On $C$-Bochner Curvature Tensor in $(LCS)_n$-Manifolds

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Abstract
The object of the present paper is to study the $C$–Bochner curvature tensor in $(LCS)_n$-manifolds.

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1. Introduction

In 2003, Shaikh [18] introduced the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) with an example, which generalizes the notion of LP-Sasakian manifolds introduced by Matsumoto [14] and also by Mihai and Rosca [15]. Then Shaikh and Baishya [19] investigated the applications of $(LCS)_n$-manifolds to the general theory of relativity and cosmology. The $(LCS)_n$-manifolds are also studied by Atçeken et. al. [1, 2, 3, 11], Hui [10], Narain and Yadav [16] many authors.

Motivated by the studies of the above authors, in this paper we classify $(LCS)_n$-manifolds, which satisfy the curvature conditions $R(\xi, X)B = 0$, $B(\xi, X)P = 0$, $B(\xi, X)S = 0$ and $C$-Bochner flat, where $B$ is the $C$-Bochner curvature tensor, $P$ is the projective curvature tensor and $S$ is the Ricci tensor.

2. Preliminaries

An $n$-dimensional Lorentzian manifold $M$ is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric $g$, that is, $M$ admits a smooth symmetric tensor field $g$ of type $(0, 2)$ such that for each point $p \in M$, the tensor $g_p : T_p M \times T_p M \rightarrow \mathbb{R}$ is non-degenerate inner product of signature $(-, +, ..., +)$, where $T_p M$ denotes the tangent vector space of $M$ at $p$ and $\mathbb{R}$ is the real number space. A non-zero vector $v \in T_p M$ is said to be timelike (resp., non-spacelike, null, spacelike) if satisfies $g_p(v, v) < 0$ (resp., $\leq 0$, $= 0$, $> 0$) [17]. The category to which a given vector falls is called its casual charater.

Definition 1. In a Lorentzian manifold $(M, g)$, a vector field $P$ defined by

$$g(X, P) = A(X)$$

for any $X \in \Gamma(TM)$ is said to be a concircular vector field if

$$(\nabla_X A)Y = \alpha \{g(X, Y) + \omega(X)A(Y)\}$$

for $Y \in \Gamma(TM)$, where $\alpha$ is a nonzero scalar function, $A$ is a 1-form, $\omega$ is also closed 1-form, and $\nabla$ denotes the Levi-Civita connection on $M$. 

Let $M$ be a Lorentzian manifold admitting a unit timelike concircular vector field $\xi$, called the characteristic vector field of the manifold. Then we have
\[ g(\xi, \xi) = -1. \]
Since $\xi$ is a unit vector field, there exists a nonzero 1-form $\eta$ such that
\[ g(X, \xi) = \eta(X). \tag{1} \]

The equation of the following form holds:
\[ (\nabla_X \eta)Y = \alpha\{g(X, Y) + \eta(X)\eta(Y)\}, \alpha \neq 0 \tag{2} \]
for all $X, Y \in \Gamma(TM)$, where $\alpha$ is nonzero scalar function satisfying
\[ \nabla_X \alpha = X(\alpha) = d\alpha(X) = \rho \eta(X), \tag{3} \]
$\rho$ being a certain scalar function given by $\rho = -\xi(\alpha)$. Let us put
\[ \nabla_X \xi = \alpha \phi X, \tag{4} \]
then from (2) and (4), we can derive
\[ \phi X = X + \eta(X)\xi \tag{5} \]
which tells us that $\phi$ is symmetric $(1, 1)$-tensor. Thus the Lorentzian manifold $M$ together with the unit timelike concircular vector field $\xi$, its associated 1-form $\eta$ and $(1, 1)$-type tensor field $\phi$ is said to be a Lorentzian concircular structure manifold. A differentiable manifold $M$ of dimension $n$ is called $(LCS)$-manifold if it admits a $(1, 1)$-type tensor field $\phi$, a covariant vector field $\eta$ and a Lorentzian metric $g$ which satisfy
\[ \eta(\xi) = g(\xi, \xi) = -1, \tag{6} \]
\[ \phi^2 X = X + \eta(X)\xi, \tag{7} \]
\[ g(X, \xi) = \eta(X)\xi, \tag{8} \]
\[ \phi \xi = 0, \eta \circ \phi = 0, \tag{9} \]
for all $X \in \Gamma(TM)$. Particulariy, if we take $\alpha = 1$, then we can obtain the $LP$-Sasakian structure of Matsumoto [14].

Also, in an $(LCS)_n$-manifold $M$, the following conditions are satisfied
\[ \eta(R(X, Y)Z) = (\alpha^2 - \rho)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)], \tag{10} \]
\[ R(\xi, X)Y = (\alpha^2 - \rho)[g(X, Y)\xi - \eta(Y)X], \tag{11} \]
\[ R(X, Y)\xi = (\alpha^2 - \rho)[\eta(Y)X - \eta(X)Y], \tag{12} \]
\[ (\nabla_X \phi)Y = \alpha[g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \tag{13} \]
\[ S(X, \xi) = (n - 1)(\alpha^2 - \rho)\eta(X), \tag{14} \]
\[ S(\phi X, \phi Y) = S(X, Y) + (n - 1)(\alpha^2 - \rho)\eta(X)\eta(Y) \tag{15} \]
for all $X, Y, Z$ on $M$, where $R$ is the Riemannian curvature tensor and $S$ is the Ricci tensor. $Q$ is also the Ricci operator given by $S(X, Y) = g(QX, Y)$ [18].
S. Bochner [5] introduced a Kähler analogue of the Weyl conformal curvature tensor by purely formal considerations, which is now well known as the Bochner curvature tensor. A geometric meaning of the Bochner curvature tensor was given by D. E. Blair [4]. By using the Boothby-Wang fibration [6], M. Matsumoto and G. Chuman [13] constructed the C-Bochner curvature tensor from the Bochner curvature tensor.

The C-Bochner curvature tensor is given by

$$B(X, Y)Z = R(X, Y)Z + \frac{1}{n+3} [S(X, Z)Y - S(Y, Z)X + g(X, Z)QY$$

$$- g(Y, Z)QX + S(\phi X, Z)\phi Y - S(\phi Y, Z)\phi X$$

$$+ g(\phi X, Z)Q\phi Y - g(\phi Y, Z)Q\phi X + 2S(\phi X, Y)\phi Z$$

$$+ 2g(\phi X, Y)Q\phi Z - S(X, Z)\eta(Y)\xi$$

$$+ S(Y, Z)\eta(X)\xi - \eta(X)\eta(Z)QY + \eta(Y)\eta(Z)QX]$$

$$- \frac{p+n-1}{n+3} [g(\phi X, Z)Y - g(\phi Y, Z)\phi X + 2g(\phi X, Y)\phi Z]$$

$$- \frac{p-4}{n+3} [g(X, Z)Y - g(Y, Z)X]$$

$$+ \frac{p}{n+3} [g(X, Z)\eta(Y)\xi - g(Y, Z)\eta(X)\xi$$

$$+ \eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X],$$

where $S$ is the Ricci tensor of type $(0, 2)$, $Q$ is the Ricci operator defined by $g(QX, Y) = S(X, Y)$ and $p = \frac{n+3}{n+1}$, $r$ is the scalar curvature of the manifold.

The projective curvature tensor $P$ of $n$-dimensional Riemann manifold is defined by

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{(n-1)} [S(Y, Z)X - S(X, Z)Y],$$

where $S$ is the Ricci tensor of the manifold [21].

In $(LCS)_\alpha$-manifold $M$, the following conditions are satisfied

$$B(\xi, Y)Z = \left[\frac{4(\alpha^2 - \rho) + 2p - 4}{n+3}\right] [g(Y, Z)\xi - \eta(Z)Y]$$

$$+ \frac{2}{n+3} [\eta(Z)QY - S(Y, Z)\xi],$$

$$B(X, Y)\xi = \left[\frac{4(\alpha^2 - \rho) + 2p - 4}{n+3}\right] [\eta(Y)X - \eta(X)Y]$$

$$+ \frac{2}{n+3} [\eta(X)QY - \eta(Y)QX],$$

$$B(\xi, Y)\xi = \left[\frac{4(\alpha^2 - \rho) + 2p - 4}{n+3}\right] [\eta(Y)\xi + Y]$$

$$- \frac{2}{n+3} [QY + (n-1)(\alpha^2 - \rho)\eta(Y)\xi].$$

$$P(\xi, Y)Z = (\alpha^2 - \rho)g(Y, Z)\xi - \frac{1}{n-1}S(Y, Z)\xi$$

(21)

and

$$P(X, Y)\xi = P(\xi, Y)\xi = 0.$$ (22)

**Theorem 2.** If an $(LCS)_\alpha$-manifold $M$ is C-Bochner flat, then $M$ reduces to an $\eta$-Einstein Manifold.
Proof. Suppose that an \((LCS)_n\)-manifold \(M\) is \(C\)-Bochner flat. Then we have,

\[ B(X,Y)Z = 0. \]  

(23)

In (16), putting \(Z = \xi\), we have

\[
0 = R(X,Y)\xi + \frac{1}{n+3} [S(X,\xi)Y - S(Y,\xi)X \\
+ g(X,\xi)QY - g(Y,\xi)QX - S(X,\xi)\eta(Y)\xi \\
+ S(Y,\xi)\eta(X)\xi + \eta(X)QY - \eta(Y)QX] \\
- \frac{p-4}{n+3} [g(X,\xi)Y - g(Y,\xi)X] \\
+ \frac{2}{n+3} [g(X,\xi)\eta(Y)\xi - g(Y,\xi)\eta(X)\xi \\
+ \eta(Y)X - \eta(X)Y].
\]  

(24)

In (24), by using the equations (6),(8),(9),(12) and (14), we obtain

\[
0 = [\alpha^2 - \rho + \frac{2p-4}{n+3} - \frac{(n-1)(\alpha^2 - \rho)}{n+3}] [\eta(Y)X - \eta(X)Y] \\
+ \frac{2}{n+3} [\eta(X)QY - \eta(Y)QX].
\]  

(25)

Putting \(X = \xi\) in (25) and by using (14), we obtain

\[
\frac{2}{n+3} QY = [\alpha^2 - \rho + \frac{2p-4}{n+3} - \frac{3(n-1)(\alpha^2 - \rho)}{n+3}] \eta(Y)\xi \\
+ [\alpha^2 - \rho + \frac{2p-4}{n+3} - \frac{(n-1)(\alpha^2 - \rho)}{n+3}] Y,
\]  

(26)

which is equivalent to

\[ QY = [2p - 4 + 4(\alpha^2 - \rho)]Y + [2p - 4 - (\alpha^2 - \rho)(2n - 6)] \eta(Y)\xi. \]  

(27)

Inner product both sides of the equation by \(W \in \chi(M)\) and taking into account \(p = \frac{n+1}{n+1}\), we conclude

\[
S(Y,W) = [2(\alpha^2 - \rho) - (1 + \frac{r}{n+1})] g(Y,W) \\
+ [(3-n)(\alpha^2 - \rho) - (1 + \frac{r}{n+1})] \eta(Y)\eta(W).
\]

\[ \square \]

Theorem 3. Let \(M\) be an \((LCS)_n\)-manifold. Then, \(R(\xi,Y)B\) is always identically zero, for any \(Y \in \chi(M)\).

Proof. For any \(X,Y,U,W,Z \in \chi(M)\) on \(M\), we have

\[
- B(U,R(X,Y)W)Z - B(U,W)R(X,Y)Z.
\]  

(28)

In (28), for \(X = \xi\), we have

\[
- B(U,R(\xi,Y)W)Z - B(U,W)R(\xi,Y)Z.
\]  

(29)
By using (11) in (29), we obtain
\[
(R(\xi, Y)B)(U, W, Z) = (\alpha^2 - \rho) \left[ g(Y, B(U, W)Z)\xi - \eta(B(U, W)Z)Y \right. \\
- B \left( g(Y, U)\xi - \eta(U)Y, W \right)Z \\
- B(U, g(Y, W)\xi - \eta(W)Y)Z \\
- B(U, W) (g(Y, Z)\xi - \eta(Z)Y). \tag{30}
\]

Now, by using (18),(19) and choosing \(U = Z = \xi\), we obtain
\[
(R(\xi, Y)B)(\xi, W, \xi) = g(Y, A\eta(W) + AW - \frac{2}{n+3}QW - D\eta(W)\xi)\xi \\
- \eta(A\eta(W)\xi + AW - \frac{2}{n+3}QW - D\eta(W))Y \\
- 2\eta(Y) (A\eta(W)\xi + AW - \frac{2}{n+3}QW - D\eta(W)) \\
- A\eta(W)Y + 2A\eta(Y)W - \frac{4}{n+3}\eta(Y)QW + \frac{2}{n+3}\eta(W)QY \\
+ \eta(W) [A\eta(Y)\xi + AW - \frac{2}{n+3} - D\eta(Y)] \\
- Ag(W, Y)\xi + \frac{2}{n+3}S(W, Y)\xi, \tag{31}
\]

where, \(A = \frac{4(\alpha^2 - \rho)+2p-4}{n+3}\) and \(D = \frac{2(n-1)(\alpha^2 - \rho)}{n+3}\).

We easily obtain from (31) that
\[
(R(\xi, Y)B)(\xi, W, \xi) = 0. \tag{32}
\]


3. \((LCS)_n\)-Manifolds Satisfying Conditions \((B, \xi)P = 0\) and \((B, \xi)S = 0\)

**Theorem 4.** Let \(M\) be an \((LCS)_n\)-manifold. Then the manifold satisfies \(B(\xi, Y)P = 0\) if and only if there is the following relations

\[
\|Q\|^2 = n [(n-1)(\alpha^2 - \rho)]^2 [2(\alpha^2 - \rho) + p - 2] + r [(\alpha^2 - \rho)(n+1) + p - 2].
\]

**Proof.** In order to prove our theorem, we assume that \(B((\xi, Y)P)(U, W)Z = 0\), for all \(\xi, Y, U, W, Z \in \chi(M)\). Then we have

\[
0 = B(\xi, Y)P(U, W)Z - P(B(\xi, Y)U, W)Z \\
- P(U, B(\xi, Y)W)Z - P(U, W)B(\xi, Y)Z \tag{33}
\]

In (33), by using the equation (18) we obtain

\[
0 = \left[\frac{4(\alpha^2 - \rho)+2p-4}{n+3}\right] \left[ g(Y, P(U, W)Z)\xi - \eta(P(U, W)Z)Y \right. \\
- g(Y, U)P(\xi, W)Z + \eta(U)P(Y, W)Z \\
- g(Y, W)P(U, \xi)Z + \eta(W)P(U, Y)Z \\
- g(Y, Z)P(U, W)\xi + \eta(Z)P(U, W)Y \\
+ \frac{2}{n+3} \left[ \eta(P(U, W)Z)QY - S(Y, P(U, W)Z)\xi \\
- \eta(U)P(QY, W)Z + S(Y, U)P(\xi, W)Z \\
- \eta(W)P(U, QY)Z + S(Y, W)P(U, \xi)Z \\
- \eta(Z)P(U, W)QY + S(Y, Z)P(U, W)\xi \right]. \tag{34}
\]
Here, substituting $U = \xi$ in (34), we have

$$0 = \left[\frac{4(\alpha^2 - \rho) + 2p - 4}{n+3}\right] g(Y, P(\xi, W)Z)\xi - \eta(P(\xi, W)Z)Y$$

$$- \eta(Y)P(\xi, W)Z - P(Y, W)Z + P(\xi, Y)Z + \eta(Z)P(\xi, W)Y$$

$$+ \frac{2}{n+3} [\eta(P(\xi, W)Z)QY - S(Y, P(\xi, W)Z)]$$

$$+ P(QY, W)Z - \eta(W)P(\xi, QY)Z - \eta(Z)P(\xi, W)QY$$

$$+ \frac{2(n-1)(\alpha^2 - \rho)}{n+3} \eta(Y)P(\xi, W)Z.$$  \hspace{1cm} (35)

Let $Z = \xi$ be in (35), then also by using (6), (21) and (22), we obtain

$$\left[\frac{4(\alpha^2 - \rho) + 2p - 4}{n+3}\right] P(\xi, W)QY + \frac{2}{n+3} P(\xi, W)Y = 0.$$  \hspace{1cm} (36)

Again by using (21) in (36), we get

$$0 = \left[\frac{4(\alpha^2 - \rho) + 2p - 4}{n+3}\right] [(\alpha^2 - \rho) g(W, Y)\xi - \frac{1}{n-1} S(W, Y)]$$

$$- \frac{2}{n+3} [(\alpha^2 - \rho) g(W, QY)\xi - \frac{1}{n-1} S(W, QY)\xi]$$

which implies that

$$S(W, QY) = [(\alpha^2 - \rho)(n+1) + p - 2] S(Y, W)$$

$$- (n-1)(\alpha^2 - \rho) [2(\alpha^2 - \rho) + p - 2] g(W, Y).$$  \hspace{1cm} (37)

Now, for $[e_1, e_2, ..., e_{n-1}, \xi]$ orthonormal basis of $M$ from (37), we conclude

$$\|Q\|^2 = n [(n-1) (\alpha^2 - \rho)]^2 [2(\alpha^2 - \rho) + p - 2] + r [(\alpha^2 - \rho)(n+1) + p - 2],$$

which proves our assertion. The converse is obvious.

*Theorem 5.* Let $M$ be an $(LCS)_n$-manifold. Then $B(\xi, Y)S = 0$ if and only if there is the following relations

$$\|Q\|^2 = n [(n-1) (\alpha^2 - \rho)]^2 [2(\alpha^2 - \rho) + p - 2] + r [(\alpha^2 - \rho)(n+1) + p - 2].$$

*Proof.* We suppose that $(B(\xi, Y)S)(U, W) = 0.$ Then for all $\xi, Y, U, W \in \mathfrak{X}(M)$ we have

$$S(B(\xi, Y)U, W) + S(U, B(\xi, Y)W) = 0.$$  \hspace{1cm} (38)

In (38), by using (18) we get

$$0 = \left[\frac{4(\alpha^2 - \rho) + 2p - 4}{n+3}\right] [g(Y, U)S(\xi, W) - \eta(U)S(Y, W)$$

$$+ g(Y, W)S(U, \xi) - \eta(W)S(U, Y)]$$

$$+ \frac{2}{n+3} [\eta(U)S(QY, W) - S(Y, U)S(\xi, W)]$$

$$+ \frac{2}{n+3} [\eta(U)S(Y, QY) - S(Y, U)S(\xi, W)] .$$  \hspace{1cm} (39)

Now, in (39) substituting $U = \xi$ we obtain

$$S(QY, W) = [(\alpha^2 - \rho)(n+1) + p - 2] S(Y, W)$$

$$+ [(\alpha^2 - \rho)(n-1)] [2(\alpha^2 - \rho) + p - 2] g(Y, W).$$  \hspace{1cm} (40)

Again for $[e_1, e_2, ..., e_{n-1}, \xi]$ orthonormal basis of $M$ from (40), we conclude

$$\|Q\|^2 = n [(n-1) (\alpha^2 - \rho)]^2 [2(\alpha^2 - \rho) + p - 2] + r [(\alpha^2 - \rho)(n+1) + p - 2],$$

\hspace{1cm} \blacksquare
4. Conclusion

In the present paper, we have studied the $C$-Bochner curvature tensor of $(LCS)_n$-manifolds satisfying the conditions $C$-Bochner flat, $R.B = 0$, $B.P = 0$ and $B.S = 0$. According these cases, we classified $(LCS)_n$-manifolds. The same classification can be made for other curvature tensors.

References