Gazi University Journal of Science
GU J Sci
24(1):51-58 (2011)

GUJS
www.gujs.org

Couple Fixed Point on Cone Metric Spaces

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Received: 19/04/2010 Revised: 21/05/2010 Accepted:14/06/2010

ABSTRACT

In this article, some couple fixed point theorems are proved for the class of Banach valued metric spaces. The results are proved without any additional conditions such as normality or regularity.

Keywords: *Cone Metric Spaces, Fixed Point Theory, Couple Fixed Point* ***Mathematics Subject Classification:*** *47H10, 54H25*

1. INTRODUCTION AND PRELIMINARIES

In 1980, Bogdan Rzepecki [15], introduced a generalized metric d_E on a set X in a way that $d_E : X \times X \to S$ where E is Banach space and S is a normal cone with partial order $\preceq$. In that paper, the author generalized the fixed point theorems of Maia type. Seven years later, Shy-Der Lin [11] considered the notion K -metric spaces by replacing real numbers with cone K in the metric function, that is, $d : X \times X \to K$. In that manuscript, some results of Khan and Imdad [10] on fixed point theorems were considered for K -metric spaces. In 2007, Huang and Zhang [7] announced the notion of cone metric spaces (CMS) by replacing real numbers with an ordering Banach space. In that paper, they also discussed some properties of convergence of sequences and proved the fixed point theorems of contractive mapping for cone metric spaces: Any mapping T of a complete cone metric space X into itself that satisfies, for some $0 \le k < 1$, the inequality

$$d(Tx, Ty) \le kd(x, y) \tag{1}$$

for all $x, y \in X$, has a unique fixed point. In this manuscript, some results of Bhaskar, T.G., Lakshmikantham, V. [6] are extended to the class of cone metric spaces.

Recently, many results on fixed point theorems have been extended to cone metric spaces (see e.g. [7, 13, 16, 17, 1, 2, 3, 4, 8, 9, 5]). In [7], the authors extend some well known contraction theorems on usual complete metric space to complete cone metric spaces over regular cones. In this article, main theorem and consequent results will be proved in cone metric spaces without assuming any additional conditions, such as, regularity and normality.

Throughout this paper E stands for real Banach space. Let $P = P_E$ always be closed subset of E A subset P of E is called ***cone*** if the following conditions are satisfied:

(C1) $P \ne \emptyset$,

(C2) $ax + by \in P$ for all $x, y \in P$ and non-negative real numbers a, b,

(C3) $P \cap (-P) = \{0\}$ and $P \ne \{0\}$.

For a given cone P, one can define a partial ordering (denoted by $\le$ or $\le_P$) with respect to P by $x \le y$ if and only if $y - x \in P$. The notation $x < y$ indicate that

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$x \le y$ and $x \ne y$ while $x << y$ will show $y - x \in \operatorname{int} P$, where denotes the interior of P.
It can be easily shown that,
$\operatorname{int}P + \operatorname{int}P \subset \operatorname{int}P$ and $\lambda(\operatorname{int}P) \subset \operatorname{int}P$
where $0 < \lambda \in \mathrm{IR}$.

.
The cone P is called *(N)* ***normal*** if there is a number $K \ge 1$ such that for all $x, y \in E$:

$$0 \le x \le y \Rightarrow \|x\| \le K \|y\|. \qquad (2)$$

(R) **regular** if every increasing sequence which is bounded from above is convergent.

That is, if $\{x_n\}_{n\ge1}$ is a sequence such that $x_1 \le x_2 \le \ldots \le y$ for some $y \in E$ then there is $x \in E$ such that $\lim_{n\to\infty} \| x_n - x \| = 0$.

In ***(N)***, the least positive integer K satisfying equation (2) is called the normal constant of P. Note that, in [7] and [13], normal constant K is considered a positive real number, $(K>0)$, although it is proved that there is no normal cone for $K<1$ in (Lemma 2.1, [13]).

Lemma 1.

*(i)*Every regular cone is normal.
(ii) For each $k > 1$, there is a normal cone with normal constant $K > k$.
(iii) The cone P is regular if every decreasing sequence which is bounded from below is convergent.

Proof of *(i)* and *(ii)* are given in [13] and the last one follows from the definition.

Definition 2.(See |7|) Let X be non-empty set. Suppose the mapping $d : X \times X \to E$ satisfies:

(M1) $0 \le d(x, y)$ for all $x, y \in X$,
(M2) $d(x, y) = 0$ if and only *if* $x = y$,
(M3)
$d(x, y) \le d(x, z) + d(z, y),\ \forall\, x, y, z \in X.$

Then d is said to be quasi-cone metric on X, and the pair (X, d) is called a quasi-cone metric space ***(QCMS).*** Additionally, if d also satisfies ***(M4)*** $d(x, y) = d(y, x)$ for all $x, y \in X$ .Then d is called cone metric on X, and the pair (X, d) is called a cone metric space ***(CMS).***

Example 3. Let $E = IR^3$ and $P = \{(x, y, z) \in E : x, y, z \ge 0\}$ and $X = IR$. Define $d : X \times X \to E$ by $d(x, \tilde{x}) = (\alpha | x - \tilde{x} | + \beta | x - \tilde{x} | + \gamma | x - \tilde{x} |)$, where α, β, γ are positive constants. Then (X, d) is a CMS. Note that the cone P is normal with the normal constant $K = 1$.

Definition 4. (See [7]) Let (X, d) be a CMS, $x \in X$ and $\{x_n\}_{n\ge1}$ a sequence in X. Then,

(i) $\{x_n\}_{n\ge1}$ converges to x whenever for every $c \in E$ with $0 << c$ there is a natural number N such that $d(x_n, x) << c$ for all $n \ge N$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.
(ii) $\{x_n\}_{n\ge1}$ is a ***Cauchy sequence*** whenever for every $c \in E$ with $0 << c$ there is a natural number N such that $d(x_n, x_m) << c$ forall $n, m \ge N$.

(iii) (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Lemma 5 (See [7] Let (X, d) be a CMS, P be a normal cone with normal constant K, and $\{x_n\}$ be a sequence in X. Then, the sequence $\{x_n\}$ converges to x if and only if $d(x_n, x) \to 0$
(or equivalently $\|d(x_n, x)\| \to 0$ as $n \to \infty$.
(ii) the sequence $\{x_n\}$ Cauchy if and only if $d(x_n, x_m) \to 0$ (or equivalently $\| d(x_n, x_m)\| \to 0$ as $n \to \infty$.)

2. MAIN RESULTS

Let (X, d) be a CMS and $X^2 = X \times X$. Then the mapping $\rho : X^2 \times X^2 \to E$ such that $\rho\left((x_1, y_1), (x_2, y_2)\right) = d(x_1, x_2) + d(y_1, y_2)$ forms a cone metric on X^2.

Definition 6. Let (X, d) be a CMS. A function $f : X \to X$ is said be (sequentially) continuous if $d(x_n, x) \to 0$ implies that $d(f(x_n), f(x)) \to 0$. Analogously, A function

$F: X \times X \to X$ is (sequentially) continuous if $\rho\left(\left(x_n, y_n\right),(x, y)\right) \to 0$ implies that $d\left(F\left(x_n, y_n\right), F(x, y)\right) \to 0$ as $n \to \infty$.

Definition 7. (See [6]). Let $(X, \preceq)$ be partially ordered set and $F: X \times X \to X$. A map F is said to have the mixed monotone property if $F(x, y)$ is monotone nondecreasing in x and is monotone nonincreasing in y, that is, for any $x, y \in X$, $x_1 \preceq x_2 \Rightarrow F(x_1, y) \preceq F(x_2, y)$, $for\ x_1, x_2 \in X$ and

$y_1 \preceq y_2 \Rightarrow F(x, y_2) \preceq F(x, y_1)$,

$for\ y_1, y_2 \in X$. Note that this definition coincides with the notion of mixed monotone function IR^2 where $\preceq$ represents the usual total order $\preceq$ in IR^2.

Definition 8 (See [6]) An element $(x, y) \in X \times X$ is said to be a couple fixed point of the mapping $F: X \times X \to X$ if

$F(x, y) = x$ and $F(y, x) = y$.

Throughout this paper, let $(X, \preceq)$ be partially ordered set and d be a cone metric on X such that (X, d) is a complete CMS. Further, the product space X×X has the following ordering:

$(u, v) \preceq (x, y) \Leftrightarrow u \preceq x$ and $y \preceq v$; for all $(x, y),(u, v) \in X \times X$.

Theorem 9. Let $F: X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0,1)$ with

$$d\left(F(x, y), F(u, v)\right) \leq \frac{k}{2}\left[d(x, u) + d(y, v)\right]$$

for all $u \preceq x,\ y \preceq v$.

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$, then, there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Proof: First step of the proof is the construction of Cauchy sequences: Set $x_1 := F(x_0, y_0)$ and $y_1 := F(y_0, x_0)$. By assumptions of the theorem, $x_0 \preceq x_1$ and $y_1 \preceq y_0$. Set $x_2 := F(x_1, y_1)$ and $y_2 := F(y_1, x_1)$, and denote

$$x_2 = F(x_1, y_1) = F(F(x_0, y_0), F(y_0, x_0)) = F^2(x_0, y_0)$$

and

$$y_2 = F(y_1, x_1) = F(F(y_0, x_0), F(x_0, y_0)) = F^2(y_0, x_0).$$

Under this notation, the mixed monotone property of F yields that

$$x_1 = F(x_0, y_0) \preceq F(x_1, y_1) = F^2(x_0, y_0) = x_2$$

and

$$y_2 = F^2(y_0, x_0) = F(y_1, x_1) \preceq F(y_0, x_0) = y_1.$$

For $n = 1, 2, \ldots,$ the general term of the sequences are defined as follow:

$$x_{n+1} = F(x_n, y_n) = F(F^n(x_0, y_0), F^n(y_0, x_0)) = F^{n+1}(x_0, y_0)$$

and

$$y_{n+1} = F(y_n, x_n) = F\left(F^n(y_0, x_0), F^n(x_0, y_0)\right) := F^{n+1}(y_0, x_0).$$

Observe that

$$x_0 \preceq F(x_0, y_0) = x_1 \preceq F(x_1, y_1) = F^2(x_0, y_0) = x_2 \preceq \ldots \preceq F^{n+1}(x_0, y_0),$$

and

$$F^{n+1}(y_0, x_0) \preceq \ldots \preceq F^2(y_0, x_0) = F(y_1, x_1) = y_2 \preceq y_1 = F(y_0, x_0) \preceq y_0$$

Assert that

$$(5): d(F^{n+1}(x_0, y_0), F^n(x_0, y_0)) \leq \frac{k^n}{2}\left(d(F(x_0, y_0), x_0) + d(F(y_0, x_0), y_0)\right)$$

$$(6): d(F^{n+1}(y_0, x_0), F^n(y_0, x_0)) \leq \frac{k^n}{2}\left(d(F(y_0, x_0), y_0) + d(F(x_0, y_0), x_0)\right)$$

holds for all $n \in IN$ .This assertion can be proved by induction. For $n = 1,$ the inequalities (5) and (6) follow by (3), (4) and $x_0 \preceq F(x_0, y_0)$ and $F(y_0, x_0) \preceq y_0$. Indeed ,

$$d(F^2(x_0, y_0), F(x_0, y_0)) = d(F(F(x_0, y_0), F(y_0, x_0)), F(x_0, y_0)) \leq \frac{k}{2}\left[d(F(x_0, y_0), x_0) + d(F(y_0, x_0), y_0)\right]$$

and similarly,

$$d(F^2(y_0,x_0),F(y_0,x_0))$$
$$=d(F(F(y_0,x_0),F(x_0,y_0)),F(y_0,x_0))$$
$$=d\left[F(y_0,x_0),F(F(y_0,x_0),F(x_0,y_0))\right]$$
$$\leq \frac{k}{2}\left[d(F(y_0,x_0),y_0)+d(F(x_0,y_0),x_0)\right]$$

Now, assume that the inequalities (5) and (6) hold. By using (3), (4) and

$$F^n\left(x_0,y_0\right)\preceq F^{n+1}\left(x_0,y_0\right) \text{ and}$$
$$F^{n+1}\left(y_0,x_0\right)\preceq F^n\left(y_0,x_0\right),$$

one can obtain

$$d(F^{n+2}(x_0,y_0),F^{n+1}(x_0,y_0))$$
$$=d(F(F^{n+1}(x_0,y_0),F^{n+1}(y_0,x_0)),F(F^n(x_0,y_0),F^n(y_0,x_0)))$$
$$\leq \frac{k}{2}\left[d(F^{n+1}(x_0,y_0),F^n(x_0,y_0))+d(F^{n+1}(y_0,x_0),F^n(y_0,x_0))\right]$$
$$\leq \frac{k^{n+1}}{2}\left[d(F(x_0,y_0),x_0)+d(F(y_0,x_0),y_0)\right]$$

Analogously, one can get

$$d(\left(F^{n+2}(y_0,x_0),F^{n+1}(y_0,x_0)\right)$$
$$\leq \frac{k^{n+1}}{2}\left[d\left(F\left(y_0,x_0\right),y_0\right)+d\left(F\left(x_0,y_0\right),x_0\right)\right]$$

Let us show that $\{F^n\left(x_0,y_0\right)\}$ and $\{F^n(y_0,x_0)\}$ are Cauchy sequences in X.

Suppose $m>n$. Let $0<<c$ be given. Choose $\delta>0$ such that $c+B_\delta\left(0\right)\subset P$ where $B_\delta\left(0\right)=\{y\in E:\left\|y\right\|<\delta\}$. Now, choose a natural number N_0 such that $\frac{k^n}{2(1-k)}[d(F\left(x_0,y_0\right),x_0)+d(F(y_0,x_0),y_0)]\in B_\delta\left(0\right)$ for all $n\geq N_0$. Then

$$\frac{k^n}{2(1-k)}\left[d(F(x_0,y_0),x_0)+d(F(y_0,x_0),y_0)\right]$$
$$<<c.$$

Thus,

$$d(F^m(x_0,y_0),F^n(x_0,y_0))$$
$$\leq d(F^m(x_0,y_0),F^{m-1}(x_0,y_0))$$
$$+\ldots+d(F^{n+1}(x_0,y_0),F^n(x_0,y_0))$$
$$\leq \frac{(k^{m-1}+\ldots+k^n)}{2}\left[d(F(x_0,y_0),x_0)+d(F(y_0,x_0),y_0)\right]$$
$$=\frac{k^n(1+k+\ldots+k^{m-n-1})}{2}\begin{bmatrix}d(F(x_0,y_0),x_0)\\+d(F(y_0,x_0),y_0)\end{bmatrix}$$
$$=\frac{k^n-k^m}{2(1-k)}\left[d(F(x_0,y_0),x_0)+d(F(y_0,x_0),y_0)\right]$$
$$\leq \frac{k^n}{2(1-k)}\left[d(F(x_0,y_0),x_0)+d(F(y_0,x_0),y_0)\right]<<c$$

For all $m>n\geq N_0$. Thus, $\left\{F^n\left(x_0,y_0\right)\right\}$ is a Cauchy sequence in X.

Analogously, one can show that $\left\{F^n(y_0,x_0)\right\}$ is Cauchy sequence in X. Since X is complete CMS, there exist $x,y\in X$ such that $x^n=F^n\left(x_0,y_0\right)\to x$ and $y^n=F^n\left(y_0,x_0\right)\to y$ as $n\to\infty$.

To conclude the proof, we show $F\left(x,y\right)=x$ and $F\left(y,x\right)=y.$ Let $0<<c.$ Choose a natural number N_1 such that $d\left(x_{n+1},x\right)=d\left(F^{n+1}\left(x_0,y_0\right),x\right)<<\frac{c}{2}$, for all $n\geq N_1.$ Since F is continuous, there exists N_2 such that, for all $n>N_2$, one has $\left(x_n,y_n\right)\to\left(x,y\right)$ implies that $d\left(F\left(x_n,y_n\right),F\left(x,y\right)\right)<<\frac{c}{2}$, for all $n\geq N_2$. By triangle inequality,

$$d(F(x,y),x)\leq d(F(x,y),x_{n+1})+d(x_{n+1},x)$$
$$=d(F(x,y),F^{n+1}(x_0,y_0))+d(F^{n+1}(x_0,y_0),x)$$
$$=d(F(x,y),F(F^n(x_0,y_0),F^n(y_0,x_0)))$$
$$+d(F^{n+1}(x_0,y_0),x)$$

Hence, choose $N_0=max\left\{N_1,N_2\right\}$, for all $n>N_0$:

$d(F(x,y),x)$

$\leq d(F(x,y),F(F^n(x_0,y_0),F^n(y_0,x_0)))+d(F^{n+1}(x_0,y_0),x)$

$<< c$.

Thus, $d(F(x,y),x) << \frac{c}{l}$ for all $l \geq 1$. Thus,

$\frac{c}{l} - d(F(x,y),x) \in P$ for all $l \geq 1$.

Regarding $\frac{c}{l} \to 0$ as $l \to \infty$. One can conclude that $-d(F(x,y),x) \in P$.

On account of $d(F(x,y),x) \in P$, one can obtain that $d(F(x,y),x) = 0$. This yields that $F(x,y) = x$ .Analogously, one can show $F(y,x) = y$.

Theorem 10. Let $F: X \times X \to X$ be a mapping having the mixed monotone property on X. Suppose that X has the following properties:

(i) if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \preceq x$, for all n,

(ii) if a non-increasing sequence $\{y_n\} \to y$, then $y \preceq y_n$, for all n.

Assume that there exists a $k \in [0,1)$ with

$$d(F(x,y),F(u,v)) \leq \frac{k}{2}\left[d(x,u)+d(y,v)\right]$$
for all $u \preceq x,\ y \preceq v$.

If there exist $x_0, y_0 \in X$ such that $x_0 \preceq F(x_0,y_0)$ and $F(y_0,x_0) \preceq y_0$, then there exist $x, y \in X$ such that $x = F(x,y)$ and $y = F(y,x)$.

Proof: Regarding the proof of Theorem 9, it is sufficient to show that $x = F(x,y)$ and $y = F(y,x)$.

Let $0 << c$. Since $x_n = F^n(x_0,y_0) \to x$ and $y_n = F^n(y_0,x_0) \to y$, then there exists N_0, such that $d(F^n(x_0,y_0),x) << \frac{c}{2}$ and $d(F^m(y_0,x_0),y) << \frac{c}{2}$ for all $n,\ m > N_0$.

Using triangle inequality and regarding $F^n(x_0,y_0) = x_n \preceq x$ and $y \preceq y_n = F^n(x_0,y_0)$ one can get

$$\begin{aligned}
d(F(x,y),x) &\leq d(F(x,y),x_{n+1}) + d(x_{n+1},x)\\
&= d(F(x,y),F^{n+1}(x_0,y_0)) + d(F^{n+1}(x_0,y_0),x)\\
&= d(F(x,y),F(F^n(x_0,y_0),F^n(y_0,x_0)))\\
&\quad + d(F^{n+1}(x_0,y_0),x)\\
&\leq \frac{k}{2}[d(x,F^n(x_0,y_0)) + d(y,F^n(y_0,x_0))]\\
&\quad + d(F^{n+1}(x_0,y_0),x)\\
&<< \frac{k}{2}[\frac{c}{2}+\frac{c}{2}] + \frac{c}{2} << c \quad (\text{since } 0 \leq k < 1),
\end{aligned}$$

for all $n > N$. This yields that $F(x,y) = x$. Analogously, one can show $F(y,x) = y$. ■

The couple fixed point is unique if the product space $X \times X$ endowed with the partial order mentioned earlier has one of the equivalent conditions:

(8): Every pair of elements has either a lower bound or an upper bound.

(9): For every (x,y), $(x^*,y^*) \in X \times X$, there exists a $(z,w) \in X \times X$ which is comparable with (x,y) *and* (x^*,y^*).

Equivalence of these conditions is proved in [12]. Notice that Theorem 9 can not guarantee the uniqueness of the fixed point. But under the condition (8) or (9), it yield the uniqueness of the fixed point. The following theorem will clarify and explain this consideration.

Theorem 11: Under the hypothesis of Theorem 9, the uniqueness of the couple fixed point of F is obtained by the condition (9).

Proof: By Theorem 9, there exists $(x,y) \in X \times X$ such that $F^n(x_0,y_0) \to x$ and $F^m(y_0,x_0) \to y$.

Let $(x^*,y^*) \in X \times X$ be another couple fixed point of F. Consider two cases:

Case (i): Suppose that (x,y) *and* (x^*,y^*) are comparable with respect to the ordering in $X \times X$.

Then for each $n = 0,1,2,...,$ the pair $\left(F^n\left(x,y\right),F^n\left(y,x\right)\right)=\left(x,y\right)$ is comparable to $(F^n\left(x^*,y^*\right),F^n\left(y^*,x^*\right))=\left(x^*,y^*\right)$. Furthermore, by (4),

$$\rho\left(\left(x,y\right),\left(x^*,y^*\right)\right)=d\left(x,x^*\right)+d\left(y,y^*\right)$$
$$=d(F^n(x,y),F^n\left(x^*,y^*\right))$$
$$+d(F^n(y,x),F^n\left(y^*,x^*\right))$$
$$\leq k^n[d\left(x,x^*\right)+d\left(y,y^*\right)]$$
$$= k^n\ \rho\left(\left(x,y\right)+\left(x^*,y^*\right)\right)$$

which yields that $\rho\left(\left(x,y\right),\left(x^*,y^*\right)\right)=0.$ That is, couple fixed point is unique.

Case (ii): Suppose that $\left(x,y\right)$ and $\left(x^*,y^*\right)$ are not comparable with respect to the ordering in $X\times X$. Then, there exists an upper bound or lower bound $\left(z_1,z_2\right)\in X\times X$ of $\left(x,y\right)$ and $\left(x^*,y^*\right)$. Thus, for each $n=0,1,2,...,$ the pair $\left(F^n\left(z_1,z_2\right),F^n\left(z_2,z_1\right)\right)=\left(x,y\right)$ is comparable with the pair $\left(F^n\left(x,y\right),F^n\left(y,x\right)\right)$ and $\left(F^n\left(x^*,y^*\right),F^n\left(y^*,x^*\right)\right)=\left(x^*,y^*\right)$. Hence, by (4) and triangle inequality,

$$\rho\left((x,y),\left(x^*,y^*\right)\right)$$
$$\leq\rho\left(\left(F^n(x,y),F^n(y,x)\right),\left(F^n\left(x^*,y^*\right),F^n\left(y^*,x^*\right)\right)\right)$$
$$\leq\rho\left(\left(F^n(x,y),F^n(y,x)\right),\left(F^n(z_1,z_2),F^n(z_2,z_1)\right)\right)$$
$$+\rho\left(\left(F^n(z_1,z_2),F^n(z_2,z_1)\right),\left(F^n\left(x^*,y^*\right),F^n\left(y^*,x^*\right)\right)\right)$$
$$\leq k^n\ \left[d\left(x,z_1\right)+d(y,z_2)+d(z_1,x^*)+d(z_2,y^*)\right]$$

whose right hand side tends to zero whenever $n\to\infty$.

That is equivalent to saying that

$$\rho\left(\left(x,y\right),\left(x^*,y^*\right)\right)=0\ .\blacksquare$$

Theorem 12. Under the hypothesis of Theorem 9, suppose that each pair of elements of X has an upper bound or lower bound in X. Then $x=y.$

Proof: By Theorem 9, there exists couple fixed point $\left(x,y\right)\in X\times X$ such that $F\left(x,y\right)=x$ and $F\left(y,x\right)=y.$ Consider two cases:

Case (i): Suppose x is comparable with y. That is, $x=F\left(x,y\right)$ is comparable with $y=F\left(y,x\right).$ Thus, by (4), $d\left(x,y\right)=d\left(F\left(x,y\right),F\left(y,x\right)\right)\leq\ k\ d\left(x,y\right).$ Since $0\ \leq\ k\ <1,$ then $d\left(x,y\right)=0.$ That is, $x=y.$

Case (ii): Suppose x is not comparable with y. Then, there exists an upper bound or lower bound of x and y. That is, there exists $z\ \in X$ which is comparable with $x=F\left(x,y\right)$ and $y=F(y,x)$.

Suppose that $x\preceq z$ and $y\ \preceq\ z$ holds. Then, it follows

$$F\left(x,y\right)\ \preceq F\left(z,y\right)\text{ and }F\left(x,z\right)\ \preceq\ F\left(x,y\right)$$
$$F\left(y,x\right)\ \preceq F\left(z,x\right)\text{ and }F\left(y,z\right)\ \preceq\ F\left(y,x\right)$$

By the mixed monotone property of F, one can obtain

(A)

$$F^2\left(x,y\right)=F\left(F\left(x,y\right),F\left(y,x\right)\right)$$
$$\preceq F\left(F\left(z,y\right),F\left(y,z\right)\right)=F^2\left(z,y\right)$$

which implies that $F^2\left(x,y\right)\ \preceq\ F^2\left(z,y\right)$

(B)

$$F^2\left(y,x\right)=F\left(F\left(y,x\right),F\left(x,y\right)\right)$$
$$\preceq F\left(F\left(z,x\right),F\left(x,z\right)\right)=F^2\left(z,x\right)$$

which implies that $F^2\left(y,x\right)\ \preceq\ F^2\left(z,x\right)$

(C)

$$F^2\left(x,z\right)=F\left(F\left(x,z\right),F\left(z,x\right)\right)$$
$$\preceq F\left(F\left(x,y\right),F\left(y,x\right)\right)=F^2\left(x,y\right)$$

which implies that $F^2\left(x,z\right)\preceq F^2\left(x,y\right)$

(D)

$$F^2\left(y,z\right)=F\left(F\left(y,z\right),F\left(z,y\right)\right)$$
$$\preceq F\left(F\left(y,x\right),F\left(x,y\right)\right)=F^2(y,x)$$

which implies that $F^2\left(y,z\right)\preceq F^2\left(y,x\right).$

For $n>2,$ one can obtain the analogous form of **(A), (B), (C)** and **(D)**. Consider the following:

$$d(x,y)=d(F^{n+1}(x,y),F^{n+1}(y,x))$$
$$=d\left(F\left(F^n(x,y),F^n(y,x)\right),F\left(F^n(y,x),F^n(x,y)\right)\right)$$
$$\leq d\left(F\left(F^n(x,y),F^n(y,x)\right),F\left(F^n(x,z),F^n(z,x)\right)\right)$$
$$+d\left(F\left(F^n(x,z),F^n(z,x)\right),F\left(F^n(y,x),F^n(x,y)\right)\right)$$
$$\leq d\left(F\left(F^n(x,y),F^n(y,x)\right),F\left(F^n(x,z),F^n(z,x)\right)\right)$$
$$+d\left(F(F^n(x,z),F^n(z,x)),F\left(F^n(z,x),F^n(x,z)\right)\right)$$
$$+d\left(F\left(F^n(z,x),F^n(x,z)\right),F\left(F^n(y,x),F^n(x,y)\right)\right).$$

The contractivity condition on F yields that

$$d(x,y)\leq\frac{k}{2}[d(F^n(x,y),F^n(x,z))$$
$$+d(F^n(y,x),F^n(z,x))$$
$$+d(F^n(x,z),F^n(z,x))$$
$$+d(F^n(z,x),F^n(x,z))$$
$$+d(F^n(z,x),F^n(y,x))$$
$$+d(F^n(x,z),F^n(x,y))]$$
$$=k[d(F^n(x,y),F^n(z,x))$$
$$+d(F^n(x,z),F^n(z,x))$$
$$+d(F^n(z,x),F^n(x,y))]$$

Simple calculations imply that $d(x,y)=k^{n+1}[d(x,z)+d(z,y)]$. The right-hand side tends to as $n\to\infty$. That is, $d(x,y)=0$. ■

Theorem 13: Under the hypothesis of Theorem 9, suppose that the normal constant of the cone $K=1$ and $x_0,\ y_0\in X$ are comparable. Then $x=y$.

Proof: By Theorem 9, $x_0\in X$ satisfies that $x_0\preceq F(x_0,y_0)$. Consider the case $x_0\preceq y_0$. We assert that $x_n\preceq y_n$ for all $n\in$ IN. For n=1 , it is followed by the mixed property of F, that is,

$$x_1=F(x_0,y_0)\preceq F(y_0,x_0)=y_1.$$

Suppose that $x_n\preceq y_n$. Consider

$$x_{n+1}=F^{n+1}(x_0,y_0)$$
$$=F\left(F^n(x_0,y_0),F^n(y_0,x_0)\right)$$
$$=F(x_n,y_n)\preceq F(y_n,x_n)=y_{n+1}$$

which implies that our assertion is true, that is, $x_n\preceq y_n$ for all $n\in$ IN. . For a given $0<<c,$ there exists a $N_0\in IN$ such that

$$d\left(x,F^n(x_0,y_0)\right)<<\frac{c}{3} \text{ and}$$

$d\left(F^n(y_0,x_0),y\right)<<\frac{c}{3}$ for all $n\geq N_0$. By (4) and triangle inequality,

$$d(x,y)\leq d(x,F^{n+1}(x_0,y_0))$$
$$+d(F^{n+1}(x_0,y_0),y)$$
$$\leq d(x,F^{n+1}(x_0,y_0))$$
$$+d(F^{n+1}(x_0,y_0),F^{n+1}(y_0,x_0))$$
$$+d(F^{n+1}(y_0,x_0),y)$$
$$\leq d(x,F^{n+1}(x_0,y_0))$$
$$+d\begin{pmatrix}F(F^n(x_0,y_0),F^n(y_0,x_0)),\\ F(F^n(y_0,x_0),F^n(x_0,y_0))\end{pmatrix}$$
$$+d(F^{n+1}(y_0,x_0),y)$$
$$\leq d(x,F^{n+1}(x_0,y_0))$$
$$+\frac{k}{2}d(F^n(y_0,x_0),F^n(x_0,y_0)$$
$$+d(F^{n+1}(y_0,x_0),y)$$
$$\leq d(x,F^{n+1}(x_0,y_0))$$
$$+\frac{k}{2}(d(F^n(y_0,x_0),y)+d(x,y)$$
$$+d(x,F^n(x_0,y_0))+d(F^{n+1}(y_0,x_0),y)$$

Thus,

$$(1-k)d(x,y)\leq d(x,F^{n+1}(x_0,y_0))$$
$$+\frac{k}{2}(d(F^n(y_0,x_0),y)$$
$$+d(x,F^n(x_0,y_0))$$
$$+d(F^{n+1}(y_0,x_0),y)$$
$$<<\frac{c}{3}+\frac{k}{2}(\frac{c}{3}+\frac{c}{3})+\frac{c}{3}$$
$$<<c\ \ (\text{since } 0\leq k<1)$$

which turns leads to $d(x,y)=0$. Thus $x=y$. Analogously, the other case $y_0\preceq x_0$ is obtained that conclude the proof. ■

Remark 14: If we assume that $x_0, y_0 \in X$ are comparable in addition to the hypothesis of **Theorem 9,** then $x = y$.

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