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The proximal point algorithm in complete geodesic spaces with negative curvature

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Abstract

The proximal point algorithm is an approximation method for finding a minimizer of a convex function. In this paper, using the properties of the resolvent which was proposed by the authors, we show the proximal point algorithm using a suitable notion of weak convergence in complete geodesic spaces with negative curvature.

Keywords: CAT(−1) space; proximal point algorithm; resolvent; convex function; geodesic space.
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1. Introduction

The proximal point algorithm is an approximation method for finding a minimizer of a proper lower semicontinuous convex function. The resolvent of this function plays an important role in this algorithm.

Let X be a complete CAT(0) space and f a proper lower semicontinuous convex function of X into $]-\infty,\infty]$. Then the resolvent of λf is defined by

$$J_\lambda x = \operatorname*{argmin}_{y\in X}\left\{f(y) + \frac{1}{\lambda}d(y,x)^2\right\}$$

for all $x \in X$ and $\lambda > 0$. In 1998, Mayer [10] showed that it is well-defined as a single valued mapping. The proximal point algorithm is one of the most famous methods for approximating a minimizer of a convex function. This algorithm was originally proposed by Martinet [9] and Rockafellar [11] considered more general settings. In a complete CAT(0) space, the following theorem was shown by Bačák [1] in 2013.

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Theorem 1.1 (Bačák [1]). *Let X be a complete* CAT(0) *space, f a proper lower semicontinuous convex function of X into $]-\infty, \infty]$ such that* $\operatorname{argmin}_X f$ *is nonempty, and $\{\lambda_n\}$ a sequence of positive real numbers with $\sum_{n=1}^{\infty} \lambda_n = \infty$. If a sequence $\{x_n\}$ of X is defined by $x_1 \in X$ and $x_{n+1} = J_{\lambda_n} x_n$ for $n \in \mathbb{N}$, then $\{x_n\}$ is Δ-convergent to an element of* $\operatorname{argmin}_X f$.

Let X be a complete CAT(1) space satisfying $d(u, v) < \pi/2$ for all $u, v \in X$ and f a proper lower semicontinuous convex function of X into $]-\infty, \infty]$. Then the resolvent of λf is defined by

$$Q_\lambda x = \operatorname*{argmin}_{y \in X} \left\{ f(y) + \frac{1}{\lambda} \tan d(y, x) \sin d(y, x) \right\}$$

for all $x \in X$ and $\lambda > 0$. Kimura and Kohsaka [6] showed its well-definedness and the following theorem.

Theorem 1.2 (Kimura and Kohsaka [7]). *Let X be a complete* CAT(1) *space satisfying $d(u, v) < \pi/2$ for all $u, v \in X$, f a proper lower semicontinuous convex function of X into $]-\infty, \infty]$ satisfying* $\operatorname{argmin}_X f \neq \emptyset$, *and $\{\lambda_n\}$ a sequence of positive real numbers satisfying $\sum_{n=1}^{\infty} \lambda_n = \infty$. If a sequence $\{x_n\}$ of X is defined by $x_1 \in X$ and $x_{n+1} = Q_{\lambda_n} x_n$ for $n \in \mathbb{N}$, then $\{x_n\}$ is Δ-convergent to an element of* $\operatorname{argmin}_X f$.

In this paper, we propose the proximal point algorithm in a complete CAT(−1) space. In Section 2, we introduce the definition of CAT(κ) spaces and resolvents in a complete CAT(−1) space. In Section 3, we give several results which are necessary to prove the main theorem. In Section 4, we show the proximal point algorithm in a complete CAT(−1) space and prove Δ-convergence of the generated sequence.

2. Preliminaries

Let X be a metric space with metric d. We denote by $\mathcal{F}(T)$ the set of all fixed points of a mapping T of X into itself. For $x, y \in X$, a continuous mapping $c : [0, l] \to X$ is called geodesic joining x and y if c satisfies $c(0) = x, c(l) = y$ and $d(c(s), c(t)) = |s - t|$ for all $s, t \in [0, 1]$. Its image, which is denoted by $[x, y]$, is called a geodesic segment with endpoints x and y. X is said to be a geodesic space if there exists a geodesic joining any two points in X. In this paper, when X is a geodesic metric space, a geodesic joining any two points of X is always assumed to be unique.

Let X be a geodesic metric space. For all $x, y \in X$ and $\alpha \in [0, 1]$, there exists a unique point $z \in X$ such that $d(x, z) = (1 - \alpha) d(x, y)$ and $d(z, y) = \alpha d(x, y)$. This point is called a convex combination of x and y which is denoted by $\alpha x \oplus (1 - \alpha) y$. A subset $C \subset X$ is called convex if $[x, y] \subset X$ for all $x, y \in C$. A geodesic triangle with vertices $x, y, z \in X$ is defined by $[x, y] \cup [y, z] \cup [z, x]$, which is denoted by $\triangle(x, y, z)$.

Let M_κ^2 be a two dimensional model space for all $\kappa \in \mathbb{R}$. For example, $M_0^2 = \mathbb{R}^2$, M_1^2 is two-dimensional unit sphere $\mathbb{S}^2$, and M_{-1}^2 is two-dimensional hyperbolic space $\mathbb{H}^2$. A comparison triangle to $\triangle(x, y, z) \subset X$ with vertices $\bar{x}, \bar{y}, \bar{z} \in M_\kappa^2$ is defined by $[\bar{x}, \bar{y}] \cup [\bar{y}, \bar{z}] \cup [\bar{z}, \bar{x}]$ with $d(x, y) = d(\bar{x}, \bar{y}), d(y, x) = d(\bar{y}, \bar{z})$, and $d(z, x) = d(\bar{z}, \bar{x})$, which is denoted by $\bar{\triangle}(\bar{x}, \bar{y}, \bar{z})$. $\bar{w} \in [\bar{x}, \bar{y}]$ is called a comparison point of $w \in [x, y]$ if $d(x, w) = d(\bar{x}, \bar{w})$ holds. For $\kappa \in \mathbb{R}$, X is called a CAT(κ) space if $d(p, q) \leqq d(\bar{p}, \bar{q})$ holds whenever $\bar{p}$ and $\bar{q} \in \bar{\triangle}$ are the comparison points for p and $q \in \triangle$, respectively. In general, if $\kappa < \kappa'$, then the CAT(κ) spaces are CAT(κ') spaces [3]. We know that the following lemma holds, which is called the midpoint theorem.

Lemma 2.1 ([3]). *Let X be a* CAT(−1) *space, $x, y, z \in X$ and $\alpha \in [0, 1]$. Then*

$$\begin{aligned} &\cosh d(\alpha x \oplus (1 - \alpha) y, z) \sinh d(x, y) \\ &\qquad \leqq \cosh d(x, z) \sinh \alpha d(x, y) + \cosh d(y, z) \sinh (1 - \alpha) d(x, y). \end{aligned}$$

Corollary 2.2 ([5]). *Let X, x, y and z be the same as in Lemma 2.1. Then*

$$\cosh d\left(\frac{1}{2} x \oplus \frac{1}{2} y, z\right) \cosh \frac{1}{2} d(x, y) \leqq \frac{1}{2} \cosh d(x, z) + \frac{1}{2} \cosh d(y, z).$$

Let X be a metric space and $\{x_n\}$ a sequence in X. An asymptotic center of $\{x_n\}$ is defined by

$$\left\{ u \in X \,\middle|\, \limsup_{n\to\infty} d(u, x_n) = \inf_{y\in X} \limsup_{n\to\infty} d(y, x_n) \right\},$$

which is denoted by $\mathcal{A}(\{x_n\})$. A sequence $\{x_n\}$ Δ-converges to a point u in X if

$$\mathcal{A}(\{x_{n_i}\}) = \{u\}$$

for all subsequences $\{x_{n_i}\}$ of $\{x_n\}$, which is denoted by $x_n \stackrel{\Delta}{\rightharpoonup} u$. In this case, u is called a Δ-limit of $\{x_n\}$. A subset C of X is said to be Δ-closed if $u \in C$ whenever $\{x_n\} \subset C$ and $x_n \stackrel{\Delta}{\rightharpoonup} u$. For a sequence $\{x_n\}$ in X, we denoted by $\omega_\Delta(\{x_n\})$ the set of all $u \in X$ such that there exists a subsequence of $\{x_n\}$ which is Δ-convergent to u. We know that the following fundamental properties hold.

Lemma 2.3 ([4])**.** *Let X be a complete* CAT(0) *space and $\{x_n\}$ a bounded sequence in X. Then $\mathcal{A}(\{x_n\})$ consists of one point and $\omega_\Delta(\{x_n\})$ is nonempty.*

Lemma 2.4 ([2])**.** *Let X be a complete* CAT(0) *space and $\{x_n\}$ a bounded sequence in X. If $\{d(z, x_n)\}$ is convergent for all $z \in \omega_\Delta(\{x_n\})$, then $\{x_n\}$ is Δ-convergent.*

Let X be a geodesic metric space and f a function of X into $]-\infty, \infty]$. We say that f is lower semicontinuous if the set $\{x \in X \mid f(x) \leqq a\}$ is closed for all $a \in \mathbb{R}$. If f is continuous, then it is lower semicontinuous. The function f is said to be Δ-lower semicontinuous if

$$f(u) \leqq \liminf_{n\to\infty} f(x_n)$$

whenever $\{x_n\}$ is Δ-convergent to u. The domain of f is defined by $\{x \in X \mid f(x) \in \mathbb{R}\}$, which is denoted by $\mathrm{dom} f$. The function f is said to be proper if $\mathrm{dom} f$ is nonempty. f is said to be convex if

$$f(\alpha x \oplus (1-\alpha)y) \leqq \alpha f(x) + (1-\alpha) f(y)$$

holds for all $x, y \in X$ and $\alpha \in]0, 1[$.

Lemma 2.5 ([2])**.** *Let X be a complete* CAT(0) *space and f a proper lower semicontinuous convex function of X into $]-\infty, \infty]$. Then f is Δ-lower semicontinuous.*

Let X be a complete CAT(-1) space and f a proper lower semicontinuous convex function of X into $]-\infty, \infty]$. Then the resolvent of f is defined by

$$R_f x = \operatorname*{argmin}_{y\in X} \{f(y) + \tanh d(y, x) \sinh d(y, x)\}$$

for all $x \in X$. The authors [5] showed that it is well-defined and the following important properties hold:

- R_f is firmly hyperbolically vicinal in the sense that,

$$\begin{aligned}(C_x^2(1+C_y^2)C_y + C_y^2(1+C_x^2)C_x)\cosh d(R_f x, R_f y) \\ \leqq C_x^2(1+C_y^2)\cosh d(R_f x, y) + C_y^2(1+C_x^2)\cosh d(x, R_f y)\end{aligned} \tag{1}$$

  for all $x, y \in X$, where $C_z = \cosh d(R_f z, z)$ for $z \in X$. See also [8];

- $\mathcal{F}(R_f) = \mathrm{argmin}_X f$;

- if $\mathcal{F}(R_f)$ is nonempty, then R_f is quasi-nonexpansive, that is, $d(R_f x, z) \leqq d(x, z)$ for $x \in X$ and $z \in \mathcal{F}(R_f)$.

3. Fundamental properties of the resolvents

In this section, we show some important properties which are needed to show the proximal point algorithm in complete CAT(-1) spaces.

Theorem 3.1. *Let X be a complete* CAT(-1) *space, f a proper lower semicontinuous convex function of X into $]-\infty, \infty]$ and R_λ a resolvent of λf for $\lambda > 0$. Set $C_{\lambda,z} = \cosh d(R_\lambda z, z)$ for all $z \in X$. Then the inequalities*

$$\begin{aligned}&\lambda\left(f(R_\lambda x) - f(R_\mu y)\right)\sinh d(R_\lambda x, R_\mu y)\\&\leqq \left(\frac{1}{C_{\lambda,x}^2} + 1\right) d(R_\lambda x, R_\mu y)\left(\cosh d(R_\mu y, x) - C_{\lambda,x}\cosh d(R_\lambda x, R_\mu y)\right)\end{aligned}\tag{2}$$

and

$$\begin{aligned}&\left(\lambda C_{\lambda,x}^2(1 + C_{\mu,y}^2)C_{\mu,y} + \mu C_{\mu,y}^2(1 + C_{\lambda,x}^2)C_{\lambda,x}\right)\cosh d(R_\lambda x, R_\mu y)\\&\leqq \lambda C_{\lambda,x}^2(1 + C_{\mu,y}^2)\cosh d(R_\lambda x, y) + \mu C_{\mu,y}^2(1 + C_{\lambda,x}^2)\cosh d(R_\mu y, x)\end{aligned}\tag{3}$$

hold for all $x, y \in X$ and $\lambda, \mu > 0$.

Proof. Let $\lambda, \mu > 0$ and $x, y \in X$. Set $z_t = tR_\mu y \oplus (1-t)R_\lambda x$ for $t \in]0, 1[$ and let $D = d(R_\lambda x, R_\mu y)$. By the definition of R_λ and the convexity of f, we have

$$\begin{aligned}&\lambda f(R_\lambda x) + \tanh d(R_\lambda x, x)\sinh d(R_\lambda x, x)\\&\leqq \lambda f(z_t) + \tanh d(z_t, x)\sinh d(z_t, x)\\&\leqq t\lambda f(R_\mu y) + (1-t)\lambda f(R_\lambda x) + \tanh d(z_t, x)\sinh d(z_t, x)\end{aligned}$$

and hence

$$\begin{aligned}&\lambda t\left(f(R_\lambda x) - f(R_\mu y)\right)\\&\leqq \tanh d(z_t, x)\sinh d(z_t, x) - \tanh d(R_\lambda x, x)\sinh d(R_\lambda x, x)\\&= \left(\frac{1}{\cosh d(R_\lambda x, x)\cosh d(z_t, x)} + 1\right)\left(\cosh d(z_t, x) - \cosh d(R_\lambda x, x)\right).\end{aligned}$$

Then multiplying $(\sinh D)/t$ and using Lemma 2.1, we get

$$\begin{aligned}&\lambda\left(f(R_\lambda x) - f(R_\mu y)\right)\sinh D\\&\leqq \left(\frac{1}{\cosh d(R_\lambda x, x)\cosh d(z_t, x)} + 1\right)\\&\qquad \times \frac{1}{t}\left(\cosh d(z_t, x)\sinh D - \cosh d(R_\lambda x, x)\sinh D\right)\\&\leqq \left(\frac{1}{\cosh d(R_\lambda x, x)\cosh d(z_t, x)} + 1\right)\\&\qquad \times \frac{1}{t}\left(\cosh d(R_\mu y, x)\sinh tD - \cosh d(R_\lambda x, x)(\sinh D - \sinh(1-t)D)\right)\\&= \left(\frac{1}{\cosh d(R_\lambda x, x)\cosh d(z_t, x)} + 1\right)\frac{2}{t}\sinh\left(\frac{t}{2}D\right)\\&\qquad \times \left[\cosh d(R_\mu y, x)\cosh\left(\frac{t}{2}D\right) - \cosh d(R_\lambda x, x)\cosh\left(\left(1 - \frac{t}{2}\right)D\right)\right].\end{aligned}$$

Letting $t \downarrow 0$, we have

$$\lambda\left(f(R_\lambda x) - f(R_\mu y)\right)\sinh D \leqq \left(\frac{1}{C_{\lambda,x}^2} + 1\right) D\left(\cosh d(R_\mu y, x) - C_{\lambda,x}\cosh D\right)$$

and this inequality is (2).

From (2), we have

$$\mu\lambda\left(f(R_\lambda x) - f(R_\mu y)\right)\sinh D \leqq \mu D\left(\frac{1}{C_{\lambda,x}^2} + 1\right)\left(\cosh d(R_\mu y, x) - C_{\lambda,x}\cosh D\right)$$

and

$$\mu\lambda\left(f(R_\mu y) - f(R_\lambda x)\right)\sinh D \leqq \lambda D\left(\frac{1}{C_{\mu,y}^2} + 1\right)\left(\cosh d(R_\lambda x, y) - C_{\mu,y}\cosh D\right).$$

Adding these inequalities, we get

$$\begin{aligned} 0 \leqq \mu D&\left(\frac{1}{C_{\lambda,x}^2} + 1\right)\left(\cosh d(R_\mu y, x) - C_{\lambda,x}\cosh D\right) \\ &+ \lambda D\left(\frac{1}{C_{\mu,y}^2} + 1\right)\left(\cosh d(R_\lambda x, y) - C_{\mu,y}\cosh D\right) \end{aligned}$$

and hence we obtain (3). □

The inequality (3) is a generalization of (1). In fact, if $\lambda = \mu = 1$, then (3) becomes (1). Using Theorem 3.1, we obtain the following corollary.

Corollary 3.2. *Let X, f, λ and R_λ be the same as in Theorem 3.1. If $\operatorname{argmin}_X f$ is nonempty, then the following hold:*

(i) $\lambda(f(R_\lambda x) - f(u)) \leqq 2(\cosh d(u, x) - C_{\lambda,x}\cosh d(u, R_\lambda x))$;

(ii) $C_{\lambda,x}\cosh d(u, R_\lambda x) \leqq \cosh d(u, x)$

for all $x \in X$ and $u \in \operatorname{argmin}_X f$.

Proof. We first show (i). Let $x \in X$ and $u \in \operatorname{argmin}_X f$. Since $\mathcal{F}(R_f) = \operatorname{argmin}_X f$, it follows from (2) that

$$\begin{aligned} &\lambda(f(R_\lambda x) - f(u))\sinh d(u, R_\lambda x) \\ &\leqq \left(\frac{1}{C_{\lambda,x}^2} + 1\right) d(u, R_\lambda x)(\cosh d(u, x) - C_{\lambda,x}\cosh d(u, R_{\lambda,x})). \end{aligned}$$

Suppose that $u \neq R_\lambda x$. Since $0 < t/\sinh t < 1$ and $\cosh t \geqq 1$ for all $t > 0$, we get

$$\begin{aligned} &\lambda(f(R_\lambda x) - f(u)) \\ &\leqq \left(\frac{1}{C_{\lambda,x}^2} + 1\right)\frac{d(u, R_\lambda x)}{\sinh d(u, R_\lambda x)}(\cosh d(u, x) - C_{\lambda,x}\cosh d(u, R_{\lambda,x})) \\ &< 2(\cosh d(u, x) - C_{\lambda,x}\cosh d(u, R_{\lambda,x})). \end{aligned}$$

If $u = R_\lambda x$, then it is obvious to hold with equality. Thus we obtain (i).

We next show (ii). Since $u \in \operatorname{argmin}_X f$, (i) implies that

$$0 \leqq 2(\cosh d(u, x) - C_{\lambda,x}\cosh d(u, R_\lambda x))$$

and hence we get the conclusion. □

4. The proximal point algorithm

In this section, we show the proximal point algorithm in complete CAT(-1) spaces. We remark that the following important properties hold:

- $\mathcal{F}(R_\lambda) = \operatorname{argmin}_X f$;
- if $\mathcal{F}(R_\lambda)$ is nonempty, then R_λ is quasi-nonexpansive.

Theorem 4.1. *Let X be a complete* CAT(-1) *space, $\{z_n\}$ a bounded sequence in X, $\{\beta_n\}$ a sequence of positive real numbers with $\sum_{n=1}^{\infty} \beta_n = \infty$ and*

$$g(y) = \limsup_{n\to\infty} \frac{1}{\sum_{l=1}^{n} \beta_l} \sum_{k=1}^{n} \beta_k \cosh d(y, z_k)$$

for all $y \in X$. Then $\operatorname{argmin}_X g$ *consists of one point.*

Proof. Fix $y \in X$. Since $\{z_n\}$ is bounded, there exists $K > 0$ such that

$$\cosh d(y, z_k) < \cosh K$$

and hence $g(y) < \infty$. Therefore it follows that

$$1 \leqq \inf_{y\in X} g(y) < \infty.$$

Let $\{y_n\}$ be a sequence in X with $g(y_{n+1}) \leqq g(y_n)$ and $\lim_n g(y_n) = l$, where $l = \inf g(X)$. From Corollary 2.2, we get

$$\cosh d\left(\frac{1}{2}y_n \oplus \frac{1}{2}y_m, z_k\right) \cosh d\left(\frac{1}{2}d(y_n, y_m)\right) \leqq \frac{1}{2}\cosh d(y_n, z_k) + \frac{1}{2}\cosh d(y_m, z_k)$$

and hence

$$l \cosh\left(\frac{1}{2}d(y_n, y_m)\right) \leqq g\left(\frac{1}{2}y_n \oplus \frac{1}{2}y_m\right) \cosh\left(\frac{1}{2}d(y_n, y_m)\right) \leqq \frac{1}{2}g(y_n) + \frac{1}{2}g(y_m).$$

Suppose that $m \geqq n$. Then, by the definition of $\{y_n\}$ and $1 \leqq l < \cosh K$, we have

$$\cosh\left(\frac{1}{2}d(y_n, y_m)\right) \leqq \frac{g(y_n)}{l} \to 1.$$

This implies that $\{y_n\}$ is a Cauchy sequence. Further, since X is complete, we know that $\{y_n\}$ is convergent to some $p \in X$. By the continuity of g, we have $g(p) = \lim_n g(y_n) = l$. Thus we have $p \in \operatorname{argmin}_X g$.

We next show the uniqueness of p. Let $p, q \in \operatorname{argmin}_X g$. Then, from the proof above, we know that the inequality

$$l \cosh\left(\frac{1}{2}d(p, q)\right) \leqq \frac{1}{2}g(p) + \frac{1}{2}g(q) = l$$

holds. This inequality implies that $p = q$. Thus $\operatorname{argmin}_X g$ consists of one point. □

Theorem 4.2. *Let X be a complete* CAT(-1) *space, f a proper lower semicontinuous convex function of X into $]-\infty, \infty]$, and $\{\lambda_n\}$ a sequence of positive real numbers with $\sum_{n=1}^{\infty} \lambda_n = \infty$. If a sequence $\{x_n\}$ of X is defined by $x_1 \in X$ and*

$$x_{n+1} = \operatorname*{argmin}_{y\in X} \left\{ f(y) + \frac{1}{\lambda_n} \tanh d(y, x_n) \sinh d(y, x_n) \right\}$$

for all $n \in \mathbb{N}$, then $\operatorname{argmin}_X f$ *is nonempty if and only if $\{x_n\}$ is bounded.*

Proof. We first suppose that $\operatorname{argmin}_X f$ is nonempty and show that $\{x_n\}$ is bounded. Let $u \in \operatorname{argmin}_X f$. Then, since R_{λ_n} is quasi-nonexpansive, we have

$$d(u, x_{n+1}) \leqq d(u, x_n) \leqq \cdots \leqq d(u, x_1).$$

This inequality implies the conclusion.

We next show the other direction. Suppose that $\{x_n\}$ is bounded. Put

$$\beta_n = \frac{\lambda_n C^2_{\lambda_n, x_n}}{1 + C^2_{\lambda_n, x_n}}$$

for all $n \in \mathbb{N}$. Then it is obvious that $\beta_n > 0$. Further, since

$$\beta_n = \frac{\lambda_n C^2_{\lambda_n, x_n}}{1 + C^2_{\lambda_n, x_n}} \geqq \frac{\lambda_n C^2_{\lambda_n, x_n}}{2C^2_{\lambda_n, x_n}} = \frac{\lambda_n}{2} \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n = \infty,$$

we get $\sum_{n=1}^{\infty} \beta_n = \infty$. Thus, by Theorem 4.1, we know that $\operatorname{argmin}_X g$ is nonempty, where

$$g(y) = \limsup_{n\to\infty} \frac{1}{\sum_{l=1}^{n} \beta_l} \sum_{k=1}^{n} \beta_k \cosh d(y, x_{k+1})$$

for all $y \in X$.

Let $p \in \operatorname{argmin}_X g$ and $\mu > 0$. Using Lemma 3.1, we have

$$\begin{aligned}
&\left(\lambda_k C^2_{\lambda_k, x_k}(1 + C^2_{\mu,p}) + \mu C^2_{\mu,p}(1 + C^2_{\lambda_k, x_k})\right) \cosh d(x_{k+1}, R_\mu p) \\
&\leqq \lambda_k C^2_{\lambda_k, x_k}(1 + C^2_{\mu,p}) \cosh d(x_{k+1}, p) + \mu C^2_{\mu,p}(1 + C^2_{\lambda_k, x_k}) \cosh d(R_\mu p, x_k)
\end{aligned}$$

and hence

$$\begin{aligned}
&\frac{\lambda_k C^2_{\lambda_k, x_k}}{1 + C^2_{\lambda_k, x_k}} \cosh d(x_{k+1}, R_\mu p) \\
&\leqq \frac{\lambda_k C^2_{\lambda_k, x_k}}{1 + C^2_{\lambda_k, x_k}} \cosh d(x_{k+1}, p) + \frac{\mu C^2_{\mu,p}}{1 + C^2_{\mu,p}} \left(\cosh d(R_\mu p, x_k) - \cosh d(R_\mu p, x_{k+1})\right).
\end{aligned}$$

Put $\sigma_n = \sum_{l=1}^{n} \beta_l$. Adding both sides of the inequality above from $k = 1$ to $k = n$, we get

$$\begin{aligned}
&\frac{1}{\sigma_n} \sum_{k=1}^{n} \beta_k \cosh d(x_{k+1}, R_\mu p) \\
&\leqq \frac{1}{\sigma_n} \sum_{k=1}^{n} \beta_k \cosh d(x_{k+1}, p) + \frac{\mu C^2_{\mu,p}}{1 + C^2_{\mu,p}} \times \frac{\cosh d(R_\mu p, x_1) - 1}{\sigma_n}.
\end{aligned}$$

Since $\lim_n \sigma_n = \infty$, we obtain

$$\begin{aligned}
g(p) \leqq g(R_\mu p) &= \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} \beta_k \cosh d(x_{k+1}, R_\mu p) \\
&\leqq \limsup_{n\to\infty} \frac{1}{\sigma_n} \sum_{k=1}^{n} \beta_k \cosh d(x_{k+1}, p) = g(p).
\end{aligned}$$

This implies that $R_\mu p = p$. Further, since $\mathcal{F}(R_\lambda) = \operatorname{argmin}_X f$, p is an element of $\operatorname{argmin}_X f$. □

We finally show the proximal point algorithm in complete CAT(−1) spaces.

Theorem 4.3. *Let X be a complete* CAT(-1) *space, f a proper lower semicontinuous convex function of X into $]-\infty,\infty]$ with* $\operatorname{argmin}_X f$ *is nonempty, and $\{\lambda_n\}$ a sequence of positive real numbers with $\sum_{n=1}^{\infty}\lambda_n=\infty$. If a sequence $\{x_n\}$ of X is defined by $x_1\in X$ and*

$$x_{n+1}=\operatorname*{argmin}_{y\in X}\left\{f(y)+\frac{1}{\lambda_n}\tanh d(y,x_n)\sinh d(y,x_n)\right\},$$

then the following hold:

(i) *If $u\in\operatorname{argmin}_X f$, then*

$$(f(x_{n+1})-\min f(X))\leqq\frac{2}{\sum_{k=1}^{n}\lambda_n}(\cosh d(u,x_1)-1)$$

for all $n\in\mathbb{N}$;

(ii) *$\{x_n\}$ is Δ-convergent to an element of* $\operatorname{argmin}_X f$.

Proof. We first show (i). Let $u\in\operatorname{argmin}_X f$. By the definition of R_{λ_n}, we have

$$\begin{aligned} f(u)&\leqq f(x_{n+1}) \\ &\leqq f(x_{n+1})+\frac{1}{\lambda_n}\tanh d(x_{n+1},x_n)\sinh d(x_{n+1},x_n) \\ &\leqq f(x_n). \end{aligned}\tag{4}$$

On the other hand, from Corollary 3.2, we know that the inequality

$$\lambda_n(f(x_{n+1}-f(u)))\leqq 2(\cosh d(u,x_n)-\cosh d(u,x_{n+1}))\tag{5}$$

holds. Using (4) and (5), we get

$$\begin{aligned} \lambda_k(f(x_{n+1})-f(u))&\leqq\lambda_k(f(x_{k+1})-f(u)) \\ &\leqq 2(\cosh d(u,x_k)-\cosh d(u,x_{k+1})) \end{aligned}$$

for all $n\in\mathbb{N}$ and $k\in\{1,2,\dots,n\}$. Adding both sides from $k=1$ to $k=n$, we have

$$\begin{aligned} (f(x_{n+1})-\min f(X))\sum_{k=1}^{n}\lambda_k&\leqq 2\sum_{k=1}^{n}(\cosh d(u,x_k)-\cosh d(u,x_{k+1})) \\ &=2(\cosh d(u,x_1)-\cosh d(u,x_{n+1})) \\ &\leqq 2(\cosh d(u,x_1)-1) \end{aligned}$$

and hence we obtain (i).

We next show (ii). From (i), we have

$$\lim_{n\to\infty}f(x_n)=\inf f(X).$$

Further, since R_{λ_n} is quasi-nonexpansive, we get

$$d(u,x_{n+1})\leqq d(u,x_n)\leqq\cdots\leqq d(u,x_1)\tag{6}$$

for all $u\in\operatorname{argmin}_X f$ and hence $\{x_n\}$ is bounded. Therefore $\omega_\Delta(\{x_n\})$ is nonempty and let $z\in\omega_\Delta(\{x_n\})$. Then we can find a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ whose Δ-limit is z. By Lemma 2.5, we have

$$f(z)\leqq\liminf_{i\to\infty}f(x_{n_i})=\lim_{n\to\infty}f(x_n)=\inf f(X)$$

and hence $z\in\operatorname{argmin}_X f$. From the inequality (6), we know that $\{d(z,x_n)\}$ is convergent. Further, from Lemma 2.4, we know that $\{x_n\}$ is Δ-convergent and hence it is obvious that its Δ-limit is an element of $\operatorname{argmin}_X f$. Thus we get the conclusion. □

Using Theorems 4.2 and 4.3, we obtain the following corollary.

Corollary 4.4. *Let X be a complete* CAT(-1) *space, f a proper lower semicontinuous convex function of X into $]-\infty,\infty]$ and R_f a resolvent of f. Then the following hold:*

(i) $\operatorname{argmin}_X f$ *is nonempty if and only if $\{R_f^n x\}$ is bounded for some $x \in X$;*
(ii) *if* $\operatorname{argmin}_X f$ *is nonempty then $\{R_f^n x\}$ is Δ-convergent to an element of* $\operatorname{argmin}_X f$ *for each $x \in X$.*

Proof. Put $\lambda_n = 1$ for all $n \in \mathbb{N}$. Then, we have $\lambda_n > 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Thus, using Theorems 4.2 and 4.3, we get (i) and (ii). □

We may unify Theorems 1.1, 1.2, and 4.3 by using the following notation and function. For $\kappa \in \mathbb{R}$, let X be a CAT(κ) space. We denote by D_κ the diameter of the model space M_κ^2, that is,

$$D_\kappa = \begin{cases} \dfrac{\pi}{\sqrt{\kappa}} & (\kappa > 0); \\ \infty & (\kappa \leqq 0). \end{cases}$$

Next we define a function $\varphi_\kappa \colon \mathbb{R} \to \mathbb{R}$ as follows:

$$\begin{aligned} \varphi_\kappa(t) &= t^2 + \frac{\kappa}{6}t^4 + \frac{31\kappa^2}{360}t^6 + \frac{173\kappa^3}{5040}t^8 + \frac{25261\kappa^4}{1814400}t^{10} + \cdots \\ &= \begin{cases} \dfrac{1}{\kappa}\tan(\sqrt{\kappa}t)\sin(\sqrt{\kappa}t) & (\kappa > 0); \\ t^2 & (\kappa = 0); \\ \dfrac{1}{-\kappa}\tanh(\sqrt{-\kappa}t)\sinh(\sqrt{-\kappa}t) & (\kappa < 0). \end{cases} \end{aligned}$$

Using the notation and function, we unify Theorems 1.1, 1.2, and 4.3 by the following result.

Theorem 4.5. *For $\kappa \in \mathbb{R}$, let X a* CAT(κ) *space satisfying $d(u,v) < D_\kappa/2$ for all $u,v \in X$ and f a proper lower semicontinuous convex function of X into $]-\infty,\infty]$ satisfying* $\operatorname{argmin}_X f \neq \emptyset$, *and $\{\lambda_n\}$ a sequence of positive real numbers satisfying $\sum_{n=1}^{\infty} \lambda_n = \infty$. If a sequence $\{x_n\}$ of X is defined by $x_1 \in X$ and*

$$x_{n+1} = \operatorname*{argmin}_{y \in X} \left\{ f(y) + \frac{1}{\lambda_n}\varphi_\kappa(d(y,x_n)) \right\},$$

then $\{x_n\}$ is Δ-convergent to an element of $\operatorname{argmin}_X f$.

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