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Different Approximation to Fuzzy Ring Homomorphisms

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Abstract:

In this study we approach the definition of TL −ring homomorphism. In the literature, the definition of fuzzy ring homomorphism is given by Malik and Mordeson by using their fuzzy function definition. In this study, we give the definition of fuzzy ring homomorphism by using the definition of Mustafa Demirci's fuzzy function. Some definition and theorems of ring homomorphism in classic algebra are adapted to fuzzy algebra and proved.

Keywords: Fuzzy sets, Fuzzy Relations, Fuzzy Functions, Fuzzy Ring Homomorphisms.

1. INTRODUCTION

The theory of fuzzy sets was introduced by Zadeh [10]. Fuzzy sets gives opportunity to constitute the uncertain problems in real life to mathematical models. Most of the problems in engineering, economics, medical science etc, have various uncertainties. The fuzzy set theory helps to modelling and solving these problems. Many mathematician tried to transfer the classic set theory to use the definition of Zadeh' s fuzzy set. Rosenfeld [12] gave the definition of fuzzy groups and fuzzy grupoids. Liu [9,10] gave the definition of fuzzy subrings and fuzzy ideals of a ring. Fuzzy relations are playing an important role in fuzzy modelling, fuzzy control and significant applications in relational databases, approximate reasoning, medical diagnosis. Malik and Mordeson gave some conditions to fuzzy relations to define fuzzy function [11]. With this definition, they introduced fuzzy ring homomorphism. In these studies, they used fuzzy subsets $\mu: X \to [0,1]$ and they used infimum for operation on [0,1]. In literature there isn't a certain fuzzy function definition and therefore there isn't a certain fuzzy ring homomorphism definition. In this study, we gave a different definition of fuzzy ring homomorphism. To give this definition, we used the fuzzy function definition of Demirci [2,3] and we used L-subsets $\mu: X \to L$ which L is a complete lattice and T-norms as operation of L.

In this study, we used the definition of fuzzy subrings and fuzzy ideals of a ring from Wang [14,15]. Some definitions and theorems of ring homomorphism in the classic algebra are adapted to fuzzy algebra with this definition and proved.

2. PRELIMINARY

In this section, we have presented the basic definitions and results of fuzzy algebra which may be found in the earlier studies.

Definition 2.1. [1] Let (L, ≤) be a complete lattice with top and bottom elements 1, 0, respectively. *A triangular norm* (briefly t-norm) is a binary operation T on L which is commutative, associative, monotone and has 1 as a neutral element, i.e., it is a function.

$T: L^2 \to L$ such that for all $x, y, z \in L$

$(T1)$ $T(x,y) = T(y,x)$.

$(T2)$ $T(x, T(y,z)) = T(T(x,y), z)$.

$(T3)$ $T(x,y) \leq T(x,z)$ $whenever$ $y \leq z$.

$(T4)$ $T(x,1) = x$.

Definition 2.2. [1]

a) A t-norm T on a lattice L is called ⋁-*distributive* if

$T(a, b_1 \vee b_2) = T(a, b_1) \vee T(a, b_2)$.

b) A t-norm T on a complete lattice L is called *infinitely* ⋁-*distributive* if

$$T\left(a, \bigvee_{Q} b_\tau\right) = \bigvee_{Q} T(a, b_\tau)$$

for any subset $\{a, b_\tau \in L, \tau \in Q\}$ of L.

Theorem 2.3. [1] Let L be a complete lattice. If T is a infinitely ⋁-distributive t-norm then

$$\bigvee_{i \in I} \bigvee_{j \in J} T(a_i, b_j) = T\left(\bigvee_{i \in I} a_i, \bigvee_{j \in J} b_j\right).$$

Definition 2.4. [17] Let L be a complete lattice. With *a L-subset* of X we mean a function from X into L. We denote all L-subsets set by $F(X, L)$. In particular, when L is [0,1], the L-subsets of X are called *fuzzy subsets*.

Definition 2.5. [3] If X and Y are sets then the function $f: X \times Y \to L$ is called *a L-relation* and the set of all L-relations is denoted by $F(X \times Y, L)$.

Definition 2.6. [2] Let L be a complete lattice. $E: X \times X \to L$ a L-relation E on a set X is *a TL-equivalence relation* if and only if for all $a, b, c \in X$ the following properties are satisfies;

$(E1)$ $E(a,a) = 1$.

$(E2)$ $E(a,b) = E(b,a)$.

$(E3)$ $T(E(a,b), E(b,c)) \leq E(a,c)$.

E is called *a separable TL- equivalence relation* or a *TL- equality* if in addition,

$(E4)$ $E(a,b) = 1$ implies a=b.

If E is a TL-equivalence relation on X it is shown by (X, E).

Example. [2] Let X be non-empty set and $\alpha \in L$. Then

$i)$ $EX_M(x,y) = 1$

$ii)$ $EX_L(x,y) = \begin{cases} 1 & x = y \\ 0 & x \neq y \end{cases}$

$iii)$ $EX_\propto(x,y) = \begin{cases} 1 & x = y \\ \propto & x \neq y \end{cases}$

are TL-equivalence relations of X.

Theorem 2.7. [2] Let (X,E) and (Y,F) be two equivalence relations. Then TL-subset $E \times F$

$E \times F: (X \times Y) \times (X \times Y) \to L$ defined by

$(E \times F)((x,y),(x',y')) = T(E(x,x'), F(y,y'))$

is a TL-equivalence relation.

Definition 2.8. [2] Let E be a TL-equivalence relation on a set X. A L-subset μ of X is *extensional* or *observable* w.r.t. E if and only if

$T(\mu(b), E(a,b)) \leq \mu(a) \quad \forall a, b \in X$.

Definition 2.9. [13] Let X,Y and Z are sets and $f: X \times Y \to L$ and $g: Y \times Z \to L$ be L-relations. Then $g \circ_T f: X \times Z \to L$ L-relation is called composition of f and g such that

for $(x,z) \in X \times Z$

$$g \circ_T f(x,z) = \bigvee_{y \in Y} T(f(x,y), g(y,z))$$

Definition 2.10. [3] Let $f: X \times Y \to L$ be a L-relation then we call the function

$f^{-1}: Y \times X \to L$ defined by

$f^{-1}(y,x) = f(x,y)$ the inverse of f L-relation.

Definition 2.11. [3] Let $f: X \times Y \to L$ be a L-relation and $A \in F(X,L)$ and $B \in F(Y,L)$. The L-subsets $f(A)$, $f^{-1}(B)$ defined by for all $x \in X, y \in Y$

$f(A)(y) = \vee_{x \in X} T(A(x), f(x,y))$ and

$$f^{-1}(B)(x) = \bigvee_{y \in Y} T(B(y), f(x,y))$$

are respectively *the image of A* and *the inverse image of B*.

Definition 2.12. [2] Let (X, E), (Y, F) be two TL-equivalence relations and $f \in F(X \times Y, L)$. Then;

a) f is called *an E-extensional* if the inequality
T(f(x,y), E(x,x'))≤f(x',y)
is satisfied $for\ all\ x, x' \in X\ \ and\ for\ all\ y \in Y$. We call all E-extensional L-relations set as $F(X \times Y, E, L)$.
b) f is called a *F-extensional* if the inequality
T(f(x,y), F(y,y')) ≤ f(x,y')
Is satisfied $for\ all\ x \in X\ and\ for\ all\ y, y' \in Y$. We denote all F-extensional L-relations set by $F(X \times Y, F, L)$.
c) A L-relation such as f is called *E-F-extensional* if f is E-extensional and F-extensional and denote all E-F- extensional relations set by $F(X \times Y, E, F, L)$..

Definition 2.13. [2] Let (X,E) and (Y,F) be two TL-equivalence relations and $f \in F(X \times Y, E, F, L)$ then;

a) f is called *partial TL-function* if
T(f(x,y), f(x,y'))≤F(y,y') is satisfied for all $x \in X$ and for all $y, y' \in Y$.
b) f is called *fully defined* if f fulfills the condition $\bigvee_{z\in Y} f(x,z) = 1$ for all $x \in X$.
c) A fully defined partial TL- function is called *a TL-function*.

Definition 2.14. [3] Let $f \in F(X \times Y, E, F, L)$ be a TL-function;

a) f is called surjective if and only if $\bigvee_{x\in X} f(x,y) = 1$ for all $y \in Y$.
b) f is called injective if and only if $T\big(f(x,y), f(x',y)\big) \le E(x,x')$ for all $x, x' \in X$ and $y \in Y$.

Proposition 2.15. Let (X,E) and (Y,F) be two TL-equivalence relations, $f \in F(X \times Y, E, F, L)$ and $A \in F(X, L)$. If T is a infinitely ⋁- distributive t-norm then $f(A)$ is F-extensional L-subset.

Proof. $For\ x, x' \in X, y, y' \in Y$ and

$$T\big(f(A)(y), F(y,y')\big)$$
$$= T\left(\bigvee_{x\in X} T\big(A(x), f(x,y)\big), F(y,y')\right)$$
$$= \bigvee_{x\in X} T\Big(T\big(A(x), f(x,y)\big), F(y,y')\Big)$$
$$= \bigvee_{x\in X} T\Big(A(x), T\big(f(x,y), F(y,y')\big)\Big)$$
$$\le \bigvee_{x\in X} T\big(A(x), f(x,y')\big) = f(A)(y').$$

Proposition 2.16. Let (X,E) and (Y,F) be two TL-equivalence relations, $f \in F(X \times Y, E, F, L)$ and $B \in F(Y, L)$. If T is a infinitely ⋁-distributive t-norm then $f^{-1}(B)$ is E-extensional L-subset.

Proof. $For\ x, x' \in X\ y, y' \in Y$ and

$$T\big(f^{-1}(B)(x), E(x,x')\big)$$
$$= T\left(\bigvee_{y\in Y} T\big(B(y), f(x,y)\big), E(x,x')\right)$$
$$= \bigvee_{y\in Y} T\Big(T\big(B(y), f(x,y)\big), E(x,x')\Big)$$
$$= \bigvee_{y\in Y} T\Big(B(y), T\big(f(x,y), E(x,x')\big)\Big)$$
$$\le \bigvee_{y\in Y} T\big(B(y), f(x',y)\big) = f^{-1}(B)(x').$$

Theorem 2.17. Let $f \in F(X \times Y, E, F, L)$, $g \in F(Y \times Z, F, G, L)$ be TL-functions and T be a infinitely ⋁-distributive t-norm. Then $g \circ_T f \in F(X \times Z, E, G, L)$ is a TL-function.

Proof. $For\ x, x' \in X\ y, y' \in Y, z, z' \in Z$

$$T\big((g \circ_T f)(x,z), E(x,x')\big)$$
$$= T\left(\bigvee_{y\in Y} T\big(f(x,y), g(y,z)\big), E(x,x')\right)$$
$$= \bigvee_{y\in Y} T\Big(T\big(f(x,y), E(x,x')\big), g(y,z)\Big)$$
$$\le \bigvee_{y\in Y} T\big(f(x',y), g(y,z)\big) = (g \circ_T f)(x',z)$$

and so $g \circ_T f$ is an E-extensional.

$$T\big((g \circ_T f)(x,z), G(z,z')\big)$$
$$= T\left(\bigvee_{y\in Y} T\big(f(x,y), g(y,z)\big), G(z,z')\right)$$
$$= \bigvee_{y\in Y} T\Big(f(x,y), T\big(G(z,z'), g(y,z)\big)\Big)$$
$$\leq \bigvee_{y\in Y} T\big(f(x,y), g(y,z')\big) = (g \circ_T f)(x,z')$$

Hence $g \circ_T f$ is a G-extensional.

$$T\big((g \circ_T f)(x,z), (g \circ_T f)(x,z')\big)$$
$$= T\left(\bigvee_{y\in Y} T\big(f(x,y), g(y,z)\big), \bigvee_{y'\in Y} T\big(f(x,y'), g(y',z')\big)\right)$$
$$= \bigvee_{y\in Y}\bigvee_{y'\in Y} T\Big(T\big(f(x,y), f(x,y')\big), T\big(g(y,z), g(y',z')\big)\Big)$$
$$\leq \bigvee_{y\in Y}\bigvee_{y'\in Y} T\Big(T\big(F(y,y'), g(y,z)\big), g(y',z')\Big)$$
$$\leq \bigvee_{y'\in Y} T\big(g(y',z), g(y',z')\big)$$
$$\leq \bigvee_{y'\in Y} G(z,z') = G(z,z').$$

Hence $g \circ_T f$ is a partial TL-function.

$$\bigvee_{z\in Y} (g \circ_T f)(x,z) = \bigvee_{z\in Y}\bigvee_{y\in Y} T\big(f(x,y), g(y,z)\big)$$
$$= \bigvee_{z\in Y} T\left(\bigvee_{y\in Y} f(x,y), g(y,z)\right)$$
$$= \bigvee_{z\in Y} T\big(1, g(y,z)\big) = \bigvee_{z\in Y} g(y,z) = 1.$$

Hence $g \circ_T f$ is full defined.

Finally $g \circ_T f$ is TL-function.

Theorem 2.18. Let (R_1, E_1), (R_2, E_2), (S_1, F_1) and (S_2, F_2) be TL-equivalence relations and

$f: R_1 \times S_1 \to L$

$g: R_2 \times S_2 \to L$ be TL-functions. Then the TL-equivalence relation is defined by

$g \times f: (R_1 \times R_2) \times (S_1 \times S_2) \to L$ such that

$$(g \times f)\big((x_1,x_2),(y_1,y_2)\big)$$
$$= T\big(f(x_1,y_1), g(x_2,y_2)\big)$$

is a TL-function.

Proof.

i) *For* $x_1,\ x_1^* \in R_1\ x_2,\ x_2^* \in R_2\ \ y_1 \in S_1,\ \ y_2 \in S_2$

$$T\Big((g \times f)\big((x_1,x_2),(y_1,y_2)\big), (E_1 \times E_2)\big((x_1,x_2),(x_1^*,x_2^*)\big)\Big)$$
$$= T\Big(T\big(f(x_1,y_1), g(x_2,y_2)\big), T\big(E_1(x_1,x_1^*), E_2(x_2,x_2^*)\big)\Big)$$
$$= T\Big(T\big(f(x_1,y_1), E_1(x_1,x_1^*)\big), T\big(g(x_2,y_2), E_2(x_2,x_2^*)\big)\Big)$$
$$\leq T\big(f(x_1^*,y_1), g(x_2^*,y_2)\big)$$

$= (g \times f)\big((x_1^*,x_2^*),(y_1,y_2)\big)$. Then, we prove $(g \times f)$ is $(E_1 \times E_2)$ extensional.

ii) *For* $x_1 \in R_1,\ x_2 \in R_2,\ y_1,\ y_1^* \in S_1,\ y_2,\ y_2^* \in S_2$

$$T\Big((g \times f)\big((x_1,x_2),(y_1,y_2)\big), (F_1 \times F_2)\big((y_1,y_2),(y_1^*,y_2^*)\big)\Big)$$
$$= T\Big(T\big(f(x_1,y_1), g(x_2,y_2)\big), T\big(F_1(y_1,y_1^*), F_2(y_2,y_2^*)\big)\Big)$$
$$= T\Big(T\big(f(x_1,y_1), F_1(y_1,y_1^*)\big), T\big(g(x_2,y_2), F_2(y_2,y_2^*)\big)\Big)$$
$$\leq T\big(f(x_1,y_1^*), g(x_2,y_2^*)\big)$$

$= (g \times f)\big((x_1,x_2),(y_1^*,y_2^*)\big)$.Then, we prove $(g \times f)$ is $(F_1 \times F_2)$ extensional.

iii) *For* $x_1 \in R_1,\ x_2 \in R_2,\ y_1, y_1^* \in S_1\ \ y_2,\ y_2^* \in S_2$

$$T\Big((g \times f)\big((x_1,x_2),(y_1,y_2)\big), (g \times f)\big((x_1,x_2),(y_1^*,y_2^*)\big)\Big)$$
$$= T\Big(T\big(f(x_1,y_1), g(x_2,y_2)\big), T\big(f(x_1,y_1^*), g(x_2,y_2^*)\big)\Big)$$
$$= T\Big(T\big(f(x_1,y_1), f(x_1,y_1^*)\big), T\big(g(x_2,y_2), g(x_2,y_2^*)\big)\Big)$$
$$\leq T\big(F_1(y_1,y_1^*), F_2(y_2,y_2^*)\big)$$

$= (F_1 \times F_2)\big((y_1,y_2),(y_1^*,y_2^*)\big)$. Then, we prove $(g \times f)$ is partial TL-function.

iv) For each $(x_1,x_2) \in (R_1 \times R_2)$

$$\bigvee_{(y_1,y_2)\in S_1\times S_2} (g\times f)\big((x_1,x_2),(y_1,y_2)\big)$$
$$= \bigvee_{(y_1,y_2)\in S_1\times S_2} T\big(f(x_1,y_1), g(x_2,y_2)\big)$$
$$= T\left(\bigvee_{y_1\in S_1} f(x_1,y_1), \bigvee_{y_2\in S_2} g(x_2,y_2)\right)$$
$= T(1,1) = 1$ (because f and g are fully defined)

Then, we prove $(g\times f)$ is fully defined and so $(g\times f)$ is TL-function.

Definition 2.19. [14] Let R be a ring and $\mu: R \to L$, $\vartheta: R \to L$ are L-subsets. Define $\mu +_T \vartheta, -\mu, \mu -_T \vartheta, \mu ._T \vartheta \in F(R, L)$ as follows.

$(\mu +_T \vartheta)(x)$
$$= \bigvee \{T(\mu(y),\vartheta(z)) | y,z\in R, y+z=x\}$$
$(-\mu)(x) = \mu(-x)$

$(\mu -_T \vartheta)(x)$
$$= \bigvee \{T(\mu(y),\vartheta(z)) | y,z\in R, y-z=x\}$$
$(\mu ._T \vartheta)(x)$
$$= \bigvee \{T(\mu(y),\vartheta(z)) | y,z\in R, y.z=x\}$$

Where x is any element of R. $\mu +_T \vartheta, \mu -_T \vartheta, \mu ._T \vartheta$ are called the T-sum, T-difference, and T-. product of μ and ϑ, respectively, and $-\mu$ is called the negative of μ.

Definition 2.21. [14] Let $\mu: R \to L$ such that μ satisfies conditions (R1), (R2) and (R3). Then μ is called *a TL-left ideal of R* if it also satisfies the condition

$(R5)_l \quad \mu(xy) \geq \mu(y)\ \forall x,y\in R;$

a TL-right ideal of R if it also satisfies the condition

$(R5)_r \quad \mu(xy) \geq \mu(x)\ \forall x,y\in R;$

and a TL-two sided ideal or *TL-ideal of R* if it is also satisfies the condition

$(R5)\ \mu(xy) \geq \mu(x) \vee \mu(y)\ \forall x,y\in R.$

In particular, when $T = \wedge$, a TL-left ideal, TL-right ideal and TL-ideal of R are referred to as an L-left ideal, L-right ideal and L-ideal of R, respectively we denote by $TLI_l(R)$, $TLI_r(R)$ and TLI(R), respectively, the set of all TL-left ideals of R, the set of all TL-right ideals of R and the set of all TL-ideals of R.

3. TL-Ring Homomorphisms

In this section we gave the definition of TL-ring homomorphism and carry with this definition the most of the theorems and definitions about ring homomorphisms.

Definition 3.1. Let R and S be rings, $(R,E), (S,F)$ be TL-equivalence relations and $f\in F(R\times S,E,F,L)$ be TL-function. f is called *TL-ring homomorphism* if it satisfies the following conditions

$for\ all\ x,x'\in R\ and\ y,y'\in S.$

(H1) $f(x+x', y+y') \geq T\big(f(x,y), f(x',y')\big)$

(H2) $f(x.x', y.y') \geq T\big(f(x,y), f(x',y')\big)$

Definition 3.2. Let $f\in F(R\times S,E,F,L)$ be a TL-ring homomorphism

a) If $f(0,0) = 1$ then f is called *a perfect TL-ring homomorphism*.

b) If $f(-x,-y) \geq f(x,y)\ for\ all\ x\in R\ and\ y\in S$, then f is called *a strong TL-ring homomorphism*.

c) If f is both perfect and strong TL-ring homomorphism, then it is called *as a complete TL-ring homomorphism.*

Theorem 3.3. Let $f\in F(R\times S,E,F,L)$ and $g\in F(S\times K,F,G,L)$ be TL-ring homomor-phisms and T be a infinitely ⋁-distributive t-norm. Then $g\circ_T f\in F(R\times K,E,G,L)$ TL-function is a TL-ring homomorphism.

Proof From theorem 2.17 $g\circ_T f$ is TL-function. Now we show that only $g\circ_T f$ satisfies the TL-ring homomorphism conditions.

i) $For\ x,x'\in R\ \ y,y'\in S\ and\ z,z'\in K$

$(g\circ_T f)(x+x', z+z')$
$$= \bigvee_{y\in S} T\big(f(x+x',y), g(y,z+z')\big)$$
$$= \bigvee_{y,y'\in S} T\big(f(x+x',y+y'), g(y+y',z+z')\big)$$

$$\geq \bigvee_{y,y'\in S} T\Big(T\big(f(x,y),f(x',y')\big),T\big(g(y,z),g(y',z')\big)\Big)$$

$$= T\left(\bigvee_{y\in S} T\big(f(x,y),g(y,z)\big),\bigvee_{y'\in S} T\big(f(x',y'),g(y',z')\big)\right)$$

$$= T\big((g\circ_T f)(x,z),(g\circ_T f)(x',z')\big).$$

ii) $For\ x,x'\in R\ \ y,y'\in S\ and\ z,z'\in K$

$$(g\circ_T f)(x.x',z.z')$$
$$= \bigvee_{y,y'\in S} T\big(f(x.x',y),g(y,z.z')\big)$$

$$= \bigvee_{y,y'\in S} T\big(f(x.x',y.y'),g(y.y',z.z')\big)$$

$$\geq \bigvee_{y,y'\in S} T\Big(T\big(f(x,y),f(x',y')\big),T\big(g(y,z),g(y',z')\big)\Big)$$

$$= T\left(\bigvee_{y\in S} T\big(f(x,y),g(y,z)\big),\bigvee_{y'\in S} T\big(f(x',y'),g(y',z')\big)\right)$$
$$= T\big((g\circ_T f)(x,z),(g\circ_T f)(x',z')\big).$$

Theorem 3.4. Let $f\in F(R\times S,E,F,L)$ be a perfect TL-ring homomorphism and T be a infinitely ⋁-distributive t-norm. If $A:R\to L$ is a TL-subring of R, then $f(A):S\to L$ is a TL-subring of S.

Proof. $For\ x,\ x'\in R\ \ y,y'\in S$

$\boldsymbol{R1})\ Since\ for\ x=0$

$T\big(A(0),f(0,0)\big)=1\ then$

$$f(A)(0)=\bigvee_{x\in R} T\big(A(x),f(x,0)\big)\ =1$$

R2) $f(A)(-y)=\big(-f(A)\big)(y)$

$$= -\bigvee_{x\in R} T\big(A(x),f(x,y)\big)$$

$$= \bigvee_{x\in R} T\big((-A)(x),f(x,y)\big)$$

$$= \bigvee_{x\in R} T\big(A(-x),f(x,y)\big)$$

$$\geq \bigvee_{x\in R} T\big(A(x),f(x,y)\big)=f(A)(y)$$

R3)$f(A)(y+y')=\bigvee_{x\in R} T\big(A(x),f(x,y+y')\big)$

$$= \bigvee_{x,x'\in R} T\big(A(x+x'),f(x+x',y+y')\big)$$

$$\geq \bigvee_{x,x'\in R} T\Big(T\big(A(x),A(x')\big),T\big(f(x,y),f(x',y')\big)\Big)$$

$$= T\left(\bigvee_{x\in R} T\big(A(x),f(x,y)\big),\bigvee_{x'\in R} T\big(A(x'),f(x',y')\big)\right)$$
$$= T\big(f(A)(y),f(A)(y')\big)$$

R4)$f(A)(y.y')=\bigvee_{x\in R} T\big(A(x),f(x,y.y')\big)$

$$= \bigvee_{x,x'\in R} T\big(A(x.x'),f(x.x',y.y')\big)$$

$$\geq \bigvee_{x,x'\in R} T\Big(T\big(A(x),A(x')\big),T\big(f(x,y),f(x',y')\big)\Big)$$

$$= T\left(\bigvee_{x\in R} T\big(A(x),f(x,y)\big),\bigvee_{x'\in R} T\big(A(x'),f(x',y')\big)\right)$$

$$= T\big(f(A)(y),f(A)(y')\big)$$

Theorem 3.5. Let $f\in F(R\times S,E,F,L)$ be a perfect TL-ring homomorphism and T be a infinitely ⋁-distributive t-norm. If $B:S\to L$ is a TL-subring of S, then $f^{-1}(B):R\to L$ is a TL-subring of R.

Proof. $For\ x,\ x'\in R\ \ y,y'\in S$

R1)$f^{-1}(B)(0)=\bigvee_{y\in S} T\big(B(y),f(0,y)\big)$

for $y=0$ $\ T\big(B(0),f(0,0)\big)=1$ then

$f^{-1}(B)(0)=1.$

R2)$f^{-1}(B)(-x)=-f^{-1}(B)(x)$

$$= -\bigvee_{y\in S} T\big(B(y),f(x,y)\big)$$

$$= \bigvee_{y\in S} T\big((-B)(y),f(x,y)\big)$$

$$= \bigvee_{y\in S} T\big(B(-y),f(x,y)\big)$$

$$\geq \bigvee_{y\in S} T\big(B(y),f(x,y)\big)=f^{-1}(B)(x).$$

R3) $f^{-1}(B)(x+x') = \bigvee_{y\in S} T\big(B(y), f(x+x',y)\big)$

$= \bigvee_{y,y'\in S} T\big(B(y+y'), f(x+x', y+y')\big)$

$\geq \bigvee_{y,y'\in S} T\Big(T\big(B(y),B(y')\big), T\big(f(x,y), f(x',y')\big)\Big)$

$= T\left(\bigvee_{y\in S} T\big(B(y), f(x,y)\big), \bigvee_{y'\in S} T\big(B(y'), f(x',y')\big)\right)$

$= T\big(f^{-1}(B)(x), f^{-1}(B)(x')\big).$

R4) $f^{-1}(B)(x.x') = \bigvee_{y\in S} T\big(B(y), f(x.x',y)\big)$

$= \bigvee_{y,y'\in S} T\big(B(y.y'), f(x.x', y.y')\big)$

$\geq \bigvee_{y,y'\in S} T\Big(T\big(B(y),B(y')\big), T\big(f(x,y), f(x',y')\big)\Big)$

$= T\left(\bigvee_{y\in S} T\big(B(y), f(x,y)\big), \bigvee_{y'\in S} T\big(B(y'), f(x',y')\big)\right)$

$= T\big(f^{-1}(B)(x), f^{-1}(B)(x')\big).$

Theorem 3.6. Let $f \in F(R\times S, E, F, L)$ be a surjective perfect TL-ring homomorphism and T be a infinitely ⋁-distributive t-norm. If $A: R \to L$ is a TL-ideal of R, then $f(A): S \to L$ is a TL-ideal of S.

Proof. The conditions R1, R2 and R3 are provided by Theorem 3.4.

R5$)_l$ $f(A)(y.y') = \bigvee_{x\in R} T\big(A(x), f(x, y.y')\big)$

$= \bigvee_{x,x'\in R} T\big(A(x.x'), f(x.x', y.y')\big)$

$\geq \bigvee_{x,x'\in R} T\Big(A(x), T\big(f(x,y), f(x',y')\big)\Big)$

$= T\left(\bigvee_{x\in R} T\big(A(x), f(x,y)\big), \bigvee_{x'\in R} f(x',y')\right)$

(f is surjective then $\bigvee_{x'\in R} f(x',y') = 1$)

$= \bigvee_{x\in R} T\big(A(x), f(x,y)\big) = f(A)(y).$

R5$)_r$ $f(A)(y.y') = \bigvee_{x\in R} T\big(A(x), f(x, y.y')\big)$

$= \bigvee_{x,x'\in R} T\Big(A\big(x.x', f(x.x', y.y')\big)\Big)$

$\geq \bigvee_{x,x'\in R} T\Big(A(x'), T\big(f(x,y), f(x',y')\big)\Big).$

Theorem 3.7. Let $f \in F(R\times S, E, F, L)$ be a perfect TL-ring homomorphism and T be a infinitely ⋁-distributive t-norm. If $B: S \to L$ is a TL-ideal of S, then $f^{-1}(B): R \to L$ is a TL-ideal of R.

Proof. The conditions R1, R2 and R3 are provided by Theorem 3.5.

For $x, x' \in R \quad y, y' \in S$

R5$)_l$ $f^{-1}(B)(x.x') = \bigvee_{y\in S} T\big(B(y), f(x.x',y)\big)$

$= \bigvee_{y,y'\in S} T\big(B(y.y'), f(x.x', y.y')\big)$

$= T\left(\bigvee_{y\in S} T\big(B(y), f(x,y)\big), \bigvee_{y'\in S} f(x',y')\right)$

$\left(\bigvee_{y'\in S} f(x',y') = 1 \text{ for all } x' \in R \text{ because f is TL} - \text{function. Then}\right)$

$= T\left(\bigvee_{y\in S} T\big(B(y), f(x,y)\big), 1\right)$

$= \bigvee_{y\in S} T\big(B(y), f(x,y)\big) = f^{-1}(B)(x).$

R5$)_r$ $f^{-1}(B)(x.x') = \bigvee_{y\in S} T\big(B(y), f(x.x',y)\big)$

$= \bigvee_{y,y'\in S} T\big(B(y.y'), f(x.x', y.y')\big)$

$\geq \bigvee_{y,y'\in S} T\Big(B(y'), T\big(f(x,y), f(x',y')\big)\Big)$

$= T\left(\bigvee_{y\in S} f(x,y), \bigvee_{y'\in S} T\big(B(y'), f(x',y')\big)\right)$

$$\left(\bigvee_{y\in S} f(x,y)=1\, for\ all\ x\in R\right)$$

because f is TL-function. Then

$$=\bigvee_{y'\in S} T\big(B(y'), f(x',y')\big)=f^{-1}(B)(x').$$

Definition 3.8. Let $f\in F(R\times S,E,F,L)$ be a TL-ring homomorphism. Define Kerf $\in F(R,L)$ and Imf $\in F(S,L)$ as follows.

$(Kerf)(x)=f(x,0)$ and

$$(Imf)(y)=\bigvee_{x\in R} f(x,y)$$

where $x\in R\ and\ y\in S$. Kerf is called *the kernel of f* and Imf is called *the image of f.*

Theorem 3.9. Let $f\in F(R\times S,E,F,L)$ be a complete TL-ring homomorphism then Kerf $\in F(R,L)$ is a TL-subring of R.

Proof. $For\ x,y\in R$

R1) $Kerf(0)=f(0,0)=1$ because f is complete TL-ring homomorphism.

R2) $Kerf(-x)=f(-x,0)\geq f(x,0)$

$=Kerf(x).$

R3) $Kerf(x+y)=f(x+y,0)$

$=f(x+y,0+0)\geq T\big(f(x,0),f(y,0)\big)$

$=T\big(Kerf(x),Kerf(y)\big).$

R4) $Kerf(xy)=f(xy,0)=f(xy,00)$

$\geq T\big(f(x,0),f(y,0)\big)=T\big(Kerf(x),Kerf(y)\big).$

Theorem 3.10. Let $f\in F(R\times S,E,F,L)$ be a complete TL-ring homomorphism and T be a infinitely $\bigvee$-distributive t-norm. Then Imf $\in F(S,L)$ is a TL-subring of S.

Proof.$For\ x,x'\in R\quad y,y'\in S$

R1) It follows $(Imf)(0)$

$$=\bigvee_{x\in R} f(x,0)=1$$ because for x = 0

$f(0,0)=1.$

R2)$(Imf)(-y)=\bigvee_{x\in R} f(x,-y)\geq\bigvee_{x\in R} f(x,y)$

$=(Imf)(y)$ (because f is complete).

R3)$(Imf)(y+y')=\bigvee_{x\in R} f(x,y+y')$

$$=\bigvee_{x,x'\in R} f(x+x',y+y')$$

$$\geq\bigvee_{x,x'\in R} T\big(f(x,y),f(x',y')\big)$$

$$=T\left(\bigvee_{x\in R} f(x,y),\bigvee_{x'\in R} f(x',y')\right)$$

$=T\big(Imf(y),Imf(y')\big).$

R4)$(Imf)(yy')=\bigvee_{x\in R} f(x,yy')$

$$=\bigvee_{x,x'\in R} f(xx',yy')$$

$$\geq\bigvee_{x,x'\in R} T\big(f(x,y),f(x',y')\big)$$

$$T\left(\bigvee_{x\in R} f(x,y),\bigvee_{x'\in R} f(x',y')\right)$$

$=T\big(Imf(y),Imf(y')\big).$

Theorem 3.11. Let R and S be rings, (R,ER_L) and (S,FS_L) be TL-equalities and $f:R\to S$ function be a ring homomorphism. Then TL-function $\bar{f}\in F(R\times S,E,F,L)$ $\bar{f}:R\times S\to L$ defined by

$$\bar{f}(x,y)=\begin{cases}1 & f(x)=y\\ 0 & f(x)\neq y\end{cases}$$

is a TL-ring homomorphism.

Proof. It is clear that $\bar{f}$ is a TL-function. Let obtain the TL-ring homomorphism conditions.

a) $\bar{f}(x+x',y+y')$

$$=\begin{cases}1 & f(x+x')=y+y'\\ 0 & f(x+x')\neq y+y'\end{cases}$$

$$=\begin{cases}1 & f(x)+f(x')=y+y'\\ 0 & f(x)+f(x')\neq y+y'\end{cases}$$

i) If $f(x+x')=y+y'$ then

$\bar{f}(x+x',y+y')=1$ and the inequality

$\bar{f}(x+x',y+y')\geq T\left(\bar{f}(x,y),\bar{f}(x',y')\right)$

is satisfied.

ii) If $f(x+x') \neq y+y'$ then $f(x)=y$ and $f(x')=y'$ can't be both because if it can

$f(x)+f(x')=y+y' \Rightarrow f(x+x')=y+y'$

so one of them doesn't exist then

$T\left(\bar{f}(x,y),\bar{f}(x',y')\right)=T(0,1)=0$

and the inequality

$\bar{f}(x+x',y+y') \geq T\left(\bar{f}(x,y),\bar{f}(x',y')\right)$

is satisfied.

b) $\bar{f}(xx',yy')=\begin{cases}1 & f(xx')=yy'\\ 0 & f(xx')\neq yy'\end{cases}$

$=\begin{cases}1 & f(x)f(x')=yy'\\ 0 & f(x)f(x')\neq yy'\end{cases}$

i) If $f(xx')=yy'$ then $\bar{f}(xx',yy')=1$ and $\bar{f}(xx',yy') \geq T\left(\bar{f}(x,y),\bar{f}(x',y')\right)$ is satisfied.

ii) If $f(xx')\neq yy'$ then $f(x)=y$ and $f(x')=y'$ can't be both because if it can

$f(x)f(x')=yy' \Rightarrow f(xx')=yy'$

so one of them doesn't exist then

$T\left(\bar{f}(x,y),\bar{f}(x',y')\right)=T(0,1)=0$ and the inequality

$\bar{f}(xx',yy') \geq T\left(\bar{f}(x,y),\bar{f}(x',y')\right)$ is satisfied.

Theorem 3.12. Let $f \in F(R\times S,E,F,L)$ and $g \in F(S\times K,F,G,L)$ be TL-ring homomorphisms. Then

a) $Ker(g\circ_T f)=f^{-1}(Kerg)$.

b) $Im(g\circ_T f)=g(Imf)$.

Proof. a) $Ker(g\circ_T f)=(g\circ_T f)(x,0)$

$=\bigvee_{y\in S} T\big(f(x,y),g(y,0)\big)$

$=\bigvee_{y\in S} T\big(f(x,y),Kerg(y)\big)=f^{-1}(Kerg)(x)$.

b) $Im(g\circ_T f)(z)=\bigvee_{x\in R}(g\circ_T f)(x,z)$

$=\bigvee_{x\in R}\left(\bigvee_{y\in S} T\big(f(x,y),g(y,z)\big)\right)$

$=\bigvee_{y\in S} T\left(\bigvee_{x\in R} f(x,y)\,,g(y,z)\right)$

$=\bigvee_{y\in S} T\big((Imf)(y),g(y,z)\big)=g(Imf)(z)$.

Theorem 3.13. Let $f \in F(R\times S,E,F,L)$ be a strong TL-ring homomorphism and T be a infinitely ∨-distributive t-norm. Then for $A \in F(R,L)$,

$f^{-1}\big(f(A)\big) \leq A+Kerf$.

Proof.

$f^{-1}\big(f(A)\big)(x)=\bigvee_{y\in S} T\big(f(A)(y),f(x,y)\big)$

$=\bigvee_{y\in S}\left(T\left(\bigvee_{x'\in R} T\big(A(x'),f(x',y)\big)\right),f(x,y)\right)$

$=\bigvee_{y\in S}\bigvee_{x'\in R} T\left(T\big(A(x'),f(x',y)\big),f(x,y)\right)$

$\leq \bigvee_{y\in S}\bigvee_{x'\in R} T\left(T\big(A(x'),f(x',y)\big),f(-x,-y)\right)$

$=\bigvee_{y\in S}\bigvee_{x'\in R} T\left(A(x'),T\big(f(x',y),f(-x,-y)\big)\right)$

$\leq \bigvee_{y\in S}\bigvee_{x'\in R} T\big(A(x'),f(x'-x,y-y)\big)$

$\leq \bigvee T\big(A(x'),f(x-x',0)\big)$

$=\bigvee T\big(A(x'),(Kerf)(x-x')\big)$

$(by\ x'+(x-x')=x)=(A+Kerf)(x)$.

Theorem 3.14. Let (R_1, E_1), (R_2, E_2), (S_1, F_1) and (S_2, F_2) be TL-equivalence relations and

$f\colon R_1\times S_1\to L$ $g\colon R_2\times S_2\to L$ be TL-ring homomorphisms. Then the TL-function defined by

$g\times f\colon (R_1\times R_2)\times(S_1\times S_2)\to L$

$(g\times f)\big((x_1,x_2),(y_1,y_2)\big)=T(f(x_1,y_1),g(x_2,y_2))$

is a TL-ring homomorphism.

Proof. We proved that $(g\times f)$ is a TL-function in Theorem 2.18. Now we will prove the TL-ring homomorphism conditions.

H1. $x_1,x_1^*\in R_1, x_2,x_2^*\in R_2, y_1,y_1^*\in S_1$

$, y_2, y_2^* \in S_2$

$$(g \times f)\big((x_1, x_2) + (x_1^*, x_2^*), (y_1, y_2) + (y_1^*, y_2^*)\big)$$
$$= (g \times f)\big((x_1 + x_1^*, x_2 + x_2^*), (y_1 + y_1^*, y_2 + y_2^*)\big)$$
$$= T\big(f(x_1 + x_1^*, y_1 + y_1^*), g(x_2 + x_2^*, y_2 + y_2^*)\big)$$
$$\geq T\Big(T\big(f(x_1, y_1), f(x_1^*, y_1^*)\big), T\big(g(x_2, y_2), g(x_2^*, y_2^*)\big)\Big)$$
$$= T\Big(T\big(f(x_1, y_1), g(x_2, y_2)\big), T\big(f(x_1^*, y_1^*), g(x_2^*, y_2^*)\big)\Big)$$
$$= T\Big((g \times f)\big((x_1, x_2), (y_1, y_2)\big), (g \times f)\big((x_1^*, x_2^*), (y_1^*, y_2^*)\big)\Big).$$

H2. $x_1, x_1^* \in R_1, x_2, x_2^* \in R_2, y_1, y_1^* \in S_1,$

$y_2, y_2^* \in S_2$

$$(g \times f)\big((x_1, x_2).(x_1^*, x_2^*), (y_1, y_2).(y_1^*, y_2^*)\big)$$
$$= (g \times f)\big((x_1.x_1^*, x_2.x_2^*), (y_1.y_1^*, y_2.y_2^*)\big)$$
$$= T\big(f(x_1.x_1^*, y_1.y_1^*), g(x_2.x_2^*, y_2.y_2^*)\big)$$
$$\geq T\Big(T\big(f(x_1, y_1), f(x_1^*, y_1^*)\big), T\big(g(x_2, y_2), g(x_2^*, y_2^*)\big)\Big)$$
$$= T\Big(T\big(f(x_1, y_1), g(x_2, y_2)\big), T\big(f(x_1^*, y_1^*), g(x_2^*, y_2^*)\big)\Big)$$
$$= T\Big((g \times f)\big((x_1, x_2), (y_1, y_2)\big), (g \times f)\big((x_1^*, x_2^*), (y_1^*, y_2^*)\big)\Big).$$

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