}_{n2}\quad \bar{C}_{n3}\quad \bar{C}_{n4}]^T \tag{9}$$

where

$$y_{n0}=y_n(z=0),\theta_{n0}=\theta_n(z=0),y_{n1}=y_n(z=1),\theta_{n1}=\theta_n(z=1)$$

Eqs. (8) and (9) can be rewritten in the form below:

$$\begin{bmatrix} y_{n0}\\ \theta_{n0}\\ y_{n1}\\ \theta_{n1}\end{bmatrix}=\begin{bmatrix} 1 & 1 & 1 & 1\\ K_{n1} & K_{n2} & K_{n3} & K_{n4}\\ e^{is_{n1}} & e^{is_{n2}} & e^{is_{n3}} & e^{is_{n4}}\\ K_{n1}e^{is_{n1}} & K_{n2}e^{is_{n2}} & K_{n3}e^{is_{n3}} & K_{n4}e^{is_{n4}}\end{bmatrix}\begin{bmatrix}\bar{C}_{n1}\\ \bar{C}_{n2}\\ \bar{C}_{n3}\\ \bar{C}_{n4}\end{bmatrix} \tag{10}$$

The closed form of Eq. (10) is given in Eq. (11):

$$\delta_n=\Delta_n\bar{C}_n \tag{11}$$

where

$$\Delta_n=\begin{bmatrix} 1 & 1 & 1 & 1\\ K_{n1} & K_{n2} & K_{n3} & K_{n4}\\ e^{is_{n1}} & e^{is_{n2}} & e^{is_{n3}} & e^{is_{n4}}\\ K_{n1}e^{is_{n1}} & K_{n2}e^{is_{n2}} & K_{n3}e^{is_{n3}} & K_{n4}e^{is_{n4}}\end{bmatrix}$$

The end forces of the beam is given in vector form in Eq. (12):

$$F=[Q_{n0}\quad M_{n0}\quad Q_{n1}\quad M_{n1}]^T \tag{12}$$

where

$$Q_{n0}=Q_n(z=0),M_{n0}=M_n(z=0),Q_{n1}=Q_n(z=1),M_{n1}=M_n(z=1)$$

Eqs. (12) and (9) can be written in the following form:

$$\begin{bmatrix} Q_{n0} \\ M_{n0} \\ Q_{n1} \\ M_{n1} \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \Lambda_1 & \Lambda_2 & \Lambda_3 & \Lambda_4 \\ -\lambda_1 e^{is_{n1}} & -\lambda_2 e^{is_{n2}} & -\lambda_3 e^{is_{n3}} & -\lambda_4 e^{is_{n4}} \\ -\Lambda_1 e^{is_{n1}} & -\Lambda_2 e^{is_{n2}} & -\Lambda_3 e^{is_{n3}} & -\Lambda_4 e^{is_{n4}} \end{bmatrix} \begin{bmatrix} \bar{C}_{n1} \\ \bar{C}_{n2} \\ \bar{C}_{n3} \\ \bar{C}_{n4} \end{bmatrix} \tag{13}$$

where $\lambda_j = \frac{AG}{\bar{k}L} is_{nj} - \frac{AG}{\bar{k}} K_{nj}; \Lambda_j = \frac{EI}{L} is_{nj} K_{nj}; j = 1,2,3,4$

The closed form of Eq. (13) can be written as:

$$F_n = \kappa_n \bar{C}_n \tag{14}$$

where

$$\kappa_n = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \Lambda_1 & \Lambda_2 & \Lambda_3 & \Lambda_4 \\ -\lambda_1 e^{is_{n1}} & -\lambda_2 e^{is_{n2}} & -\lambda_3 e^{is_{n3}} & -\lambda_4 e^{is_{n4}} \\ -\Lambda_1 e^{is_{n1}} & -\Lambda_2 e^{is_{n2}} & -\Lambda_3 e^{is_{n3}} & -\Lambda_4 e^{is_{n4}} \end{bmatrix}$$

Eqs.(11) and (14) are used to construct the dynamic stiffness matrix of the n th span of the Timoshenko beam on elastic foundation.

$$F_n = \kappa_n \Delta_n^{-1} \bar{C}_n \tag{15}$$

In Eq.(15), $\kappa_n \Delta_n^{-1}$ represents the dynamic stiffness matrix of the n th span. The natural frequencies of the beam are calculated by equating the determinant of assembly of $\kappa_1 \Delta_1^{-1}, \kappa_2 \Delta_2^{-1}$ and $\kappa_3 \Delta_3^{-1}$ to zero. It should be noted that the related rows and columns of dynamic stiffness matrix of the beam are erased according to boundary conditions.

4. NUMERICAL ANALYSIS AND DISCUSSION

A three-span Timoshenko beam on Winkler foundation is used for numerical analysis with the following properties: E=2x10^7 kN/m^2, G=7692308 kN/m^2, $\bar{k}$ =1.2, ρ=25 kN/m^3, L=6 m, b=1 m.

The boundary conditions are same for each span and given below.

$$y_n(z=0)=0, M_n(z=0)=0, y_n(z=1)=0, M_n(z=1)=0$$

The analyses are performed for constant k_{s2} and k_{s3} with varying k_{s1}, constant k_{s1} and k_{s3} with varying k_{s2}, constant spring stiffnesses with various beam height values. The first five natural frequencies of the beam are presented in Tables (1-3). It should be noted that SAP2000 results are obtained by dividing spans into 1 cm segments for accuracy.

Table.1. First five natural frequencies (h=0.75 m, k_{s2}=10000 kN/m, k_{s3}=10000 kN/m)

	k_{s1} (kN/m)									
Natural Frequency (Hz)	5000		10000		15000		20000		25000	
	DSM	SAP2000	DSM	SAP2000	DSM	SAP2000	DSM	SAP2000	DSM	SAP2000
1st Mode	28.0633	27.9550	28.4933	28.3837	28.8479	28.737	29.1383	29.026	29.3761	29.263
2nd Mode	34.2849	34.1590	34.7388	34.6111	35.2293	35.100	35.7462	35.615	36.2791	36.146
3rd Mode	47.7167	47.5440	47.8373	47.6645	47.9680	47.795	48.1097	47.936	48.2637	48.089
4th Mode	97.9198	98.8130	98.0322	98.9265	98.1410	99.036	98.2463	99.142	98.3481	99.245
5th Mode	108.2491	109.1420	108.3922	109.2864	108.5373	109.433	108.6843	109.582	108.8329	109.732

Table.2. First five natural frequencies (h=0.75 m, k_{s1}=10000 kN/m, k_{s3}=10000 kN/m)

Natural Frequency (Hz)	k_{s2} (kN/m) 5000		10000		15000		20000		25000	
	DSM	SAP2000	DSM	SAP2000	DSM	SAP2000	DSM	SAP2000	DSM	SAP2000
1st Mode	28.0891	27.9805	28.4933	28.3837	28.8707	28.7591	29.2232	29.1088	29.5528	29.4346
2nd Mode	34.7134	34.5983	34.7388	34.6111	34.7642	34.624	34.7894	34.6364	34.8145	34.6488
3rd Mode	47.3841	47.2243	47.8373	47.6645	48.2986	48.1141	48.7671	48.5716	49.2421	49.0365
4th Mode	97.9209	98.8141	98.0322	98.9265	98.1421	99.0373	98.2505	99.1466	98.3576	99.2545
5th Mode	108.3738	109.2747	108.3922	109.2864	108.4108	109.2981	108.4295	109.3099	108.448	109.3218

Table.3. First five natural frequencies (k_{s1}= k_{s2}= k_{s3}=10000 kN/m)

Natural Frequency (Hz)	h(m) 0.55		0.65		0.75		0.85		0.95	
	DSM	SAP2000	DSM	SAP2000	DSM	SAP2000	DSM	SAP2000	DSM	SAP2000
1st Mode	23.5876	23.4357	25.8912	25.7561	28.4933	28.3837	31.2525	31.1793	34.0842	34.0593
2nd Mode	27.9915	27.8159	31.2701	31.1127	34.7388	34.6111	38.2623	38.1773	41.7629	41.7336
3rd Mode	37.6891	37.4618	42.8127	42.5751	47.8373	47.6645	52.7395	52.6141	57.4279	57.3597
4th Mode	75.5356	75.6747	87.0604	87.5351	98.0322	98.9265	108.3847	109.7625	118.095	119.9983
5th Mode	84.3307	84.4696	96.7702	97.2515	108.3922	109.2864	119.1587	120.5124	129.078	130.9151

It is seen from Table 1 that, the natural frequencies are increased with increasing spring stiffness of an outer span of three-span beam. Table 2 shows that there is also an augmentation in natural frequencies when the spring stiffness of middle span is increased. There is no significant difference between the particular increment of spring stiffness of middle span and an outer span on natural frequencies. Table 3 reveals that the natural frequencies are increased due to increasing beam height and higher modes are more sensitive to this effect.

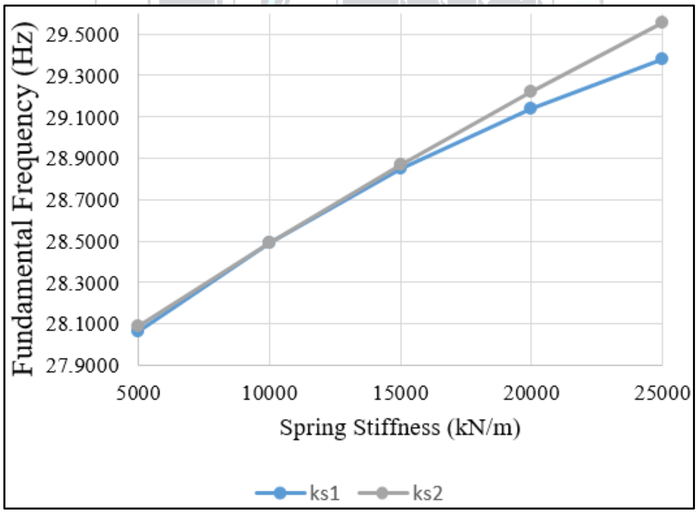

Figure.2. Fundamental frequencies for different k_{s1} and k_{s2} values

Figure 2 implies that fundamental frequency of three-span Timoshenko beam on Winkler foundation is more sensitive to spring stiffness of middle span in comparison with outer span especially for high stiffness values. Figure 3 represents the variation of first three natural frequencies with different beam height values.

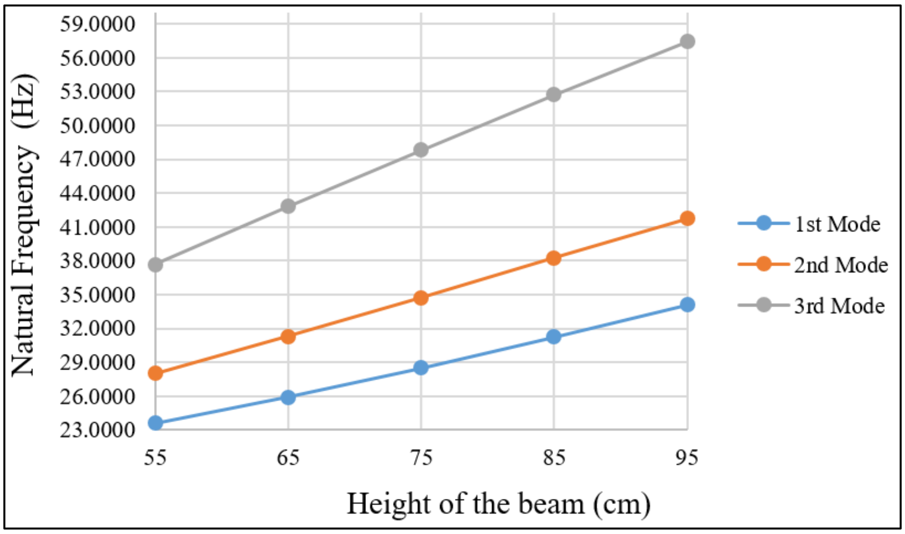

Figure.3. First three natural frequencies for different h values ($k_{s1}=k_{s2}=k_{s3}=10000$ kN/m)

6. CONCLUSIONS

The first five exact natural frequencies of three-span Timoshenko beams on Winkler foundation are obtained using dynamic stiffness approach. The spring stiffnesses of middle span is more effective than outer span. Different beam height values are used in the numerical analysis and effects on natural frequencies are observed. SAP2000 provides fairly well results when segment number increased sufficiently. The DSM can be used for calculating exact natural frequencies of multi-span Timoshenko beams on elastic foundation with different support conditions and foundation models.

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