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SOME RELATIONS BETWEEN FUNCTIONALS ON BOUNDED REAL SQUENCES

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Abstract. In this paper, we mainly concern with the functionals L^{**} and l^{**} defined on bounded real sequences and give some inequalities between these functionals.

1. Introducton

If $T = (t_{nk})$ is an infinite matrix with real entries, and if $x = (x_k)$ is a sequence of real numbers, then Tx denotes the transformed sequence whose n-th term is given by $(Tx)_n = \sum_{k=1}^{\infty} t_{nk}x_k$. In order to investigate the effect of such transformations upon the derived set, Knopp [5] introduced the idea of the core ($\mathcal{K}$-core) of a sequence and proved the well-known Core Theorem. That theorem asserts that $\mathcal{K}$-core$\{Tx\} \subseteq \mathcal{K}$-core$\{x\}$, whenever Tx exists for the nonnegative regular matrix T. Some variants of the Core Theorem may be found in [2], [9], [10], [12].

Considering the method of almost convergence Loone [6] and Das [2] introduced the Banach core ($\mathcal{B}-$core) of a bounded sequence and proved some analogues of the assertions for the $\mathcal{K}$-core (see also [4], [10], [12], [13]).

Before proceeding further we recall some notation and terminology. By l^{∞} and c we denote the spaces of all bounded and convergent real sequences, respectively.

Let $T = (t_{nk})$ be an infinite matrix, and let X and Y be two sequence spaces. If Tx exists for each $x \in X$ and $Tx \in Y$ then we say that T maps X into Y. The set of matrices which map X into Y is denoted by (X, Y). The set of matrices which map X into Y and leave the limit or sum invariant is denoted by $(X, Y; p)$. For example, if $T \in (c, c; p)$, then $\lim Tx = \lim x$ for every $x \in c$. In this case T is called regular (see [1], [11]). If it is regular and satisfies

$$\lim_n \sum_k |t_{nk} - t_{n,k+1}| = 0,$$

then T is called strongly regular [11].

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It is well-known [7], [11] that the functional

$$q(x) = \inf_{n_1, n_2, \ldots n_r} \limsup_k \frac{1}{r} \sum_{i=1}^{r} x_{k+n_i}$$

is sublinear on l^∞. We consider the following functionals on l^∞ :

$$\begin{aligned} L(x) &= \limsup x_n \ , \\ l^*(x) &= \liminf_r \sup_k \tfrac{1}{r} \sum_{i=0}^{r} x_{k+i} \ , \\ L^*(x) &= \limsup_r \sup_k \tfrac{1}{r} \sum_{i=0}^{r} x_{k+i}. \end{aligned}$$

It follows from the Corollary of Theorem 1 in [3] that $q(x) = L^*(x)$.
If $q(x) = -q(-x) = s$, then x is called almost convergent to s [7], and in this case we write $F - \lim x = s$. By F we denote the set of all almost convergent sequences.

The Banach core ($\mathcal{B}$-core) of a bounded sequence x is defined to be the closed interval $[-q(-x), q(x)]$ (see [2], [6]). Since $q(x) \leq L(x)$ for every $x \in l^\infty$, it follows that $\mathcal{B}$-core $\{x\} \subseteq \mathcal{K}$-core $\{x\}$ where $\mathcal{K}$-core $\{x\}$ is the Knopp core and it is given by $\mathcal{K}$-core $\{x\} = [\liminf x, \limsup x]$. It is shown in [6], [10] that

$$\mathcal{K} - core\{Ax\} \subseteq \mathcal{B} - core\{x\} \text{ (for every } x \in l^\infty) \tag{1}$$

if and only if A is strongly regular and $\lim\limits_n \sum\limits_k |a_{nk}| = 1$.

With this terminology the Knopp core theorem gives the conditions on the matrix A so that the inequality $LA \leq L^*$, on l^∞, holds. Hence (1) yields the inequality $LA \leq L^*$ on l^∞.

Also it is well-known [8], [3] that the functional

$$Q(x) = \inf_{n_1, n_2, \ldots n_r} \limsup_k \frac{1}{r} \sum_{i=1}^{r} |x_{k+n_i}|$$

is sublinear on l^∞. Define for $x \in l^\infty$,

$$L^{**}(x) = \limsup_r \sup_k \frac{1}{r} \sum_{i=0}^{r} |x_{k+i}| \, .$$

Then substituting $|x| = (|x_n|)_{n \geq 0}$ for $x = (x_n)$, in Corollary of Theorem 1 in [3], we obtain $Q(x) = L^{**}(x)$

Throughout the paper we consider only real matrices and real bounded sequences.

In this paper we will give a reception between functionals L and L^{**}, than some inequalities.

2. THE FUNCTIONALS L^{**} AND l^{**} AND SOME INEQUALITIES

If we take the sequence $x = (x_n)$ defined by $x_n = (-1)$ for all n, it follows that,

$$L(x) = -1,\ L^{**}(x) = 1$$

hence

$$L^{**}(x) > L(x).$$

Now, if we define sequence $x = (x_n)$ by $x_n = (-1)^n$ for all n, it follows that,

$$L(x) = 1,\ L^{**}(x) = 1,$$

hence

$$L^{**}(x) = L(x).$$

Finally, if we consider the bounded sequence $x = (x_n)$ given by $x_n \geq 0$ for all n, then

$$L^{**}(x) = L^{*}(|x|) = L^{*}(x) \leq L(x),$$

hence

$$L^{**}(x) \leq L(x).$$

In this paper we mainly compare LA with L^{**}.

Theorem 2.1. *If* $L^{**}(x-y) = 0$ *on* l^{∞}, *then* $L^{**}(x) = L^{**}(y)$

Proof. We know that

$$L^{**}(x-y) = \lim_r \sup_k \sup \tfrac{1}{r} \sum_{i=0}^{r} |x_{k+i} - y_{k+i}| .$$

Now

$$\begin{aligned} L^{**}(x) &= \lim_r \sup_k \sup \tfrac{1}{r} \sum_{i=0}^{r} |x_{k+i}| . \\ &= \lim_r \sup_k \sup \tfrac{1}{r} \sum_{i=0}^{r} |x_{k+i} - y_{k+i} + y_{k+i}| \\ &\leq \lim_r \sup_k \sup \tfrac{1}{r} \sum_{i=0}^{r} |x_{k+i} - y_{k+i}| + \lim_r \sup_k \sup \tfrac{1}{r} \sum_{i=0}^{r} |y_{k+i}| \\ &= 0 + \lim_r \sup_k \sup \tfrac{1}{r} \sum_{i=0}^{r} |y_{k+i}| \\ &= L^{**}(y) \end{aligned}$$

If we interchange the roles of x and y,then we also get

$$L^{**}(y) \leq L^{**}(x),$$

which implies that

$$L^{**}(x) = L^{**}(y).$$

□

Theorem 1 in[5] is valid if we write $(|x_n|)$ in place of $x \in l^\infty$. Using this result we get,

$$\limsup_n \sup_i \frac{1}{n+1} \sum_{r=i}^{i+n} |x_r| \leq L(|x|).$$

If we define sublinear functional P on l^∞ by

$$P(x) = \limsup_k |x_k|,$$

we can give the following

Corollary 1. *On l^∞,*

$$L^{**} \leq P.$$

Let us define the functional Z on l^∞ by

$$Z(x) = \frac{|x_1| + |x_2| + ... + |x_k|}{k}.$$

We recall that the matrix B called normal if it is lower semi triangular matrix with non-zero diagonal entries.

Theorem 7 of Yardimci [13] gives us the necessary and sufficient conditions for $L^*(Ax) \leq L(Bx)$, whenever B is a normal matrix and Bx is bounded. This theorem is valid if we take C_1 Cesáro matrix instead of B and $|x| = (|x_n|)$. Thus we get the following

Corollary 2. *On l^∞,*

$$L^{**}A(x) \leq Z(x).$$

The following result compares LA with L^{**}.

Theorem 2.2. *If A is a strogly regular matrix and*

$$\lim_n \sum_k |a_{nk}| = 1,$$

then

$$LA \leq L^{**}.$$

on l^∞.

Proof. Let A be a strongly regular matrix and $\lim_n \sum_k |a_{nk}| = 1$. Then Theorem 6 [10] implies that

$$LA\,(x) \leq L^*\,(x)$$

for all $x \in l^\infty$. Also we know that,

$$L^*\,(x) \leq L^*\,(|x|) = L^{**}(x)$$

on l^∞. So,

$$LA\,(x) \leq L^{**}(x)$$

on l^∞. This proves the theorem. □

The following theorem also gives some sufficient conditions for this inequality.

Theorem 2.3. *Let A be a strongly regular matrix. Then If there exist a nonnegative strongly regular matrix B, which is absolutely equivalent to A on l^∞, then*

$$LA(x) \leq L^{**}(x),\ \ (\text{for every } x \in l^\infty).$$

Proof. By absolute equivalence of A and B, for every $x \in l^\infty$,

$$\lim_n \{(Ax)_n - (Bx)_n\} = 0. \tag{2}$$

Now Theorem 6.5.I of Cooke [1] implies that

$$L(Ax) \leq L(x), (\text{for every } x \in l^\infty).$$

Since B is non-negative strongly regular matrix, it follows from Theorem 3 in [10] that, for every $x \in l^\infty$,

$$L(Bx) \leq L^*(x). \tag{3}$$

Since (2) holds, Theorem 6.3.II of Cooke [1] implies that

$$L(Ax) = L(Bx). \tag{4}$$

Now (3) and (4) imply

$$L(Ax) \leq L^*(x) \leq L^{**}(x).$$

□

Define the functionals l^{**} on l^∞ by

$$l^{**}(x) = \liminf_r \sup_k \frac{1}{r} \sum_{i=0}^{r} |x_{k+i}|.$$

With this definition we have

Theorem 2.4. *Let A be any matrix such that $\sup_n \sum_k |a_{nk}| < \infty$. If*

$$\limsup_n \sup_i \frac{1}{n+1} \sum_{r=i}^{i+n} |a_{rk}| = 0,$$

then we have $L^{**}A \leq l^{**}$ on l^∞.

Proof. By hypothesis, Ax exist for every $x \in l^\infty$. Then,

$$\begin{aligned} L^{**}(Ax) &= \limsup_n \sup_i \tfrac{1}{n+1} \sum_{r=i}^{i+n} \left| \sum_k a_{rk} x_k \right| \\ &\leq \|x\| \limsup_n \sup_i \tfrac{1}{n+1} \sum_{r=i}^{i+n} \sum_k |a_{rk}| \\ &= \|x\| \limsup_n \sup_i \sum_k \tfrac{1}{n+1} \sum_{r=i}^{i+n} |a_{rk}| = 0. \end{aligned}$$

Also we know that

$$L^{**}(Ax) \geq 0,$$

Hence we get

$$L^{**}(Ax) = 0.$$

From the definition of l^{**} we can write

$$l^{**}(x) \geq 0.$$

So we get

$$L^{**}(Ax) \leq l^{**}(x).$$

□

ÖZET: Bu çalışmada temel amacımız, sınırlı reel diziler üzeinde tanımlı L^{**}ve l^{**}fonksiyonellerini incelemek ve bunlar arasındaki bazı eşitsizlikleri vermektir.

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