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ON A NEW SEQUENCE SPACE DEFINED BY ORLICZ FUNCTIONS

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Abstract. The sequence space BV_σ was introduced and studied by Mursaleen [9]. In this paper we extend BV_σ to $BV_\sigma(M,p,r)$ and study some properties and inclusion relations on this space.

1. Introduction

Let l_∞ and c denote the Banach spaces of bounded and convergent sequences $x=(x_k)_{k=1}^\infty$ respectively. Let σ be an injection of the set of positive integers $\mathbb{N}$ into itself having no finite orbits and T be the operator defined on l_∞ by $T((x_n)_{n=1}^\infty)=(x_{\sigma(n)})_{n=1}^\infty$.

A positive linear functional ϕ , with $||\phi||=1$, is called a σ - mean or an invariant mean if $\phi(x)=\phi(Tx)$ for all $x\in l_\infty$.

A sequence x is said to be σ - convergent , denoted by $x\in V_\sigma$, if $\phi(x)$ takes the same value, called $\sigma-\lim x$, for all σ- means ϕ. We have (see Schaefer [14])

$$V_\sigma=\left\{x=(x_n):\sum_{m=1}^\infty t_{m,n}(x)=L \text{ uniformly in n, } L=\sigma-\lim x\right\},$$

where for $m\geq 0$, $n>0$

$$t_{m,n}(x)=\frac{x_n+x_{\sigma(n)}+\cdots+x_{\sigma^m(n)}}{m+1}, \text{ and } t_{-1,n}=0.$$

where $\sigma^m(n)$ denotes the m th iterate of σ at n. In particular, if σ is the translation, a σ - mean is often called a Banach limit and V_σ reduces to f , the set of almost

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- convergent sequences (see Lorentz [5]). Subsequently invariant means have been studied by Ahmad and Mursaleen [1] , Mursaleen [8], Raimi [12] and many others.

The concept of paranorm is closely related to linear metric spaces. It is a generalization of that of absolute value. Let X be a linear space. A function $g : X \to \mathbb{R}$ is called paranorm, if

[P1] $g(x) \geq 0$, for all $x \in X$,
[P2] $g(-x) = g(x)$, for all $x \in X$,
[P3] $g(x + y) \leq g(x) + g(y)$, for all $x, y \in X$,
[P4] If (λ_n) is a sequence of scalars with $\lambda_n \to \lambda$ $(n \to \infty)$ and (x_n) is a sequence of vectors with $g(x_n - x) \to 0$ $(n \to \infty)$, then $g(\lambda_n x_n - \lambda x) \to 0$ $(n \to \infty)$.

A paranorm g for which $g(x) = 0$ implies $x = 0$ is called a total paranorm on X, and the pair (X, g) is called a totally paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (cf. [15, Theorem 10.4.2, p. 183]).

A map $M : \mathbb{R} \to [0, +\infty]$ is said to be an Orlicz function if M is even, convex, left continuous on $\mathbb{R}_+$, continuous at zero, $M(0) = 0$ and $M(u) \to \infty$ as $u \to \infty$. If M takes value zero only at zero we will write $M > 0$ and if M takes only finite values we will write $M < \infty$. [2,3,6,7,10,13].

W.Orlicz [11] used the idea of orlicz function to construct the space (L^M) . Lindendstrauss and Tzafriri [4] used the idea of Orlicz function to define orlicz sequence space

$$\ell_M := \left\{ x \in \omega : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}$$

in more detail . ℓ_M is a Banach space with the norm

$$||x|| := \inf\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\}$$

The space l_M is closely related to the space l_p , which is an Orlicz sequence space with $M(x) = x^p \ for \ 1 \leq p < \infty$.

The $\triangle_2$ - condition is equivalent to

$$M(Lx) \leq KLM(x), \text{ for all values of } x \geq 0, \text{ and for } L > 1.$$

An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt,$$

where η is known as the kernel of M, is right differentiable for $t \geq 0$, $\eta(0) = 0$, $\eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.Note that an Orlicz function satisfies the inequality

$$M(\lambda x) \leq \lambda M(x) \text{ for all } \lambda \text{ with } 0 < \lambda < 1.$$

Let E be a sequence space . Then E is called

(i) A sequence space E is said to be symmetric if $(x_n) \in E$ implies $(x_\pi(n)) \in E$, where $\pi(n)$ is a permutation of the elements of the elements of $\mathbb{N}$.

(ii) Solid (or normal), if $(\alpha_k x_k) \in E$, whenever $(x_k) \in E$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.

Lemma 1.1. . *A sequence space E is solid implies E is monotone.*

Mursaleen [9] defined the sequence space

$$BV_\sigma = \left\{ x \in l_\infty : \sum_m |\phi_{m,n}(x)| < \infty, \text{ uniformly in n} \right\},$$

where

$$\phi_{m,n}(x) = t_{m,n}(x) - t_{m-1,n}(x)$$

assuming that

$$t_{m,n}(x) = 0, \quad \text{for m} = -1.$$

A straightforward calculation shows that

$$\phi_{m,n}(x) = \begin{cases} \frac{1}{m(m+1)} \sum_{j=1}^{m} j(x_{\sigma^j(n)} - x_{\sigma^{j-1}(n)}) & (m \geq 1) \\ x_n, & (m = 0) \end{cases}$$

Note that for any sequence x, y and scalar λ we have

$$\phi_{m,n}(x+y) = \phi_{m,n}(x) + \phi_{m,n}(y) \text{ and } \phi_{m,n}(\lambda x) = \lambda \phi_{m,n}(x).$$

2. Main Results.

Let M be an Orlicz function, $p = (p_m)$ be any sequence of strictly positive real numbers and $r \geq 0$. Now we define the sequence space as follows :

$$BV_\sigma(M,p,r) = \left\{ \begin{array}{c} x = (x_k) : \sum\limits_{m=1}^{\infty} \frac{1}{m^r} \left[M \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right]^{p_m} < \infty, \\ \text{uniformly in n and for some } \rho > 0 \end{array} \right\}.$$

For $M(x) = x$ we get

$$BV_\sigma(p,r) = \left\{ x = (x_k) : \sum_{m=1}^{\infty} \frac{1}{m^r} |\phi_{m,n}(x)|^{p_m} < \infty, \text{ uniformly in n} \right\}.$$

For $p_m = 1$, for all m, we get

$$BV_\sigma(M,r) = \left\{ \begin{array}{c} x = (x_k) : \sum\limits_{m=1}^{\infty} \frac{1}{m^r} \left[M \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right] < \infty, \\ \text{uniformly in n and for some } \rho > 0 \end{array} \right\}.$$

For $r = 0$ we get

$$BV_\sigma(M,p) = \left\{ \begin{array}{c} x = (x_k) : \sum\limits_{m=1}^{\infty} \left[M \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right]^{p_m} < \infty, \\ \text{uniformly in n and for some } \rho > 0 \end{array} \right\}.$$

For $M(x) = x$ and $r = 0$ we get

$$BV_\sigma(p) = \left\{ x = (x_k) : \sum_{m=1}^{\infty} |\phi_{m,n}(x)|^{p_m} < \infty, \text{ uniformly in n} \right\}.$$

For $p_m = 1$, for all m and $r = 0$ we get

$$BV_\sigma(M) = \left\{ \begin{array}{c} x = (x_k) : \sum\limits_{m=1}^{\infty} \left[M \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right] < \infty, \ r \geq 0, \\ \text{uniformly in n and for some } \rho > 0 \end{array} \right\}.$$

For $M(x) = x$, $p_m = 1$, for all m, and $r = 0$ we get

$$BV_\sigma = \left\{ x = (x_k) : \sum_{m=1}^{\infty} |\phi_{m,n}(x)| < \infty, \text{uniformly in n} \right\}.$$

Theorem 2.1. *The sequence space $BV_\sigma(M,p,r)$ is a linear space over the field $\mathbb{C}$ of complex numbers.*

Proof. . Let $x, y \in BV_\sigma(M,p,r)$ and $\alpha, \beta \in \mathbb{C}$. Then there exist positive numbers ρ_1 and ρ_2 such that

$$\sum_{m=1}^{\infty} \frac{1}{m^r} \left[M \left(\frac{|\phi_{m,n}(x)|}{\rho_1} \right) \right]^{p_m} < \infty$$

and

$$\sum_{m=1}^{\infty} \frac{1}{m^r} \left[M \left(\frac{|\phi_{m,n}(y)|}{\rho_2} \right) \right]^{p_m} < \infty, \text{ uniformly in n.}$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is nondecreasing and convex we have

$$\sum_{m=1}^{\infty} \frac{1}{m^r} \left[M \left(\frac{|\alpha\phi_{m,n}(x) + \beta\phi_{m,n}(y)|}{\rho_3} \right) \right]^{p_m}$$

$$\leq \sum_{m=1}^{\infty} \frac{1}{m^r} \left[M \left(\frac{|\alpha\phi_{m,n}(x)|}{\rho_3} + \frac{|\beta\phi_{m,n}(y)|}{\rho_3} \right) \right]^{p_m}$$

$$\leq \sum_{m=1}^{\infty} \frac{1}{m^r} \frac{1}{2} \left[M \left(\frac{\phi_{m,n}(x)}{\rho_1} \right) + M \left(\frac{\phi_{m,n}(y)}{\rho_2} \right) \right] < \infty, \text{ uniformly in n.}$$

This proves that $BV_\sigma(M,p,r)$ is a linear space over the field $\mathbb{C}$ of complex numbers. □

Theorem 2.2. *For any Orlicz function M and a bounded sequence $p = (p_m)$ of strictly positive real numbers, $BV_\sigma(M,p,r)$ is a paranormed(need not be total paranormed) space with*

$$g(x) = \inf_{n \geq 1} \left\{ \rho^{\frac{p_n}{K}} : \left(\sum_{m=1}^{\infty} \frac{1}{m^r} \left[M \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right]^{p_m} \right)^{\frac{1}{K}} \leq 1, \textit{ uniformly in } n \right\}.$$

where $K = \max(1, \sup p_m)$.

Proof. It is clear that $g(x) = g(-x)$. Since $M(0) = 0$, we get

$$\inf \left\{ \rho^{\frac{p_n}{K}} \right\} = 0, \quad \text{for} \quad x = 0.$$

By using Theorem 1, for $\alpha = \beta = 1$, we get

$$g(x+y) \leq g(x) + g(y).$$

For the continuity of scalar multiplication let $l \neq 0$ be any complex number. Then by the definition we have

$$g(lx) = \inf_{n \geq 1} \left\{ \rho^{\frac{p_n}{K}} : \left(\sum_{m=1}^{\infty} \frac{1}{m^r} \left[M \left(\frac{|\phi_{m,n}(lx)|}{\rho} \right) \right]^{p_m} \right)^{\frac{1}{K}} \leq 1, \text{ uniformly in n} \right\}$$

$$g(lx) = \inf_{n\geq 1}\left\{ \begin{array}{c} (|l|s)^{\frac{p_n}{K}} : \left(\sum_{m=1}^{\infty}\frac{1}{m^r}\left[M\left(\frac{|\phi_{m,n}(lx)|}{s|l|}\right)\right]^{p_m}\right)^{\frac{1}{K}} \leq 1, \\ \text{uniformly in n} \end{array}\right\}$$

where $s = \frac{\rho}{|l|}$. Since $|l|^{p_n} \leq \max(1, |l|^H)$, we have

$$g(lx) \leq \max(1, |l|^H)\inf_{n\geq 1}\left\{ \begin{array}{c} s^{\frac{p_n}{K}} : \left(\sum_{m=1}^{\infty}\frac{1}{m^r}\left[M\left(\frac{|\phi_{m,n}(x)|}{s}\right)\right]^{p_m}\right)^{\frac{1}{K}} \leq 1, \\ \text{uniformly in n} \end{array}\right\}$$

$$= \max(1, |l|^H)g(x)$$

and therefore $g(lx)$ converges to zero when $g(x)$ converges to zero in $BV_\sigma(M,p,r)$.

□

Now let x be fixed element in $BV_\sigma(M,p,r)$. There exists $\rho > 0$ such that

$$g(x) = \inf_{n\geq 1}\left\{\rho^{\frac{p_n}{K}} : \left(\sum_{m=1}^{\infty}\frac{1}{m^r}\left[M\left(\frac{|\phi_{m,n}(x)|}{\rho}\right)\right]^{p_m}\right)^{\frac{1}{K}} \leq 1, \text{ uniformly in n}\right\}.$$

Now

$$g(lx) = \inf_{n\geq 1}\left\{\rho^{\frac{p_n}{K}} : \left(\sum_{m=1}^{\infty}\frac{1}{m^r}\left[M\left(\frac{|\phi_{m,n}(lx)|}{\rho}\right)\right]^{p_m}\right)^{\frac{1}{K}} \leq 1, \text{ uniformly in n}\right\} \to 0,$$

as $l \to 0$.

This completes the proof.

Theorem 2.3. *Suppose that* $0 < p_m \leq t_m < \infty$ *for each* $m \in \mathbb{N}$ *and* $r \geq 0$. *Then*

(*i*) $BV_\sigma(M,p) \subseteq BV_\sigma(M,t)$,

(*ii*) $BV_\sigma(M) \subseteq BV_\sigma(M,r)$.

Proof. . [i] Suppose that $x \in BV_\sigma(M,p)$. This implies that $\left[M\left(\frac{|\phi_{i,n}(x)|}{\rho}\right)\right]^{p_m} \leq 1$for sufficiently large values of i , say $i \geq m_0$ for some fixed $m_0 \in \mathbb{N}$. Since M is non decreasing, we have

$$\sum_{m=m_0}^{\infty}\left[M\left(\frac{|\phi_{i,n}(x)|}{\rho}\right)\right]^{t_m} \leq \sum_{m=m_0}^{\infty}\left[M\left(\frac{|\phi_{i,n}(x)|}{\rho}\right)\right]^{p_m} < \infty.$$

Hence $x \in BV_\sigma(M,t)$.

The proof of [ii] is trivial.

□

The following result is a consequence of the above result.

Corollary 1. *If* $0 < p_m \leq 1$ *for each* m *, then* $BV_\sigma(M,p) \subseteq BV_\sigma(M)$.

If $p_m \geq 1$ for all m , then $BV_\sigma(M) \subseteq BV_\sigma(M,p)$.

Theorem 2.4. . *The sequence space* $BV_\sigma(M,p,r)$ *is solid.*

Proof. Let $x \in BV_\sigma(M,p,r)$. This implies that

$$\sum_{m=1}^{\infty} m^{-r}\left[M\left(\frac{|\phi_{k,n}(x)|}{\rho}\right)\right]^{p_m} < \infty.$$

Let (α_m) be sequence of scalars such that $|\alpha_m| \leq 1$ for all $m \in \mathbb{N}$. Then the result follows from the following inequality

$$\sum_{m=1}^{\infty} m^{-r}\left[M\left(\frac{|\alpha_m\phi_{k,n}(x)|}{\rho}\right)\right]^{p_m} \leq \sum_{m=1}^{\infty} m^{-r}\left[M\left(\frac{|\phi_{k,n}(x)|}{\rho}\right)\right]^{p_m} < \infty.$$

Hence $\alpha x \in BV_\sigma(M,p,r)$ for all sequences of scalars (α_m) with $|\alpha_m| \leq 1$ for all $m \in \mathbb{N}$ whenever $x \in BV_\sigma(M,p,r)$.

□

From Theorem 4 and Lemma we have :

Corollary 2. . *The sequence space* $BV_\sigma(M,p,r)$ *is monotone.*

Theorem 2.5. . *Let* M_1, M_2 *be Orlicz functions satisfying* $\triangle_2$ *- condition and* $r, r_1, r_2 \geq 0$. *Then we have*

(*i*) *If* $r > 1$ *then* $BV_\sigma(M_1,p,r) \subseteq BV_\sigma(M0M_1,p,r)$,

(*ii*) $BV_\sigma(M_1,p,r) \cap BV_\sigma(M_2,p,r) \subseteq BV_\sigma(M_1+M_2,p,r)$,

(*iii*) If $r_1 \leq r_2$ then $BV_\sigma(M,p,r_1) \subseteq BV_\sigma(M,p,r_2)$.

Proof. [i] Since M is continuous at 0 from right , for $\epsilon > 0$ there exists $0 < \delta < 1$ such that $0 \leq c \leq \delta$ implies $M(c) < \epsilon$. If we define

$$I_1 = \left\{m \in \mathbb{N} : M_1\left(\frac{|\phi_{m,n}(x)|}{\rho}\right) \leq \delta \text{ for some } \rho > 0\right\},$$

$$I_2 = \left\{ m \in I\!N : M_1 \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) > \delta \text{ for some } \rho > 0 \right\},$$

then , when $M_1 \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) > \delta$ we get

$$M \left(M_1 \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right) \leq \{2M(1)/\delta\} M_1 \left(\frac{|\phi_{m,n}(x)|}{\rho} \right).$$

Hence for $x \in BV_\sigma(M_1, p, r)$ and $r > 1$

$$\begin{aligned}
\sum_{m=1}^{\infty} m^{-r} \left[M0M_1 \left(\tfrac{|\phi_{m,n}(x)|}{\rho} \right) \right]^{p_m} &= \sum_{m \in I_1} m^{-r} \left[M0M_1 \left(\tfrac{|\phi_{m,n}(x)|}{\rho} \right) \right]^{p_m} \\
&+ \sum_{m \in I_2} m^{-r} \left[M0M_1 \left(\tfrac{|\phi_{m,n}(x)|}{\rho} \right) \right]^{p_m} \\
&\leq \sum_{m \in I_1} m^{-r} [\epsilon]^{p_m} \\
&+ \sum_{m \in I_2} m^{-r} \left[\{2M(1)/\delta\} M_1 \left(\tfrac{|\phi_{m,n}(x)|}{\rho} \right) \right]^{p_m} \\
&\leq \max(\epsilon^h, \epsilon^H) \sum_{m=1}^{\infty} m^{-r} \\
&+ \max \left(\{2M(1)/\delta\}^h, \{2M(1)/\delta\}^H \right)
\end{aligned}$$

$$\text{(where } 0 < h = \inf p_m \leq p_m \leq H = \sup_m p_m < \infty).$$

[ii] The proof follows from the following inequality

$$\begin{aligned}
m^{-r} \left[(M_1 + M_2) \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right]^{p_m} &\leq Cm^{-r} \left[M_1 \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right]^{p_m} \\
&+ Cm^{-r} \left[M_2 \left(\frac{|\phi_{m,n}(x)|}{\rho} \right) \right]^{p_m}.
\end{aligned}$$

[iii]The proof is straightforward.

□

Corollary 3. . *Let M be an Orlicz function satisfying $\triangle_2$ - condition. Then we have*

(1) If $r > 1$, then $BV_\sigma(p, r) \subseteq BV_\sigma(M, p, r)$,

(2) $BV_\sigma(M, p) \subseteq BV_\sigma(M, p, r)$,

(3) $BV_\sigma(p) \subseteq BV_\sigma(p, r)$,

(4) $BV_\sigma(M) \subseteq BV_\sigma(M, r)$,

The proof is straightforward.

ÖZET: BV_σ dizi uzayı, Mursaleem tarafından tanımlanmış ve incelenmiştir [9]. Bu çalışmada BV_σ uzayını, $BV_\sigma(M,p,r)$ uzayına genişleterek bu uzaya ilişkin bazı özelikleri ve kapsama bağıntılarını elde ettik.

References

[1] Z.U. Ahmad and M. Mursaleen , An application of banach limits, Proc. Amer. Math.Soc. 103 (1983), 244 - 246.

[2] S.T. Chen, Geometry of Orlicz Spaces, Dissertationes Math. (The Institute of Mathematics, Polish Academy of Sciences) (1996).

[3] M. A. Krasnoselskii, and Rutickii, Ya. B, Convex Functions and Orlicz Spaces, (Gooningen: P. Nordhoff Ltd.) (1961) (translation).

[4] J. Lindenstrauss and L. Tzafriri, On Orlicz sequence spaces, *Israel J. Math.*, 10 : 379-390 (1971).

[5] G.G. Lorentz, A contribution to the theory of divergent series, Acta Math. 80(1948) 167-190.

[6] W. A. Luxemburg, Banach Function Spaces, Thesis (Delft) (1955)

[7] L. Maligranda, Orlicz spaces and interpolation, Seminar in Math. 5, Campinas (1989)

[8] M.Mursaleen , Matrix transformations between some new sequence spaces, Houston J. Math., 9 (1983), 505- 509.

[9] M.Mursaleen ,On some new invariant matrix methods of summability, Quart. J. Math., Oxford (2) 34 (1983), 77-86 .

[10] J. Musielak , Orlicz Spaces and Modular spaces, Lecture Notes in Math. 1034 (Springer- Verlag) (1983).

[11] W. Orlicz, Ü ber Raume (L^M), Bulletin International de l' Académie Polonaise des Sciences et des Letters, Série A, 93 - 107 (1936).

[12] R. A. Raimi, Invariant means and invariant matrix method of summability, Duke Math. J. ., 30 (1963), 81- 94.

[13] M. M . Rao and Z. D. Ren, Theory of Orlicz spaces (New York, Basel, Hong Kong: Marcel Dekker Inc.) (1991).

[14] P. Schafer , Infinite matrices and invariant means, Proc. Amer. Math.Soc. 36 (1972), 104 - 110.

[15] A. Wilansky, Summability Through Functional Analysis, North-Holland Mathematical Studies 85, 1984.

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