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Annihilator of Generalized Derivations with Power Values in Rings and Algebras

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Abstract
Let $\mathcal{F}, \mathcal{G}$ be two generalized derivations of prime ring $\mathcal{R}$ with characteristic different from 2 with associated derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively. We use the symbols $\mathcal{C} = \mathcal{Z}(\mathcal{U})$ and $\mathcal{U}$ to denote the the extended centroid of $\mathcal{R}$ and Utumi ring of quotient of $\mathcal{R}$ respectively. Let $0 \neq a \in \mathcal{R}$ and $\mathcal{F}$ and $\mathcal{G}$ satisfy $a\{(\mathcal{F}(xy) + \mathcal{G}(yx))^m - [x,y]^n\} = 0$ for all $x, y \in \mathcal{J}$, a nonzero ideal, where m and n are natural numbers. Then either $\mathcal{R}$ is commutative or there exists $c, b \in \mathcal{U}$ such that $\mathcal{F}(x) = cx$ and $\mathcal{G}(x) = bx$ for all $x \in \mathcal{R}$.

Keywords: *Semiprime rings; Generalized derivations; extended centroid; Utumi quotient ring.*

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1. Introduction

For any $x, y \in \mathcal{R}$ we use the symbol $[x, y]$ to denote the commutator $xy - yx$ and $x \circ y$ to denote ant-icommutator $xy + yx$. Recall that a ring $\mathcal{R}$ is prime if for any $a, b \in R$, $a\mathcal{R}b = \{0\}$ implies that either $a = 0$ or $b = 0$ and is semiprime if for any $a \in \mathcal{R}$, $a\mathcal{R}a = \{0\}$ implies that $a = 0$. A map $\mathcal{D} : \mathcal{R} \to \mathcal{R}$ is said to be a derivation if $\mathcal{D}$ is additive and $\mathcal{D}(xy) = \mathcal{D}(x)y + x\mathcal{D}(y)$ for all $x, y \in \mathcal{R}$. If $\mathcal{D}$ can be written as $\mathcal{D}(x) = [a, x]$ for all $x \in \mathcal{R}$, then $\mathcal{D}$ is called an inner derivation for some $a \in \mathcal{R}$. Brešar [2] brought out the definition of generalized derivation. A map $\mathcal{F} : \mathcal{R} \to \mathcal{R}$ is said to be a generalized derivation if it is additive and satisfies $\mathcal{F}(xy) = \mathcal{F}(x)y + x\mathcal{D}(y)$ for all $x, y \in \mathcal{R}$ and a derivation $\mathcal{D}$.

Daif et al. [4, Theorem 2] showed that if $\mathcal{R}$ is a semiprime ring with $\{0\} \neq \mathcal{J}$ ideal and $\mathcal{D}$ is a derivation of $\mathcal{R}$ such that $\mathcal{D}([x, y]) = [x, y]$ for any $x, y \in \mathcal{J}$, then $\mathcal{J}$ is contained in the centre of $\mathcal{R}$. Later Quadri et al. [12] discussed the commutativity of prime rings for generalized derivation instead of derivation. Further, Dhara [7] studied the result of Quadri et al. in semiprime ring. Filippis et al. [6] studied that if $\mathcal{F}$ satisfies $(\mathcal{F}([x, y]))^n = [x, y]$ for all $x, y \in \mathcal{J}$, where $\{0\} \neq \mathcal{J}$ is an ideal of a prime ring $\mathcal{R}$, n is a fixed natural number and $\mathcal{F}$ is a generalized derivations of $\mathcal{R}$, they conclude that either $\mathcal{D} = 0$, $n = 1$ and $\mathcal{F}(x) = x\ R$ for all $x \in \mathcal{R}$ or $\mathcal{R}$ is commutative.

As $\mathcal{F}$ is additive, further the above identity can be written as $(\mathcal{F}(xy) - \mathcal{F}(yx))^n = [x, y]$. Form this point of view there is a question what happen if we take two generalized derivations instead of generalized derivation. Following this line, we prove:

Theorem 1.1. *Let $\mathcal{R}$ be a prime ring with characteristic different from 2, $\mathcal{F}$ and $\mathcal{G}$ are generalized derivations of $\mathcal{R}$. Let $0 \neq a \in \mathcal{R}$ and $\mathcal{F}$ and $\mathcal{G}$ satisfy $a\{(\mathcal{F}(xy) + \mathcal{G}(yx))^m - [x, y]^n\} = 0$ for all $x, y \in \mathcal{J}$, a nonzero ideal, where m and n are natural numbers. Then we have exactly one of the following:*

1. *$\mathcal{R}$ is commutative;*

2. *there exists $c, b \in \mathcal{U}$, Utumi ring of quotient of $\mathcal{R}$ such that $\mathcal{F}(x) = cx$ and $\mathcal{G}(x) = bx$ for all $x \in \mathcal{R}$.*

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Theorem 1.2. *Let $\mathcal{R}$ be a 2-torsion free semiprime ring, $\mathcal{F}$ and $\mathcal{G}$ are generalized derivations of $\mathcal{R}$ with associated derivations $\mathcal{D}_1$, $\mathcal{D}_2$ respectively. Let $0 \neq a \in \mathcal{R}$ and $\mathcal{F}$, $\mathcal{G}$ satisfy $a\{(\mathcal{F}(xy) + \mathcal{G}(yx))^m - [x, y]^n\} = 0$ for all $x, y \in \mathcal{J}$, a nonzero ideal, where m and n are natural numbers. Then $\mathcal{R}$ contains an ideal which is central.*

2. The results on two sided ideals

Theorem 1.1 Let $\mathcal{R}$ be a prime ring with characteristic different from 2, $\mathcal{F}$ and $\mathcal{G}$ are generalized derivations of $\mathcal{R}$. Let $0 \neq a \in \mathcal{R}$ and $\mathcal{F}$ and $\mathcal{G}$ satisfy $a\{(\mathcal{F}(xy) + \mathcal{G}(yx))^m - [x, y]^n\} = 0$ for all $x, y \in \mathcal{J}$, a nonzero ideal, where m and n are natural numbers. Then we have exactly one of the following:

1. $\mathcal{R}$ is commutative;
2. there exists $c, b \in \mathcal{U}$, Utumi ring of quotient of $\mathcal{R}$ such that $\mathcal{F}(x) = cx$ and $\mathcal{G}(x) = bx$ for all $x \in \mathcal{R}$.

Proof If both $\mathcal{F}$ and $\mathcal{G}$ are zero, then $a[x, y]^n = 0$ for all $x, y \in \mathcal{J}$. Since $[x, y]$ is multilinear, by [16], we have $a[x, y] = 0$. Replacing x by zx to get $az[x, y] = 0$. Using primeness of $\mathcal{R}$ and $a \neq 0$, we have $[x, y] = 0$ and hence $\mathcal{R}$ is commutative. Suppose atleast one of $\mathcal{F}, \mathcal{G} \neq 0$, then from the hypothesis, we have

$$a\{(\mathcal{F}(xy) + \mathcal{G}(yx))^m - [x, y]^n\} = 0 \quad \text{for all} \quad x, y \in \mathcal{J}. \tag{2.1}$$

By [10], $\mathcal{F}(x) = cx + \mathcal{D}_1(x)$ and $\mathcal{G}(x) = bx + \mathcal{D}_2(x)$ for some $c, b \in \mathcal{U}$, for all $x \in \mathcal{U}$ and derivations $\mathcal{D}_1$ and $\mathcal{D}_2$. Hence $\mathcal{J}$ satisfies

$$a\{(\mathcal{F}(xy) + \mathcal{G}(yx))^m - [x, y]^n\} = 0 \tag{2.2}$$

By [3, Theorem 2], $\mathcal{U}$ satisfies this GPI, we have

$$a\{(cxy + D_1(xy)) + (byx + D_2(yx)))^m - [x, y]^n\} = 0 \ \text{ for all } \ x, y \in \mathcal{U}. \tag{2.3}$$

Now we have the following cases:
Case I: Let $\mathcal{D}_1$ and $\mathcal{D}_2$ are inner derivations of $\mathcal{U}$, i.e $\mathcal{D}_1(x) = [p, x]$ and $\mathcal{D}_2(x) = [q, x]$ for all $x \in \mathcal{U}$ and for some $p, q \in \mathcal{U}$. Then our identity $a\{(cxy + D_1(xy)) + (byx + D_2(yx)))^m - [x, y]^n\} = 0$ becomes

$$a\{(cxy + [p, xy]) + (byx + [q, yx]))^m - [x, y]^n\} = 0 \text{ for all } x, y \in \mathcal{U}. \tag{2.4}$$

Hence $a\{(axy + [p, xy]) + (byx + [q, yx]))^m - [x, y]^n\} = 0$ is a nontrivial generalized polynomial identity (GPI) for $\mathcal{U}$. Denote by $\mathcal{H}$ either the algebraic closure of $\mathcal{C}$ or $\mathcal{C}$ according as $\mathcal{C}$ is either infinite or finite respectively. By ([9, Proposition]), $a\{(cxy + [p, xy]) + (byx + [q, yx]))^m - [x, y]^n\} = 0$ is also a GPI for $\mathcal{U} \otimes_{\mathcal{C}} \mathcal{H}$. By [17, Theorem 2.5 and Theorem 3.5], $\mathcal{U} \otimes_{\mathcal{C}} \mathcal{H}$ is centrally closed prime $\mathcal{H}$-algebra, by replacing $\mathcal{R}$, $\mathcal{C}$ with $\mathcal{U} \otimes_{\mathcal{C}} \mathcal{H}$ and $\mathcal{H}$, respectively, we may assume that $\mathcal{R}$ is centrally closed and $\mathcal{C}$ is either finite or algebraically closed. By Martindale's theorem [18], $\mathcal{R}$ is then a primitive ring with nonzero socle $\mathcal{E}$ with $\mathcal{C}$ as the associated division ring. Hence by Jacobson's theorem [19, p.75] $\mathcal{R} \cong \mathcal{M}_k(\mathcal{C})$. If $k = 1$, then $\mathcal{R}$ is commutative. Now assume $\dim_{\mathcal{C}} \mathcal{V} \geq 2$.

Now we prove that for any $u \in \mathcal{V}$, u and qu are linearly $\mathcal{C}$-dependent. Let on contrary that u and qu are linearly independent for some $u \in \mathcal{V}$.

If pu is not a member of the span of $\{u, qu\}$, then $\{u, pu, qu\}$ is independent. By the density of ring $\mathbb{R}$, there exist $y, x \in \mathbb{R}$ such that

$$xqu = -u, xu = 0, ypu = u, yu = u, xpu = 0, yqu = u.$$

Then multiplying (2.4) by u from rihgt to have

$$0 = a\{(cxy + [p, xy]) - (byx + [q, yx]))^m - [x, y]^n\}u = 0 = au.$$

If for any $v \in \mathcal{V}$, $\{u, v\}$ is linearly $\mathcal{C}$-dependent, then $av = 0$. Since $a \neq 0$, there exists $w \in \mathcal{V}$ such that $aw \neq 0$ and so $\{w, v\}$ are linearly $\mathcal{C}$-independent. Also $a(w + v) = aw \neq 0$ and $a(w - v) = aw \neq 0$. By the above argument, it follows that w and cw are linearly $\mathcal{C}$-dependent, as are $\{w + v, c(w + v)\}$ and $\{w - v, c(w - v)\}$. Therefore there exist $\gamma_w, \gamma_{w+v}, \gamma_{w-v} \in \mathcal{C}$ such that

$$qw = \gamma_w w, \quad (w + v) = \gamma_{w+v}(w + v), \quad q(w - v) = \gamma_{w-v}(w - v).$$

Thus we have

$$\gamma_w w + qv = \gamma_{w+v} w + \gamma_{w+v} v \tag{2.5}$$

and

$$\gamma_w w - qv = \gamma_{w-v} w - \gamma_{w-v} v. \tag{2.6}$$

Now (2.5) and (2.6) together yields

$$(2\gamma_w - \gamma_{w+v} - \gamma_{w-v})w + (\gamma_{w-v} - \gamma_{w+v})v = 0 \tag{2.7}$$

and

$$2qv = (\gamma_{w+v} - \gamma_{w-v})w + (\gamma_{w+v} + \gamma_{w-v})v. \tag{2.8}$$

By (2.7), and since $\{w, v\}$ are $\mathcal{C}$-independent, $2\gamma_w - \gamma_{w+v} - \gamma_{w-v} = 0$ and $\gamma_{w-v} - \gamma_{w+v} = 0$. These relations imply by using char $(\mathcal{R}) \neq 2$, that $\gamma_w = \gamma_{w+v} = \gamma_{w-v}$. By (2.8) it follows $qv = \gamma_w v$. This leads to a contradiction with the fact that $\{v, qv\}$ is linear $\mathcal{C}$-independent.

In light of this, we may assume that for any $v \in \mathcal{V}$ there exists a suitable $\gamma_v \in \mathcal{V}$ such that $qv = \gamma_v v$, and standard argument shows that there is $\gamma \in \mathcal{C}$ such that $qv = \gamma v$ for all $v \in \mathcal{V}$. Then by standard argument, we have $q \in \mathcal{C}$. similarly with neccessary variation, we can show that $p \in \mathcal{C}$.

Case 2 Let $\mathcal{D}_1$ and $\mathcal{D}_2$ are not both inner derivations of $\mathcal{U}$. Then $\mathcal{D}_2(y) = [p, y] + \beta\mathcal{D}_1(y)$ for some $p \in \mathcal{U}$ and $\beta \in \mathcal{C}$. If either $\beta = 0$ or $\mathcal{D}_2$ is inner, then $\mathcal{D}_1$ is also inner which contradicts. So, $\beta \neq 0$ as wel as $\mathcal{D}_2$ is not inner. Then by (2.3), we have

$$a\{(cxy + \mathcal{D}_1(x)y + x\mathcal{D}_1(y)) + (byx + [p, yx] + \beta(\mathcal{D}_1(y)x + y\mathcal{D}_1(x)))^m - [x, y]^n\} = 0$$

for any $y, x \in \mathcal{U}$.

By the use of Kharchenko's Theorem [8], we have either $\mathcal{D}_1$ is inner or $\mathcal{U}$ satisfies $a\{(cxy + x_1y + xy_1) + (byx + [p, yx] + \beta(y_1x + yx_1)))^m - [x, y]^n\}$ i.e

$$a\{(cxy + x_1y + xy_1) + (byx + [p, yx] + \beta(y_1x + yx_1)))^m - [x, y]^n\} = 0 \tag{2.9}$$

for any $y, x, y_1, x_1 \in \mathcal{U}$.

If $\mathcal{D}_1$ is inner then $\mathcal{D}_2$ will be a inner derivation of form $\mathcal{D}_2(y) = [p + \beta q, y]$ for some $p, q \in \mathcal{U}$ which is a contradiction. In particular, putting $y = 0$ in (2.9), we have

$$a(xy_1 + \beta y_1x)^m = 0 \text{ for any } x, y_1 \in \mathcal{J}.$$

Since $xy_1 + \beta y_1x$ is multilinear polynomial, by [16], we have $a(xy_1 + \beta y_1x) = 0$. Further, this can be written as $a(xy_1 - y_1x + \beta y_1x + y_1x) = 0 = a((xy_1 - y_1x) + (y_1x + \beta y_1x)) = a(xy_1 - y_1x)$. By primeness of $\mathcal{R}$, we have $[\mathcal{R}, \mathcal{R}] = \{0\}$ and hence $\mathcal{R}$ is commutative.

Case 3 Now assume both $\mathcal{D}_1$ and $\mathcal{D}_2$ are Outer. By Kharchenko's Theorem [8], we have

$$a\{(cxy + x_1y + xy_1) + (byx + y_2x + yx_2))^m - [x, y]^n\} = 0$$

for any $y_1, x_1, y, x, y_2, x_2 \in \mathbb{U}$. For $y = 0$, we have

$$a(xy_1 + y_2x)^m = 0 \tag{2.10}$$

Since $xy_1 + y_2x$ is multilinear polynomial, by [16], we have $a(xy_1 + y_2x) = 0$. By primeness of $\mathcal{R}$, we have $\mathcal{R} \circ \mathcal{R} = \{0\}$ and hence $\mathcal{R}$ is commutative.

Now we have the following corollaries

Corollary 2.1. Let $\mathcal{F}$ and $\mathcal{G}$ be two generalized derivations of $\mathcal{R}$, a prime ring having characteristic different from 2, associated with nonzero derivations and $\{0\} \neq \mathcal{J}$ an ideal of $\mathcal{R}$. Let $0 \neq a \in \mathcal{R}$ and $\mathcal{F}$ and $\mathcal{G}$ satisfy $a\{(\mathcal{F}(xy) + \mathcal{G}(yx))^m - [x, y]^n\} = 0$ for all $x, y \in \mathcal{J}$ and for some fixed natural numbers m, n. Then $\mathcal{R}$ is commutative.

Corollary 2.2. Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two derivations of $\mathcal{R}$, a prime ring having characteristic different from 2 and $\{0\} \neq \mathcal{J}$ an ideal of $\mathcal{R}$. Let $0 \neq a \in \mathcal{R}$ and $\mathcal{D}_1$ and $\mathcal{D}_2$ satisfy $a\{(\mathcal{D}_1(xy) + \mathcal{D}_2(yx))^m - [x, y]^n\} = 0$ for all $x, y \in \mathcal{J}$ and for some fixed natural numbers m, n. Then $\mathcal{R}$ is commutative.

In the following example, we demonstrate that primeness of the ring is essential in the hypothesis of the **Theorem1.1**.

Example 2.1. Let $\mathcal{R}_1$ be any commutative ring. Let $\mathcal{R} = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathcal{R}_1 \right\}$ and $\mathcal{J} = \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathcal{R}_1 \right\}$. Define the following maps: $\mathcal{F}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & c \\ 0 & 0 \end{pmatrix}$, $\mathcal{G}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} a & a+b \\ 0 & 0 \end{pmatrix}$, $\mathcal{D}_1\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}$ and $\mathcal{D}_2\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} 0 & a-c \\ 0 & 0 \end{pmatrix}$. Then $\mathcal{F}$ and $\mathcal{G}$ are generalized derivations with associated derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively satisfying $a\{(\mathcal{F}(xy) + \mathcal{G}(yx))^m - [x, y]^n\} = 0$ for all $x, y \in \mathcal{J}$, where $m \geq 1$, $n \geq 1$ are fixed integer. Then neither $\mathcal{R}$ is commutative nor $\mathcal{F}$ and $\mathcal{G}$ can be written as $\mathcal{F}(x) = ax$ and $\mathcal{G}(x) = bx$ for all $x \in \mathcal{R}$ because of $\mathcal{D}_1$ and $\mathcal{D}_2$ are nonzero.

3. Results on semiprime rings

Theorem 1.2 Let $\mathcal{R}$ be a 2-torsion free semiprime ring, $\mathcal{F}$ and $\mathcal{G}$ are generalized derivations of $\mathcal{R}$ with associated derivations $\mathcal{D}_1$, $\mathcal{D}_2$ respectively. Let $0 \neq a \in \mathcal{R}$ and $\mathcal{F}$ and $\mathcal{G}$ satisfy $a\{(\mathcal{F}(xy) + \mathcal{G}(yx))^m - [x, y]^n\} = 0$ for all $x, y \in \mathcal{J}$, a nonzero ideal, where m and n are fixed natural numbers. Then $\mathcal{R}$ contains an ideal which is central.
Proof By [3] and [9], $\mathcal{J}$, $\mathcal{R}$ and $\mathcal{U}$ satisfy the same GPIs as well as same differential identities. By [10], $\mathcal{F}(x) = cx + \mathcal{D}_1(x)$ and $\mathcal{G}(x) = bx + \mathcal{D}_2(x)$ for some $c, b \in \mathcal{U}$ and derivations $\mathcal{D}_1$ and $\mathcal{D}_2$. Hence

$$a\{((cxy + \mathcal{D}_1(xy)) + (byx + \mathcal{D}_2(yx))^m - [x, y]^n\} = 0 \text{ for all } x, y \in \mathcal{U}. \tag{3.1}$$

Let $\mathcal{P}(\mathcal{C})$ denotes a collection of all maximal ideals of $\mathcal{C}$ and $\mathcal{M} \in \mathcal{P}(\mathcal{C})$. By the theory of orthogonal completions for semiprime rings ([9, p.31-32]), $\mathcal{MU}$ is a prime ideal of $\mathcal{U}$ which is invariant under all derivations of $\mathcal{U}$. By [1, Lemma 1 and Theorem 1], we have $\bigcap\{\mathcal{MU} \mid \mathcal{M} \in \mathcal{M}(\mathcal{C})\ \} = 0$. Set $\overline{\mathcal{U}} = \mathcal{U}/\mathcal{MU}$. Then $\mathcal{D}_1$ and $\mathcal{D}_2$ induce the following derivations $\overline{\mathcal{D}_1}$ and $\overline{\mathcal{D}_2}$ on $\overline{\mathcal{U}}$ whic is defined as $\overline{\mathcal{D}_1}(\overline{x}) = \overline{\mathcal{D}_1(x)}$ and $\overline{\mathcal{D}_2}(\overline{x}) = \overline{\mathcal{D}_2(x)}$ for all $x \in \overline{\mathcal{U}}$. Therefore,

$$\overline{a}\{((\bar{c}\bar{x}\bar{y} + \bar{\mathcal{D}}_1(\bar{x}\bar{y})) + (\bar{b}\bar{y}\bar{x} + \bar{\mathcal{D}}_2(\bar{y}\bar{x})))^m - [\overline{x}, \overline{y}]^n\} = 0$$

for all $\overline{x}, \overline{y} \in \overline{\mathcal{U}}$. Using Theorem 1.1, we have simultaneously either $[\mathcal{U}, \mathcal{U}] \subseteq \mathcal{MU}$ or $\mathcal{D}_1(\mathcal{U}) \subseteq \mathcal{MU}$ as well as either $[\mathcal{U}, \mathcal{U}] \subseteq \mathcal{MU}$ or $\mathcal{D}_2(\mathcal{U}) \subseteq \mathcal{MU}$. This gives that $\mathcal{D}_1(\mathcal{U})[\mathcal{U}, \mathcal{U}] \subseteq \mathcal{MU}$ for all $\mathcal{M} \in \mathcal{P}(\mathcal{C})$ as well as $\mathcal{D}_2(\mathcal{U})[\mathcal{U}, \mathcal{U}] \subseteq \mathcal{MU}$ for all $\mathcal{M} \in \mathcal{P}(\mathcal{C})$. In either case we have $\mathcal{D}_i(\mathcal{U})[\mathcal{U}, \mathcal{U}] \subseteq \mathcal{MU}$ for all $\mathcal{M} \in \mathcal{P}(\mathcal{C})$, $i = 1, 2$ and hence $\mathcal{D}_i(\mathcal{U})[\mathcal{U}, \mathcal{U}] = 0$. Particularly, $\mathcal{D}_i(\mathcal{R})[\mathcal{R}, \mathcal{R}] = 0 = [\mathcal{D}_i(\mathcal{R}), \mathcal{R}]\mathcal{R}[\mathcal{D}_i(\mathcal{R}), \mathcal{R}] = 0$. As $\mathcal{R}$ is a semiprime ring, we obtain that $[\mathcal{D}_i(\mathcal{R}), \mathcal{R}] = 0$. Then by [20, Theorem 3], $\mathcal{R}$ contains a nonzero central ideal.

4. Results on Banach algebras

Singer et al. [14] showed that the image of a noncommutative Banach algebra under continuous derivation is contained in radical of the algebra. Sinclair [13] proved that every primitive ideals of the algebra is invariant under continuous derivation of Banach algebra. Recently, Park [11] proved that if $\mathcal{D}$ is a continuous linear derivation of a noncommutative Banach algebra $\mathcal{A}$ satisfies $[[\mathcal{D}(x), x], \mathcal{D}(x)] \in rad(\mathcal{A})$ for all $x \in \mathcal{A}$, then $\mathcal{D}(\mathcal{A}) \subseteq rad(\mathcal{A})$. De Filippis [5] extended the Park's result to generalized derivations.

Inspire by these results we prove:

Theorem 4.1. *Let $\mathcal{A}$ be a noncommutative Banach algebra and $\mathcal{F}(x) = cx + \mathcal{D}_1(x)$ and $\mathcal{G}(x) = bx + \mathcal{D}_2(x)$ are continuous generalized derivations with associated derivations $\mathcal{D}_1(x)$ and $\mathcal{D}_2(x)$ respectively. If $\mathcal{F}$ and $\mathcal{G}$ satisfy $a\{(\mathcal{F}(xy) + \mathcal{G}(yx))^m - [x, y]^n\} \in rad(\mathcal{A}))$ for all $x, y \in \mathcal{A}$, then $\mathcal{D}_i(\mathcal{A}) \subseteq rad(\mathcal{A})$ for i=1, 2 .*

By a Banach algebra, we mean a complex normed algebra $\mathcal{A}$ whose underlying vector space is Banach space. Here Jacobson radical of $\mathcal{A}$ is defined as the intersection of all primitive ideals of $\mathcal{A}$ and we use the notation $rad(\mathcal{A})$ to denote it.

Proof of Theorem 4.1 We know that left multiplication mappings are continuous. Also $\mathcal{F}, \mathcal{G}$ are continuous by hypothesis. So $\mathcal{D}_1$ and $\mathcal{D}_2$ are continuous. By [13] it is clear that primitive ideals are invariant under continuous generalized derivations $\mathcal{F}, \mathcal{G}$. Assuming $\mathcal{A}/\mathcal{P} = \bar{\mathcal{A}}$ for any primitive ideal $\mathcal{P}$. Thus generalized derivations $\mathcal{F}_p, \mathcal{G}_q : \bar{\mathcal{A}} \to \bar{\mathcal{A}}$ is defined by $\mathcal{F}_p(\bar{x}) = \mathcal{F}_p(x + P) = \mathcal{F}(x) + P = ax + \mathcal{D}_1(x) + P$ and $\mathcal{G}_q(\bar{x}) = \mathcal{G}_q(x + P) = \mathcal{G}(x) + P = bx + \mathcal{D}_2(x) + P$ for all $\bar{x} \in \bar{\mathcal{A}}$, where $\mathcal{A}/\mathcal{P} = \bar{\mathcal{A}}$ is a factor Banach algebra. As $\mathcal{P}$ is primitive, the factor algebra $\bar{\mathcal{A}}$ is also primitive and hence it is prime and semisimple. The hypothesis $a\{(\mathcal{F}(xy) + \mathcal{G}(yx))^m - [x, y]^n\} \in rad(\mathcal{A}))$

yields that $\bar{a}\{(\mathcal{F}_p(\bar{x}\bar{y}) + \mathcal{G}_q(\bar{y}\bar{x}))^m - [\bar{x}, \bar{y}]^n\} = \bar{0}$ for all $\bar{x}, \bar{y} \in \bar{A}$. By Theorem 1.1, we have either $\bar{A}$ is commutative or $\bar{\mathcal{D}_1} = \bar{0}$ and $\bar{\mathcal{D}_2} = \bar{0}$.
Let $\bar{\mathcal{A}}$ be commutative. By [15], $\bar{\mathcal{D}_1}$ and $\bar{\mathcal{D}_2}$ are continuous in $\bar{\mathcal{A}}$. By [14], $\bar{\mathcal{D}_1} = \bar{0}$ and $\bar{\mathcal{D}_2} = \bar{0}$ in $\bar{\mathcal{A}}$. So, in both cases, we have $\bar{\mathcal{D}_1} = \bar{0}$ and $\bar{\mathcal{D}_2} = \bar{0}$ in $\bar{\mathcal{A}}$, i.e $\mathcal{D}_i(A) \subseteq \mathcal{P}$ for any primitive ideal $\mathcal{P}$ of $\mathcal{A}$ and hence $\mathcal{D}_i(A) \subseteq rad(\mathcal{A})$ for $i = 1, 2$.

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