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Influence Functions for the Moment Estimators

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ABSTRACT

Influence functions give a measure of robustness of the statistics estimated from a sample against the sample data. In this study, first, the concept of influence functions is examined, and then the influence functions for mean and variance are given. The influence functions for skewness and kurtosis are examined for both asymmetrical and symmetrical distributions and the influence function concept is generalized for scaled moments.

Key Words: Influence Functions, Skewness Measure, Kurtosis Measure.

1. INTRODUCTION

Robustness has gained a great amount of importance for the area of statistics since Huber's study in 1964 [1]. As a pure statistical concept it is the measure of the qualitative properties of any estimator. More definitely it describes the stability of an estimator under non-standard conditions. A measure of robustness can be delivered by influence functions.

Influence function (IF) is an approach used in determining the investigated model or understanding how an infinitely small error affects an estimator. The common name for this type of error is data contamination.

Influence function of an estimator is linked to the deviation occurring on the estimator because of an infinitely small data contamination. An unlimited influence function means that the deviation can be infinite. The behaviour of the influence function is related to the investigated statistics. More detailed information about this subject can be found in Huber [2] and Hampel's [3] studies.

2. INFLUENCE FUNCTIONS

Let R be the real line and let T be a real valued function that is defined on a subset of the set of all probability measures on R. For a defined T let a probability measure on R is denoted by F. For any point $x \in R$ let δ_x be the probability measure determined by the point mass 1. Mixture of F and δ_x can be expressed as $(1-\varepsilon)F + \varepsilon\delta_x$ for $0<\varepsilon<1$, and it is called the gross-error model.

If the limit is valid for every $x \in R$ then the influence function of the estimator T on the probability distribution F is,

$$IF(x,F,T) = \lim_{\varepsilon \to 0} \frac{\{T[(1-\varepsilon)F + \varepsilon\delta_x] - T(F)\}}{\varepsilon}$$

$$IF(x,F,T) = \lim_{\varepsilon \to 0} \frac{T(F_\varepsilon) - T(F)}{\varepsilon} \qquad (1)$$

[2]. As it can be seen from equation (1), the gross-error model, or in other words, the contaminated distribution can be written as

$$F_\varepsilon = (1-\varepsilon)F + \varepsilon\delta_x \qquad (2)$$

Here F is known (generally standard normally distributed), the ε part of the data in hand is composed of the gross-error coming from an unknown distribution (say G) and ε is also known.

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3. INFLUENCE FUNCTIONS FOR THE FIRST TWO MOMENT ESTIMATORS

First the estimator of the population mean is handled. The estimator of population mean of the contaminated distribution is,

$$\mu(F_\varepsilon) = (1-\varepsilon)\mu(F) + \varepsilon x \tag{3}$$

and for μ(*F*)= μ, the influence function becomes:

$$\mu(F_\varepsilon) = (1-\varepsilon)\mu + \varepsilon x \tag{4}$$

$$IF[x,F,\mu] = l\,\underset{\varepsilon\to 0}{im}\frac{(1-\varepsilon)\mu + \varepsilon x - \mu}{\varepsilon}$$

$$= x - \mu \tag{5}$$

[4], and when μ(*F*)= 0,

$$\mu(F_\varepsilon) = \varepsilon x \tag{6}$$

$$IF[x,F,\mu] = l\,\underset{\varepsilon\to 0}{im}\frac{\varepsilon x}{\varepsilon} = x$$

Second, the estimator of the population variance is handled. The estimator of population variance of the contaminated distribution is,

$$\sigma^2(F_\varepsilon) = (1-\varepsilon)\sigma^2(F) + \varepsilon x^2 \tag{8}$$

when $\sigma^2(F) = \sigma^2$ and μ(*F*)= 0, the influence function,

$$\sigma^2(F_\varepsilon) = (1-\varepsilon)\sigma^2 + \varepsilon x^2 \tag{9}$$

$$IF\left[x,F,\sigma^2\right] = l\,\underset{\varepsilon\to 0}{im}\frac{(1-\varepsilon)\sigma^2 + \varepsilon x^2 - \sigma^2}{\varepsilon}$$

$$= x^2 - \sigma^2 \tag{10}$$

[4], and when $\sigma^2(F) = 1$, the influence function becomes,

$$\sigma^2(F_\varepsilon) = (1-\varepsilon) + \varepsilon x^2 = \varepsilon(x^2 - 1) + 1 \tag{11}$$

$$IF\left[x,F,\sigma^2\right] = l\,\underset{\varepsilon\to 0}{im}\frac{(1-\varepsilon) + \varepsilon x^2 - 1}{\varepsilon}$$

$$= x^2 - 1 \tag{12}$$

If μ(*F*)= 0 then, $IF\left[x,F,\sigma^2\right] = F\left[x,F,\mu_2^{'}\right]$, where $\mu_2^{'}$ is the second moment about the origin. Hence when estimator of the second moment about the origin for the contaminated distribution is, $\sigma^2(F) = 1$, we can show that

$$\mu_2^{'}(F_\varepsilon) = \varepsilon(x^2 - 1) + 1\,. \tag{13}$$

4. INFLUENCE FUNCTIONS FOR SKEWNESS AND KURTOSIS

Let's consider the functional,

$$\gamma_1 = \frac{\mu_3}{\sigma^3} \tag{14}$$

as a unitless measure of skewness which uses the moment estimators. Moments about the arithmetic mean can be obtained by means of moments about the origin as,

$$(x-\mu)^r = \sum_{i=0}^{r}\binom{r}{i}(-\mu)^i (x)^{r-i}\ .$$

Here, $E\left(x^{r-i}\right) = \mu_{r-i}^{'}$ and $\mu = \mu_1^{'}$,

$$E\left[(x-\mu)^r\right] = E\left[\sum_{i=0}^{r}\binom{r}{i}\left(-\mu_1^{'}\right)^i (x)^{r-i}\right]$$

$$\mu_r = \sum_{i=0}^{r}\binom{r}{i}\left(-\mu_1^{'}\right)^i \mu_{r-i}^{'} \tag{15}$$

as defined in [5]. When *r*=3 in equation (15),

$$\mu_3(F_\varepsilon) = \mu_3^{'}(F_\varepsilon) - 3\mu_2^{'}(F_\varepsilon)\mu(F_\varepsilon) + 2\mu^3(F_\varepsilon)\,. \tag{16}$$

In order to obtain a result, the value of $\mu_3^{'}(F_\varepsilon)$ is required:

$$\mu_3^{'}(F_\varepsilon) = (1-\varepsilon)\mu_3^{'}(F) + \varepsilon x^3\,. \tag{17a}$$

This statement can be explained by the equation:

$$\mu_r^{'}(F_\varepsilon) = (1-\varepsilon)\mu_r^{'}(F) + \varepsilon x^r \tag{17b}$$

Beyond this point of the study it will be assumed that, μ(*F*)= 0 and $\sigma^2(F) = 1$. By rewriting the equation (16) and using equations (6), (13) and (17*a*), the following equation can be formed,

$$\mu_3(F_\varepsilon) = \varepsilon\left(x^3 - \mu_3^{'}(F)\right) + \mu_3^{'}(F) - 3\varepsilon x - 3\varepsilon^2\left(x^3 - x^2\right) + 2\varepsilon^3 x^3\,.$$

Hence μ(*F*)= 0, $\mu_3'(F)=\mu_3(F)$. Thus we have

$$\mu_3(F_\varepsilon)=\varepsilon\left(x^3-\mu_3(F)\right)+\mu_3(F)-3\varepsilon x-3\varepsilon^2\left(x^3-x^2\right)+2\varepsilon^3x^3 . \quad (18)$$

For symmetric distributions, where $\mu_3(F)=0$, the influence function of the third moment estimator is,

$$IF\left[x,F,\mu_3\right]=\lim_{\varepsilon\to 0}\frac{\mu_3(F_\varepsilon)-\mu_3(F)}{\varepsilon}=\left[\frac{d}{d\varepsilon}\left[\mu_3(F_\varepsilon)\right]\right]\Bigg|_{\varepsilon=0},$$

$$IF\left[x,F,\mu_3\right]=x^3-3x \quad (19)$$

[6]. For the asymmetric distributions, where $\mu_3(F)\neq 0$, the influence function of the third moment estimator is,

$$IF\left[x,F,\mu_3\right]=\lim_{\varepsilon\to 0}\frac{\mu_3(F_\varepsilon)-\mu_3(F)}{\varepsilon}=\left[\frac{d}{d\varepsilon}\left[\mu_3(F_\varepsilon)\right]\right]\Bigg|_{\varepsilon=0},$$

$$IF\left[x,F,\mu_3\right]=x^3-3x-\mu_3(F) \quad (20)$$

[6]. The influence function for the skewness measure γ_1 can be shown by the equation below:

$$IF\left[x,F,\gamma_1\right]=\lim_{\varepsilon\to 0}\frac{\gamma_1(F_\varepsilon)-\gamma_1(F)}{\varepsilon}=\left[\frac{d}{d\varepsilon_+}\frac{\mu_3(F_\varepsilon)}{\left[\sigma^2(F_\varepsilon)\right]^{3/2}}\right]_{\varepsilon=0}$$
$$=\frac{\left[\sigma^2(F_\varepsilon)\right]^{3/2}\left[\frac{d}{d\varepsilon}\left[\mu_3(F_\varepsilon)\right]\right]-\mu_3(F_\varepsilon)\left[\frac{d}{d\varepsilon}\left[\sigma^2(F_\varepsilon)\right]^{3/2}\right]}{\left[\sigma^2(F_\varepsilon)\right]^3}\Bigg|_{\varepsilon=0} \quad (21)$$

To get the result, the following equations can be formed by using equation (13) $\left[\sigma^2(F_\varepsilon)\right]^{3/2}=\left[\varepsilon\left(x^2-1\right)+1\right]^{3/2}$

$$\lim_{\varepsilon\to 0}\left[\sigma^2(F_\varepsilon)\right]^{3/2}=\lim_{\varepsilon\to 0}\left[\varepsilon\left(x^2-1\right)+1\right]^{3/2}=1 . \quad (22a)$$

Equation (22*a*) can be generalized for any power *r*:

$$\lim_{\varepsilon\to 0}\left[\sigma^2(F_\varepsilon)\right]^{r/2}=\lim_{\varepsilon\to 0}\left[\sigma^2(F_\varepsilon)\right]^{r}=1 \quad (22b)$$

Using

$$\lim_{\varepsilon\to 0}F_\varepsilon=\lim_{\varepsilon\to 0}(1-\varepsilon)F+\varepsilon G=F , \quad (23)$$

the following definition can be obtained,

$$\lim_{\varepsilon\to 0}\mu_3(F_\varepsilon)=\lim_{\varepsilon\to 0}(1-\varepsilon)\mu_3(F)+\varepsilon G=\mu_3(F) . \quad (24a)$$

In a similar way for any *r*'th moment, the equation (24*a*) can be generalized as

$$\lim_{\varepsilon\to 0}\mu_r(F_\varepsilon)=\mu_r(F) . \quad (24b)$$

As a result, for the symmetrical distributions where $\mu_3(F)=0$, equation (21) can be rewritten, using equations (19), (22*a*) and (24*a*). Thus the influence function for the skewness measure (γ_1) functional can be obtained as follows:

$$IF\left[x,F,\gamma_1(F_\varepsilon)\right]=x^3-3x . \quad (25)$$

The influence function of the skewness functional defined by equation (14) is more complex for an asymmetrical *F*. The reason for this is the contribution of the scale functional σ present in the denominator of γ_1, to the influence function. In order to be able to use equation (21), the contribution done by σ can be defined as,

$$IF\left[x,F,\left[\sigma^2\right]^{3/2}\right]=\left[\frac{d}{d\varepsilon_+}\left[\sigma^2(F_\varepsilon)\right]^{3/2}\right]=\frac{d}{d\varepsilon}\left[\varepsilon\left(x^2-1\right)+1\right]^{3/2}\Bigg|_{\varepsilon=0}$$
$$=\frac{3}{2}\left[\varepsilon\left(x^2-1\right)+1\right]^{1/2}\left(x^2-1\right)\Bigg|_{\varepsilon=0}=\frac{3}{2}\left(x^2-1\right) . \quad (26a)$$

This result can be generalized for odd moments as,

$$IF\left[x,F,\left[\sigma^2\right]^{r/2}\right]=\left[\frac{d}{d\varepsilon_+}\left[\sigma^2(F_\varepsilon)\right]^{r/2}\right]=\frac{d}{d\varepsilon}\left[\varepsilon\left(x^2-1\right)+1\right]^{r/2}\Bigg|_{\varepsilon=0}$$
$$=\frac{r}{2}\left[\varepsilon\left(x^2-1\right)+1\right]^{(r/2)-1}\left(x^2-1\right)\Bigg|_{\varepsilon=0}=\frac{r}{2}\left(x^2-1\right) \quad (26b)$$

and even moments as,

$$IF\left[x,F,\left[\sigma^2\right]^{r}\right]=\left[\frac{d}{d\varepsilon_+}\left[\sigma^2(F_\varepsilon)\right]^{r}\right]$$
$$=r\left(x^2-1\right) . \quad (26c)$$

As a result, equations (20), (22*a*), (24*a*) and (26*a*) can be replaced in equation (21) for the asymmetrical distributions, $\mu_3(F)\neq 0$, and the influence function for the skewness measure (γ_1) functional can be found as in [6],

$$IF\left[x,F,\gamma_1\right]=x^3-3x-\mu_3(F)-\frac{3}{2}\mu_3(F)\left(x^2-1\right) . \quad (27)$$

Now consider the unitless functional of kurtosis in equation (28),

$$\beta_2 = \frac{\mu_4}{\sigma^4} \tag{28}$$

which uses moment estimators. Assuming that $\mu(F_\varepsilon)= 0$ and $\sigma^2(F)=1$, [7] defines the symmetrical influence function,

$$SIF[x,F,\log\beta_2] = \frac{x^4-\mu_4(F)}{\mu_4(F)} - 2(x^2-1) \tag{29}$$

for the conditions where F and the contaminated distributions in equation (2) are symmetrical. The influence function of kurtosis measure which is defined in equation (28), for both the symmetrical and asymmetrical conditions can be obtained by the following approach: First using equation (15) it can be written that:

$$\mu_4(F_\varepsilon) = \mu_4^{'}(F_\varepsilon) - 4\mu(F_\varepsilon)\mu_3^{'}(F_\varepsilon) + 6\mu^2(F_\varepsilon)\mu_2^{'}(F_\varepsilon) - 3\mu^4(F_\varepsilon) \tag{30}$$

All the components of this equation except $\mu_4^{'}(F_\varepsilon)$ are defined in equations (6), (13) and (17*a*). By equation (17*b*) we have,

$$\mu_4^{'}(F_\varepsilon) = (1-\varepsilon)\mu_4^{'}(F) + \varepsilon x^4 \tag{31}$$

and when the results are replaced in equation (30), we obtain

$$\mu_4(F_\varepsilon) = \varepsilon\left[x^4 - \mu_4^{'}(F)\right] + \mu_4^{'}(F) - 4\varepsilon x\mu_3^{'}(F) - 4\varepsilon^2 x\left[x^3 - \mu_3^{'}(F)\right] + 6\varepsilon^2 x^2\left[\varepsilon(x^2-1)+1\right] - 3\varepsilon^4 x^4 \tag{32}$$

Since $\mu(F)= 0$, $\mu_4^{'}(F) = \mu_4(F)$ and $\mu_3^{'}(F) = \mu_3(F)$ the following equations can be written:

For symmetrical distributions,

$$\mu_4(F_\varepsilon) = \varepsilon\left[x^4 - \mu_4(F)\right] + \mu_4(F) - 4\varepsilon^2 x^4 + 6\varepsilon^2 x^2\left[\varepsilon(x^2-1)+1\right] - 3\varepsilon^4 x^4, \tag{33}$$

and for asymmetrical distributions,

$$\mu_4(F_\varepsilon) = \varepsilon\left[x^4 - \mu_4(F)\right] + \mu_4(F) - 4\varepsilon x\mu_3(F) - 4\varepsilon^2 x\left[x^3 - \mu_3(F)\right] + 6\varepsilon^2 x^2\left[\varepsilon(x^2-1)+1\right] - 3\varepsilon^4 x^4 \tag{34}$$

Hence the influence function for β_2 becomes

$$IF[x,F,\beta_2] = \lim_{\varepsilon\to 0}\frac{\beta_2(F_\varepsilon)-\beta_2(F)}{\varepsilon} = \left[\frac{d}{d\varepsilon_+}\frac{\mu_4(F_\varepsilon)}{\left[\sigma^2(F_\varepsilon)\right]^2}\right]$$

$$= \left.\frac{\left[\sigma^2(F_\varepsilon)\right]^2\left[\frac{d}{d\varepsilon}\left[\mu_4(F_\varepsilon)\right]\right] - \mu_4(F_\varepsilon)\left[\frac{d}{d\varepsilon}\left[\sigma^2(F_\varepsilon)\right]^2\right]}{\left[\sigma^2(F_\varepsilon)\right]^4}\right|_{\varepsilon=0} \tag{35}$$

In order to use equation (35) for symmetrical distributions,

$$IF[x,F,\mu_4] = \lim_{\varepsilon\to 0}\frac{\mu_4(F_\varepsilon)-\mu_4(F)}{\varepsilon} = \left.\left[\frac{d}{d\varepsilon_+}\mu_4(F_\varepsilon)\right]\right|_{\varepsilon=0}$$

$$= x^4 - \mu_4(F) \tag{36}$$

is obtained by equation (33) and,

$$IF\left[x,F,\left[\sigma^2\right]^2\right] = 2(x^2-1) \tag{37}$$

is obtained by equation (26*c*). By substituting the results obtained from equations (22*b*), (24*b*), (36) and (37) in equation (35) the influence function for symmetrical distributions can be defined as

$$IF[x,F,\beta_2] = \lim_{\varepsilon\to 0}\frac{\beta_2(F_\varepsilon)-\beta_2(F)}{\varepsilon}$$

$$= x^4 - \mu_4(F) - 2\mu_4(F)(x^2-1); \tag{38}$$

and for asymmetrical distributions,

$$IF[x,F,\mu_4] = \lim_{\varepsilon\to 0}\frac{\mu_4(F_\varepsilon)-\mu_4(F)}{\varepsilon} = \left.\left[\frac{d}{d\varepsilon_+}\mu_4(F_\varepsilon)\right]\right|_{\varepsilon=0}$$

$$= x^4 - \mu_4(F) - 4x\mu_3(F) \tag{39}$$

which is completely written by using equation (34). Also by replacing the results obtained from (22*b*), (24*b*), (37) and (39) in equation (35), influence function for asymmetrical distributions can be written as;

$$IF[x,F,\beta_2] = \lim_{\varepsilon\to 0}\frac{\beta_2(F_\varepsilon)-\beta_2(F)}{\varepsilon}$$

$$= x^4 - \mu_4(F) - 4x\mu_3(F) - 2\mu_4(F)(x^2-1). \tag{40}$$

The obtained results for skewness and kurtosis can also be generalized for higher order scaled moments.

5. INFLUENCE FUNCTIONS FOR SCALED MOMENTS

Scaled moments for the statistical distributions are defined by the equation,

$$\varphi_r = \frac{\mu_r}{\left(\sigma^2\right)^{r/2}} \qquad r=5,6,\dots . \qquad (41a)$$

The influence function of scaled moments are examined separately for odd and even moments for distributions which satisfies $\mu(F)= 0$ and $\sigma^2(F)=1$. As it can be seen from equation (41*a*) when $\sigma^2(F)=1$ the equation is simplified as,

$$\varphi_r = \mu_r . \qquad (41b)$$

First, the condition where the moment degree *r* is an odd value will be taken into consideration. The influence function for scaled *r*'th moment is,

$$IF[x,F,\varphi_r] = \left[\frac{d}{d\varepsilon_+}\frac{\mu_r(F_\varepsilon)}{\left[\sigma^2(F_\varepsilon)\right]^{r/2}}\right]$$

$$= \frac{\left[\sigma^2(F_\varepsilon)\right]^{r/2}\left[\frac{d}{d\varepsilon}\left[\mu_r(F_\varepsilon)\right]\right] - \mu_r(F_\varepsilon)\left[\frac{d}{d\varepsilon}\left[\sigma^2(F_\varepsilon)\right]^{r/2}\right]}{\left[\sigma^2(F_\varepsilon)\right]^r}\Bigg|_{\varepsilon=0} . \qquad (42a)$$

Using equations (22*b*), (24*b*) and (26*b*), equation (42*a*) reduces to,

$$IF[x,F,\varphi_r] = \left[\frac{d}{d\varepsilon_+}\mu_r(F_\varepsilon)\right]_{\varepsilon=0} - \mu_r(F)\frac{r}{2}\left(x^2-1\right) \cdot \qquad (42b)$$

For the condition of having an even value of *r*, using equations (22*b*), (24*b*) and (26*c*), the equation (42) can be defined as,

$$IF[x,F,\varphi_r] = \left[\frac{d}{d\varepsilon_+}\mu_r(F_\varepsilon)\right]_{\varepsilon=0} - \mu_r(F)\,r\left(x^2-1\right) \cdot \qquad (42c)$$

The first term on the right hand side of equations (42*b*) and (42*c*) can be obtained by using equation (15).

6. CONCLUSION

The influence functions for skewness measure γ_1 are defined for both symmetrical and asymmetrical distributions in the previous papers. However, the influence function of kurtosis measure β_2 is only examined for symmetrical distributions, and the asymmetrical distributions are neglected. In this study the influence function of β_2 for the asymmetrical distributions is defined. And a general approach is proposed for the influence functions for the moments of order five or more for the standard random variables.

While defining the distribution, by the method of moments, the standard error of the third or higher moments can be overestimated, especially in situations where the sample sizes are small. In these situations another way to understand how the extra data added to the original data set will affect the estimated distribution is to investigate the influence functions of the third and higher moments.

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