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On the Topological Centers of Banach Algebras

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ABSTRACT

Let A be a Banach algebra with a bounded approximate identity. Let Z_2 and $\widetilde{Z}_2$ be respectively, the topological centers of the algebras A** and (AA*)* with respect to the second Arens multiplication. In this paper, we show that $\widetilde{M}_2$ is isometrically isomorphic to $LM(A)$, where $\widetilde{M}_2$ is a closed subalgebra of $\widetilde{Z}_2$ and $LM(A)$ is the set of left multipliers operators of the Banach algebra A.

Key words: Topological center, Arens multiplication, Banach algebra, Left multiplier operator

1. INTRODUCTION, NOTATIONS AND PRELIMINARIES

Let A be a Banach algebra with a bounded approximate identity. By A^* we denote its normed dual. We always regard A as naturally embedded into its second dual A^{**}. For a in A and f in A^*, by $\langle f, a\rangle$ or $\langle a, f\rangle$ we denote the natural duality between A and A^*. The first Arens multiplication is defined in three steps as follows. For a,b$\in A$, $f \in A^*$ and $m, n \in A^{**}$, the elements $f \cdot a, m \cdot f$ of A^* and $m \cdot n$ of A^{**} are defined as follows:

$$\langle f \cdot a, b\rangle = \langle f, ab\rangle,$$
$$\langle m \cdot f, a\rangle = \langle m, f \cdot a\rangle,$$
$$\langle m \cdot n, f\rangle = \langle m, n \cdot f\rangle.$$

The second Arens multiplication is defined as follows. For a,b$\in A$, $f \in A^*$ and $m, n \in A^{**}$, the element $a\Delta f, f\Delta m$ of A^* and $m\Delta n$ of A^{**} are defined by the equalities

$$\langle a\Delta f, b\rangle = \langle f, ba\rangle,$$
$$\langle f\Delta m, a\rangle = \langle m, a\Delta f\rangle,$$
$$\langle m\Delta n, f\rangle = \langle n, f\Delta m\rangle.$$

We define the subspaces A^*A and AA^* of A^* as

$$A^*A = \{f \cdot a : f \in A^*, a \in A\},$$
$$AA^* = \{a\Delta f : a \in A, f \in A^*\}.$$

It is well-known that these subspaces are norm-closed linear subspaces of A^* Hewitt &Ross (2). On the other hand, the second dual A^{**} of A is a Banach algebra with respect to both the first and the second Arens multiplication (1). In the case where $A = L^1(G)$ and G is a locally compact Abelian group, we denote the spaces A^*A and AA^*, respectively, by $LUC(G)$ and $RUC(G)$ as in (3). In the case where $A = A(G)$, the space A^*A, which is the same as AA^*, is denoted by $UCB(\hat{G})$ as in (4). In (5), Lau and Ülger showed that $\widetilde{Z}_1 \cong RM(A)$. Terminologies and notations not explained in this section will be explained or referenced in the next section.

2. ARENS MULTIPLICATIONS AND TOPOLOGICAL CENTERS

Definition 2.1. Let A be a Banach algebra. *A left [right] approximate identity* for A is a net $\{e_\alpha : \alpha \in \Lambda\}$, where Λ is some directed system, such that for all $a \in A$,

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$\lim_{\alpha}(e_\alpha a) = a$ $[\lim_{\alpha}(ae_\alpha) = a]$ in the norm topology. The approximate identity is said to be *bounded* if $\|e_\alpha\| \leq 1$ for all. An approximate identity is said to be two-sided if it is both a right and a left one. An algebra A with a bounded two-sided approximate identity is called 'with a bounded approximate identity'. Every unital Banach algebra has an approximate identity. However, the converse is not true in general.

Definition 2.2. Let A be a Banach algebra and consider the natural duality between A and A^*. We denote the weak topology on A by $\sigma(A,A^*)$ and the weak* topology on A^* by $\sigma(A^*,A)$.

As is mentioned in the previous section, we will explain the basic properties of "$\cdot$" and "Δ" Arens multiplications:

For an element *n* fixed in A^{**}, the mapping $m \to m\cdot n$ is weak*-weak* continuous with respect to the topology $\sigma(A^{**},A^*)$ on A^{**}. However, for an element *m* fixed in A^{**}, the mapping $n \to m.n$ is in general not weak*-weak* continuous unless *m* is in A. Hence ,by making use of these explanations, the topological center of A^{**} with respect to the first Arens multiplication is defined as follows:

$Z_1 = \{m \in A^{**}:$ The mapping $n \to m.n$ is weak*-weak* continuous on A^{**} $\}$

$$= \{m \in A^{**} : m.n = m\Delta n, \text{ for all } n \in A^{**}\}$$

For m fixed in A^{**}, the mapping $n \to m\Delta n$ is weak*-weak* continuous on A^{**}. But, for *n* fixed in A^{**}, the mapping $m \to m\Delta n$ is in general not weak*-weak* continuous unless *n* is in A. Whence the topological center of A^{**} with respect to the second Arens multiplication is defined as follows:

$Z_2 = \{n \in A^{**}:$ The mapping $m \to m\Delta n$ is weak*-weak* continuous on A^{**} $\}$

Recall that the equalities $\hat{a}.m = \hat{a}\Delta m$ and $m.\hat{a} = m\Delta\hat{a}$ hold for *a* in A and *m* in A^{**}. Since the mapping $a \to \hat{a}$ ($A \to \hat{A} \subseteq A^{**}$) is an algebraic isometrical isomorphism we can write $\hat{A}$ instead of A if necessary. It is clear that $A \subseteq Z_1 \cap Z_2$ and that Z_i (i=1,2) is a closed subalgebra of A^{**}. For detailed information see (5).

Let M_1 and M_2 be two subspaces of A^{**} such that

$$M_1 = \{ m \in A^{**} : A.m \subseteq A \},$$

$$M_2 = \{ m \in A^{**} : m.A \subseteq A \}.$$

Like preceding subspaces, We define the following sets:

$$\tilde{M}_1 = \{ \mu \in (A^*A)^* : A.\mu \subseteq A \}$$

$$\tilde{M}_2 = \{ \mu \in (AA^*)^* : \mu.A \subseteq A \}$$

Let A be a Banach algebra with a bounded appoximate identity then $\tilde{M}_1$ is a closed subalgebra of $(A^*A)^*$ and $\tilde{M}_1 \subseteq \tilde{Z}_1$ (5, Proposition 4.1). An algebra A is a subalgebra of all A^{**}, $(A^*A)^*$ and $(AA^*)^*$ algebras. Note that for $a \in A$ and $\mu \in (AA^*)^*$, the multiplication element $\mu.a$ is an element of $(AA^*)^*$. Morever, if $\tilde{\mu}$ is any Hahn-Banach extension of μ to A^* then for $f \in A^*$, $a \in A$, $\mu \in (AA^*)^*$ and $\tilde{\mu} \in A^{**}$ the equalities

$$\begin{aligned} < \tilde{\mu}.a, f > &= < \hat{a}, f\Delta\hat{\mu} > = < f\Delta\hat{\mu}, a > \\ &= < \hat{\mu}, a\Delta f > = < \mu, a\Delta f > \\ &= < f\Delta\mu, a > = < \hat{a}, f\Delta\mu > \\ &= < \mu\Delta\hat{a}, f > = < \mu\Delta a, f > \\ &= < \mu.a, f > \end{aligned}$$

hold and hence we have the equalities $\mu.a = \tilde{\mu}.a$ and $\mu\Delta a = \tilde{\mu}\Delta a$. Whence we can consider $\mu.a$ as an element of A^{**}.

Let the mapping $\hat{f}.m : A^{**} \to C$ be defined by $< \hat{f}.m , n > = < f , n \Delta m >$. The functional $\hat{f}.m$ belongs to $A^{***} = A^* \oplus A^\perp$ but it does not have to be an element of A^*. Similarly, let the mapping $(\mu.f)^\sim : (AA^*)^* \to C$ be defined by $< (\mu.f)^\sim, \lambda > = < f, \lambda\Delta\mu >$. Although the functional $(\mu.f)^\sim$ belongs to $(AA^*)^{**}$ it may not be an element of A^*, see (5) for detail.

Now, the following lemma which plays an important role in our study will be given.

Lemma 2.3: Let A be a Banach algebra with a bounded appoximate identity. Let m be an element in A^{**} and μ be an element in $(AA^*)^*$. Then the following assertions hold:

a) m is in Z_2 if and only if, for each *f* in A^*, the functional $\hat{f}.m$ is in A^*. If this happens, $\hat{f}.m = m.f$ and $m.f$ is in AA^*.

b) μ is in $\widetilde{Z}_2$ if and only if, for each *g* in AA^*, the functional $(\widetilde{\mu.g})$ is in AA^*.

c) μ is in $\widetilde{Z}_2$ if and only if, for each *a* in A, $\mu.a$ is in Z_2.

Proof: a) Assume m is in Z_2, and let *f* be an element of A^*. Then, for all *n* in A^{**},

$$\begin{aligned} <\hat{f}.m, n> &= <f, n\Delta m> \\ &= <f, n.m> \\ &= <m.f, n> \end{aligned}$$

so that $\hat{f}.m = m.f$, and $\hat{f}.m$ is in A^* since $m.f$ is in A^*.

Conversely, assume that, for each *f* in A^*, the functional $\hat{f}.m$ is in A^* and let $\{n_\alpha\}_{\alpha\in\Lambda}$ be a convergent net in A^{**} that converges to some *n* in the $\sigma(A^{**}, A^*)$ topology . Then

$$<f, n_\alpha\Delta m> = <\hat{f}.m, n_\alpha> \to <\hat{f}.m, n> = <f, n\Delta m>$$

so that m is in Z_2 since, for $n \in A^{**}$, the mapping $n \to n\Delta m$ is $\sigma(A^{**}, A^*)$-continuous on A^{**}.

Now suppose m is in Z_2 and f is in A^*. Let $\{a_\alpha\}_{\alpha\in\Lambda}$ be a convergent net in A that converges to some m in the $\sigma(A^{**}, A^*)$ topology. Then, since $\hat{f}.m$ is in A^* and for each *n* in A^{**},

$$<a_\alpha\Delta f, n> = <f, n.a_\alpha> \to <f, n.m> = <f, n\Delta m> = <\hat{f}.m, n>$$

we see that the net $\{a_\alpha\Delta f\}_{\alpha\in\Lambda}$ converges weakly to the element $\hat{f}.m$ in A^*. Since AA^* is a closed subspace of A^*, we conclude that $\hat{f}.m$ is in AA^*.

b) Suppose that μ is in $\widetilde{Z}_2$, and let *g* be in AA^*. Let $\{\lambda_\alpha\}_{\alpha\in\Lambda}$ be a net in $(AA^*)^*$ that converges to some λ in $(AA^*)^*$ in the $\sigma((AA^*)^*, AA^*)$ topology . Then by the definition of $\widetilde{Z}_2$

$$<(\widetilde{\mu.g}), \lambda_\alpha> = <g, \lambda_\alpha\Delta\mu> \to <g, \lambda\Delta\mu> = <(\widetilde{\mu.g}), \lambda>$$

This shows that the functional $(\widetilde{\mu.g})$ is weak* continuous on $(AA^*)^*$. Since we have the duality $((AA^*)^*, \sigma((AA^*)^*, AA^*))^* = AA^*$, $(\widetilde{\mu.g})$ is in AA^*.

Conversely, assume that *g* and $(\widetilde{\mu.g})$ are in AA^* and let $\{\lambda_\alpha\}_{\alpha\in\Lambda}$ be a weak* convergent net in $(AA^*)^*$ converging to some λ in $(AA^*)^*$. Then,

$$<g, \lambda_\alpha\Delta\mu> = <(\widetilde{\mu.g}), \lambda_\alpha> \to <(\widetilde{\mu.g}), \lambda> = <g, \lambda\Delta\mu>$$

holds which means μ is in $\widetilde{Z}_2$.

c) Let μ is in $\widetilde{Z}_2$. Then, for each $g = a\Delta f$ in AA^*, $(\widetilde{\mu.g})$ belongs to AA^* from assertion b). Given an element *n* of A^{**}, let $\widetilde{n}$ be its restriction AA^*. Then the equality $\widetilde{n}\Delta\mu a = n\Delta\mu a$ holds and we have $(\widetilde{\mu g}) = \hat{f}.\mu a$ by the following equalities

$$\begin{aligned} <(\widetilde{\mu g}), n> &= <(\widetilde{\mu g}), \widetilde{n}> \\ &= <f, \widetilde{n}\Delta\mu a> \\ &= <\hat{f}.\mu a, n>. \end{aligned}$$

Since $\hat{f}.\mu a$ is in AA^*, we conclude, by assertion a), that $\mu.a$ is in Z_2. The converse implication also follows by the same operations.

Proposition 2.4: Let A be a Banach algebra with a bounded appoximate identity. Then $\widetilde{M}_2$ is a closed subalgebra of $(AA^*)^*$ and $\widetilde{M}_2 \subseteq \widetilde{Z}_2$.

Proof: From the definition of $\widetilde{M}_2$ we have the inclusion $\widetilde{M}_2 \subseteq (AA^*)^*$. Let (μ_n) be a sequence in $\widetilde{M}_2$. Then for all n we have $\mu_n.A \subseteq A$. Let μ be an element in $(AA^*)^*$ such that $\lim_n \|\mu_n - \mu\| = 0$. For an element a in A, $(\mu_n.a)$ is a sequence in A. Since A is closed and the multiplication is norm-continuous, we have $\|\mu_n.a - \mu.a\| \to 0$, that is, $\mu.a$ is in A. Hence $\widetilde{M}_2$ is a closed subalgebra of $(AA^*)^*$.

On the other hand, let μ be an element in $\widetilde{M}_2$. Then, for an element a in A, $\mu.a$ is in $A \subset Z_2$. By the assertion c) of Lemma 2.3, μ is in $\widetilde{Z}_2$ and hence $\widetilde{M}_2 \subseteq \widetilde{Z}_2$.

Definition 2.5: Let A be a Banach algebra with a bounded appoximate identity. A bounded linear operator $T : A \to A$ is said to be a left multiplier if $T(ab) = T(a)b$ holds for all a,b in A. The set of all left multiplier of A is denoted by LM(A).

Theorem 2.6: Let A be a Banach algebra with a bounded appoximate identity. Then the closed algebra $\widetilde{M}_2$ is isometrically isomorphic to LM(A).

Proof: For each element μ in $\widetilde{M}_2$, let $T_\mu : A \to A$ be the linear operator defined by the rule $T_\mu(a) = \mu a$, for all a in A. As, for a,b in A,

$$T_\mu(ab) = \mu(ab) = (\mu a)b = T_\mu(a)b,$$

T_μ is a left multiplier on A. Since $\|\mu a\| \le \|\mu\|.\|a\|$, it is obvious that $\|T_\mu\| \le \|\mu\|$. Actually $\|T_\mu\| = \|\mu\|$. To show the inequality $\|T_\mu\| \ge \|\mu\|$, let $(e_\alpha)_{\alpha\in\Lambda}$ be a bounded appoximate identity. As we can suppose $\|e_\alpha\| \le 1$ for all α in Λ,

$$\|T_\mu\| \ge \sup_\alpha \|T_\mu(e_\alpha)\| = \sup_\alpha \|\mu.e_\alpha\| = \sup_\alpha \sup_{\|a\Delta f\|\le 1} \|\mu.e_\alpha(a\Delta f)\|$$

.Since, for f in A^* a in A ,and

$$\|a\Delta f.e_\alpha - a\Delta f\| = \|a\Delta(f.e_\alpha - f)\| \le \|a\|.\|f.e_\alpha - f\| \to 0,$$

$$\sup_\alpha |\mu e_\alpha(a\Delta f)| \ge \lim_\alpha |\mu(e_\alpha.(a\Delta f))| = |\mu.(a\Delta f)|.$$

Hence $\sup_\alpha \sup_{\|a\Delta f\|\le 1} \|\mu.e_\alpha(a\Delta f)\| \ge \sup_{\substack{\|a\Delta f\|\le 1 \\ \alpha}} |\mu(a\Delta f)| = \|\mu\|$

so that $\|T_\mu\| \ge \|\mu\|$. Since we have the equality $\|T_\mu\| = \|\mu\|$, it follows that the mapping $S : \widetilde{M}_2 \to LM(A)$ defined by $S(\mu) = T_\mu$ is an isometry. To show that S is a Banach algebra homomorphism, let μ_1, μ_2 be in $\widetilde{M}_2$ and a in A. Indeed,

$$\begin{aligned} S(\mu_1.\mu_2)(a) &= T_{\mu_1.\mu_2}(a) = (\mu_1.\mu_2)a = \mu_1(\mu_2.a) = \mu_1(T_{\mu_2}(a)) \\ &= T_{\mu_1}(T_{\mu_2}(a)) = T_{\mu_1}.T_{\mu_2}(a) = S(\mu_1).S(\mu_2)(a). \end{aligned}$$

To complete the proof, it is enough to show that S is onto. Let T be any element in LM(A). Since we can consider A as a subalgebra of $(AA^*)^*$, the net $(T(e_\alpha))_{\alpha\in\Lambda}$ is in $(AA^*)^*$ and, for each $f.a$ in $(AA^*)^*$, it follows

$$< a\Delta f, T(e_\alpha) >=< f, T(e_\alpha).a >=< f, T(e_\alpha.a) >\to< f, T(a) >$$

This shows that the net $(T(e_\alpha))_{\alpha\in\Lambda}$ is a weak*-Cauchy in $(AA^*)^*$. Hence it converges to some element μ of (AA*)* in the weak* topology of this space. The above equalities

$$< a\Delta f, \mu) >=< f, T(a) >$$

for all f in A^* and a in A. Then $< a\Delta f, \mu) >=< f, \mu.a) >=< f, T(a) >$, which means $\mu.a = T(a)$ so that $T = T_\mu$. Since, for each T in LM(A), there is an element μ in $\widetilde{M}_2$ such that $S(\mu) = T_\mu = T$, the mapping S is onto.

Corollary 2.6: Let A be a Banach algebra with a bounded appoximate identity. If $Z_2 A \subseteq A$ then, $\widetilde{M}_2 = \widetilde{Z}_2 \cong LM(A)$.

REFERENCES

1) Arens, R., ‘‘*The adjoint of a bilinear operation*’’, ***Proc. Amer. Math. Soc.,*** 2, 839-848, (1951).

2) Hewitt, E. and Ross, K., ‘‘*Abstract Harmonic Analysis*’’, Volumes I and II, Newyork**,** (1970).

3) Lau, A.T., ‘‘*Continuity of Arens multiplication on the dual space of bounded uniformly continuous functions on locally compact groups and topological semigroups*’’, ***Math. Proc. Camb. Philos. Soc***., 99, 273-283, (1986).

4) Lau, A.T. and Losert, V., ‘‘*The C* algebra generated by operators with compact support on a locally compact group*’’, ***J. Functinal Analysis,*** 112, 1-30, (1993).

5) Lau, A.T. and Ülger, A., ‘‘*Topological centers of certain dual algebras*’’, ***Trans. Amer. Math. Soc.,*** 348 (3), 1191-1212, (1996).