THE LINEAR METHOD FOR SOLVING THE SCATTERING PROBLEM IN AN INHOMOGENEOUS MEDIUM: THE CASE OF TE POLARIZED

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Abstract

In this article the electromagnetic waves scattered from an inhomogeneous medium are considered when the electromagnetic waves are polarized in the case of transverse electric. Using the Rellich lemma, the uniqueness of the solution of the direct scattering problem is proved. In order to show the existence of the solution of this problem, the operator equations are constructed and the Riesz theory which provides the existence of the inverse operator is used. Furthermore, for solution of the inverse scattering problems, an interior boundary value problem is considered. Finally, a linear integral equation is obtained whose the solution yield the support of the scattering object.

Keywords: Electromagnetic wave, Far-field pattern, Linear method, Scattering theory.

1. Introduction

The scattering problems of time-harmonic waves which are acoustic or electromagnetic waves are the basic problems in the scattering theory. These problems have been considered by many writers as direct and indirect scattering problems [2-19, 22-24].

Before the inverse scattering problems with regard to the direct scattering problems, the most important questions are the uniqueness and the existence of the solution of the direct scattering problem. Gerlach and Kress [17] , Colton, Kress and Monk [8] have proved the uniqueness of the solution by using Green's theorems and the unique continuation property of solution. Furthermore, they have showed the existence of solution by using the jump relations of the single-layer and double-layer in the potential theory and the integral equations. For the transmission boundary value problem, this results have been proved by Colton and Piana [9] .

In the inverse scattering theory, the most importance thing is scattered far-field model. In the 1980's, the inverse scattering problem of determining the unknown scattering obstacle from information about the far- field data was considered by Angel, Colton and Kirsch [2] , Tobocman [23] and many more

mathematicians. Integral equations or Green's formulas were used to reformulate the inverse obstacle problem by these researchers.

For the solution of the inverse scattering problem, a method is the linear method which was suggested, firstly, by Colton and Kirsch [10]. Then the method is used by Colton, Kress and Monk [8], Colton and Piana [9], Colton, Piana and Potthast [11], Colton, Giebermann and Monk [12], Colton, Coyle and Monk [13], Cakoni, Colton and Monk [3], Cakoni, Colton and Haddar [4], Cakoni and Colton [5], Colton [14] and Colton and Kress [15]. This method is mathematically established by placing a network on the unknown domain by solving a linear integral equation for each point on this network and then determining the shape of the domain from the information about the solutions for this set of integral equations. To apply this method, first, the far field operatör $F : L^2(\Omega) \to L^2(\Omega)$ is defined by

$$(Fg)(\hat{x}) = \int_{\Omega} u_\infty(\hat{x}, \mathrm{d}) g(\mathrm{d}) ds(\mathrm{d}), \qquad \hat{x}, \mathrm{d} \in \Omega$$

where $\Omega = \{ x \in IR^2 : |x| = 1 \}$. Then the Regulation method [16] is used to solving of the linear integral equation $(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, y)$, where $\Phi_\infty(\hat{x}, y) = \dfrac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}.y}$ is the far-field model of the function

$\Phi(x, y) = \dfrac{i}{4} H_0^{(1)}(k|x - y|)$ for $x \neq y$ [1]. According to this method, for $\forall \varepsilon > 0$, there exists a function $g = g(., y) \in L^2(\Omega)$ such that $\|Fg - \Phi_\infty\| < \varepsilon$ and both $\|g(., y)\|$ and $\|v_g(., y)\|$ become unbounded as y approaches the boundary of the scatterer, where $v_g(x) = \int_{\Omega} e^{ikx.\mathrm{d}} g(\mathrm{d}) ds(\mathrm{d})$ is the Herlogtz wave function with kernel $g(., y)$ [16]. The Herlogtz kernel $g(., y)$ is determined from $(Fg)(\hat{x}) = \int_{\Omega} u_\infty(\hat{x}, \mathrm{d}) g(\mathrm{d}) ds(\mathrm{d})$ for y on a grid containing the scatterer. Thus, the boundary of the unknown domain can be found as the locus of points y, where $\|g(., y)\|_{L^2(\Omega)}$ begins to increase sharply.

Now, we consider the following problem:

We investigated an electromagnetic scattering problem in an inhomogenous medium when the incident wave is polarized parallel to the axis of infinite cylinder representing the scatterer and the magnetic field has only one component in the direction of the axis to the cylinder. This is the referred to as the transverse electric mode (briefly, TE mode) in scattering theory [9,22,24]. The electromagnetic waves can be obtained by using the Maxwell equations [16]. We assume that D is a simply-connected bounded domain in IR^2 with $C^2(\partial D)$ and which the domain is the cross section of the cylinder. For the time-harmonic electromagnetic waves, the scattering is defined by the Maxwell equations

$$curl E_0 - ik H_0 = 0 \qquad curl H_0 + ik E_0 = 0, \qquad \text{in } IR^2 \setminus \bar{D} \tag{1}$$

$$curlE - ikH = 0 \qquad curlH + ikn(x)E = 0, \quad in \ D \tag{2}$$

and the boundary conditions

$$v \times H_0 = v \times H, \qquad on \ \partial D \tag{3}$$

$$v \times curlE_0 + \frac{k}{\lambda}(v \times E_0) \times v = \frac{1}{n_0}(v \times curlE) + \frac{k}{\lambda}(v \times E) \times v, \ on \ \partial D \tag{4}$$

where k is positive wave number and v is outward unit normal vector on ∂D. Let (E_0, H_0) and (E, H) be electromagnetic fields outside and inside the cylinder, respectively. Let be $\lambda \in C^{0,\alpha}(\partial D)$ and $\text{Im} \lambda \geq 0$. $n(x)$ is the index of refraction defined by

$$n(x) = \frac{1}{\varepsilon_0}\left(\varepsilon(x) + \frac{i\sigma(x)}{\omega}\right)$$

where ε_0 is the constant permittivity in $IR^2 \setminus \bar{D}$, $\varepsilon(x)$ and $\sigma(x)$ are the permittivity and the conductivity of the cylinder, respectively, and ω is the frequency of the electromagnetic waves. We assume that $n(x)$ satisfies the following conditions:

(i) $n(x) \in C^2(IR^2)$ and $n(x) = n_0 > 0$ for $x \in IR^2 \setminus D$, where $n_0 (\neq 1) \in IR$

(ii) $\text{Im} n(x) \geq 0$ and $D_0 = \{x \in D : \text{Im} n(x) > 0\} \neq \varnothing$. \tag{5}

If the electromagnetic wave is polarized in the transverse electric mode, the scalar fields u_0 and u can be defined as $H_0 = (H_{0_1}, H_{0_2}, H_{0_3}) = (0, 0, u_0)$ and $H = (H_1, H_2, H_3) == (0, 0, u)$. Thus, the Maxwell equations 1-2 and the boundary conditions 3-4 are equivalent to the Helmholtz equations and the boundary conditions in the following

$$\Delta u_0 + k^2 u_0 = 0, \qquad in \ IR^2 \setminus \bar{D} \tag{6}$$

$$\nabla.\left(\frac{1}{n}\nabla u\right) + k^2 u = 0, \qquad in \ D \tag{7}$$

$$u_0 - u = 0, \qquad on \ \partial D \tag{8}$$

$$\frac{\partial u_0}{\partial v} - \frac{1}{n_0}\frac{\partial u}{\partial v} + \lambda k\left(u_0 - \frac{1}{n_0}u\right) = 0, \qquad on \ \partial D. \tag{9}$$

The exterior field u_0 can be written in the form

$$u_0(x) = u^i(x) + u^s(x),$$

where $d \in \Omega = \{x \in IR^2 : |x| = 1\}$ and $u^i(x) = e^{ikx.d}$ is the incident plane wave with incident direction d. The scattered wave u^s satisfies the Sommerfeld radiation condition

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial u^s}{\partial r} - iku^s \right) = 0 \tag{10}$$

uniformly in all directions $\hat{x} = \dfrac{x}{|x|}$ with $r = |x|$. This condition guarantees that the scattered wave has the asymptotic behaviour

$$u^s(x) = u_\infty(\hat{x}, d) \frac{e^{ikr}}{\sqrt{r}} + O\left(r^{-3/2}\right)$$

as $r = |x| \to \infty$, where u_∞ is known as the far-field pattern of the scattered wave and is defined in the form

$$u_\infty(\hat{x}, d) = \frac{e^{i\pi/4}}{\sqrt{8\pi k}} \int_{\partial D} \left[u(y) \frac{\partial e^{-ik\hat{x}.y}}{\partial \nu(y)} - \frac{\partial u}{\partial \nu}(y) e^{-ik\hat{x}.y} \right] ds(y), \hat{x} \in \Omega \quad [16].$$

2. The Direct Scattering Problem

The scattering of time-harmonic plane waves by a simply connected bounded domain $D \subset IR^2$ is formed with the following direct scattering problem. For given $f \in C^{1,\alpha}(\partial D)$ and $\lambda, g \in C^{0,\alpha}(\partial D)$ from Hölder spaces with exponent $0 < \alpha < 1$, this problem is to find a pair of functions $u_0 \in C^2(IR^2 \setminus \overline{D}) \cap C^1(IR^2 \setminus D)$ and $u \in C^2(D) \cap C^1(\overline{D})$ such that

$$\Delta u_0 + k^2 u_0 = 0, \qquad \text{in } IR^2 \setminus \overline{D} \tag{11}$$

$$\nabla.(\frac{1}{n} \nabla u) + k^2 u = 0, \quad \text{in } D \tag{12}$$

$$u_0 - u = f, \qquad \text{on } \partial D \tag{13}$$

$$\frac{\partial u_0}{\partial \nu} - \frac{1}{n_0} \frac{\partial u}{\partial \nu} + \lambda k \left(u_0 - \frac{1}{n_0} u \right) = g, \qquad \text{on } \partial D, \tag{14}$$

where k is positive wave number and ν is the unit outward to ∂D. n_0 and n are defined in the conditions (i)-(ii) of (5). u_0 satisfies the Sommerfeld radiation condition (10), i.e.

$$\lim_{r \to \infty} \sqrt{r}\left(\frac{\partial u_0}{\partial r} - iku_0 \right) = 0 , \tag{15}$$

where $r = |x|$. For simplicity, we will always suppose that $\operatorname{Im} \lambda \geq 0$ on ∂D.

Theorem 2.1. The solution of the boundary value problem 11-15 is unique.

Proof. We suppose that the solution of the problem 11-15 is not unique. Let $u_0 = u_{0_1} - u_{0_2}$ and $u = u_1 - u_2$. Thus u_0 and u satisfy the homogeneous boundary value problem 6-9.
We first show that

$$\lim_{r \to \infty} \int_{\Omega_r} |u_0|^2 \, ds = 0 , \tag{16}$$

where Ω_r denotes the circle with the radius r and centered in the origin and ds is the arc lenght. To achieve this, from the Sommerfeld radiation condition 15, we have

$$\lim_{r \to \infty} \int_{\Omega_r} \left[\left| \frac{\partial u_0}{\partial \nu} \right|^2 + k^2 |u_0|^2 + 2k \operatorname{Im}\left(u_0 \frac{\partial \overline{u_0}}{\partial \nu} \right) \right] ds = \lim_{r \to \infty} \int_{\Omega_r} \left| \frac{\partial u_0}{\partial \nu} - iku_0 \right|^2 ds = 0 . \tag{17}$$

We take r large enough such that $D \subset \Omega_r$. Applying Green's theorem [16] in the domain $D_r = \left\{ y \in IR^2 \setminus \overline{D} : |y| < r \right\}$, we have

$$\int_{D_r} -k^2 |u_0|^2 \, dy + \int_{D_r} |grad u_0|^2 \, dy = -\int_{\Omega_r} u_0 \frac{\partial \overline{u_0}}{\partial \nu} ds(y) + \int_{\partial D} u_0 \frac{\partial \overline{u_0}}{\partial \nu} ds(y) .$$

Taking imaginary parts of this equation, from the equation 17, we obtain

$$\lim_{r \to \infty} \int_{\Omega_r} \left[\left| \frac{\partial u_0}{\partial \nu} \right|^2 + k^2 |u_0|^2 \right] ds = -2k \operatorname{Im} \int_{\partial D} u_0 \frac{\partial \overline{u_0}}{\partial \nu} ds . \tag{18}$$

Applying Divergence theorem [18] to the function $u \left(\frac{1}{\overline{n}} \nabla \overline{u} \right)$, from the condition (i) of (5) and boundary conditions 8-9, we get

$$\int_D \left[-k^2 |u|^2 + \frac{1}{n} |\nabla u|^2 \right] dy = \int_{\partial D} \left[u_0 \frac{\partial \overline{u_0}}{\partial \nu} + \overline{\lambda} k |u_0|^2 - \frac{\overline{\lambda} k}{n_0} |u_0|^2 \right] ds(y).$$

Taking imaginary parts of this equation, we have

$$\text{Im} \int_D \frac{1}{n} |\nabla u|^2 \, dy = \text{Im} \int_{\partial D} u_0 \frac{\partial \overline{u_0}}{\partial \nu} ds(y) + \text{Im} \int_{\partial D} \left(1 - \frac{1}{n_0} \right) \overline{\lambda} k |u_0|^2 \, ds(y). \tag{19}$$

From the condition (ii) of 5, the left-hand of equation 19 is positive or zero. Since $\text{Im}\,\overline{\lambda} \leq 0$ on ∂D, again from the condition (i) of 5, the last integral in the right-hand of equation 19 is negative or zero. Thus we obtain

$$\text{Im} \int_{\partial D} u_0 \frac{\partial \overline{u_0}}{\partial \nu} ds(y) \geq 0.$$

The equation 18 becomes

$$\lim_{r \to \infty} \int_{\Omega_r} \left[\left| \frac{\partial u_0}{\partial \nu} \right|^2 + k^2 |u_0|^2 \right] ds \leq 0. \tag{20}$$

Since the left-hand of equation 20 is positive or zero, we get the equation 16. From Rellich's lemma [16], $u_0 = 0$ in $IR^2 \setminus \overline{D}$ and so $u_0 = \frac{\partial u_0}{\partial \nu} = 0$ in $IR^2 \setminus D$ from the Theorem 3.12 in [7]. From the conditions 8-9, we obtain $u = \frac{\partial u}{\partial \nu} = 0$ on ∂D. From the unique continuation principle (see : Theorem 8.6 in [16]), we obtain $u = 0$ in D.

We will now apply the Riesz's theory (the inverse operator's existence) for compact operators [7,18] to demonstrate the existence of solution to the boundary value problem 11-15. With the change of variables $u(x) = \sqrt{n(x)} w(x)$, the boundary value problem 11-15 takes form

$$\Delta u_0 + k^2 u_0 = 0 , \text{ in } IR^2 \setminus \overline{D} \tag{21}$$

$$\Delta w + \left(k^2 n + p \right) w = 0 , \quad \text{ in } D \tag{22}$$

$$u_0 - \sqrt{n_0} w = f , \quad \text{ on } \partial D \tag{23}$$

$$\frac{\partial u_0}{\partial \nu} - \frac{1}{\sqrt{n_0}} \frac{\partial w}{\partial \nu} + \lambda k \left(u_0 - \frac{1}{\sqrt{n_0}} w \right) = g , \quad \text{ on } \partial D \tag{24}$$

where

$$p(x) = -\sqrt{n(x)}\Delta\frac{1}{\sqrt{n(x)}} \ . \tag{25}$$

Then for $\psi, \phi \in C(\partial D)$ and $\psi_1 \in C(D)$, let's define the following functions

$$u_0(x) = \int_{\partial D} \left[\frac{\partial\Phi_0(x, y)}{\partial\nu(y)}\psi(y) + \Phi_0(x, y)\phi(y) \right]ds(y), \qquad x \in IR^2 \setminus \partial D \tag{26}$$

$$w(x) = \int_{\partial D} \left[\sqrt{n_0}\frac{\partial\Phi(x, y)}{\partial\nu(y)}\psi(y) + \Phi(x, y)\phi(y) \right]ds(y) + \int_D \Phi(x, y)\rho(y)\psi_1(y)dy, x \in IR^2 \setminus \partial D \tag{27}$$

where $\rho(x) = k^2 n_0 - [k^2 n(x) + p(x)]$ and   the   functions $\Phi_0(x, y) = \frac{i}{4}H_0^{(1)}(k|x - y|)$ and

$\Phi(x, y) = \frac{i}{4}H_0^{(1)}(k\sqrt{n_0}|x - y|)$, $x \neq y$ in IR^2 are the fundemantal solutions of the Helmholtz

equations which are $\Delta u + k^2 u = 0$ and $\Delta u + k^2 n_0 u = 0$, respectively, where $H_0^{(1)}$ is the Hankel function

of the first kind and the zero order. The functions u_0 defined by equation 26 and w defined by equation

27 satisfies the problem 21-24 and the Sommerfeld radiation condition 15.

   We introduce the following integral operators:

The operators K, S, T and K^t are defined from $C(\partial D)$ to $C(\partial D)$, such that

$$(K\psi)(x) = 2\int_{\partial D} \frac{\partial\Phi(x, y)}{\partial\nu(y)}\psi(y)ds(y), \qquad x \in \partial D \tag{28}$$

$$(S\phi)(x) = 2\int_{\partial D} \Phi(x, y)\phi(y)ds(y), \qquad x \in \partial D \tag{29}$$

$$(T\psi)(x) = 2\frac{\partial}{\partial\nu(x)}\int_{\partial D} \frac{\partial\Phi(x, y)}{\partial\nu(y)}\psi(y)ds(y), \quad x \in \partial D \tag{30}$$

$$(K^t\phi)(x) = 2\frac{\partial}{\partial\nu(x)}\int_{\partial D} \Phi(x, y)\phi(y)ds(y), \quad x \in \partial D \tag{31}$$

The operators $K^\wedge$ and $S^\wedge$ are defined from $C(\partial D)$ to $C(D)$, such that

$$(K^{\wedge}\psi)(x) = 2\int_{\partial D}\frac{\partial\Phi(x,y)}{\partial\nu(y)}\psi(y)ds(y), \quad x \in D \tag{32}$$

$$(S^{\wedge}\phi)(x) = 2\int_{\partial D}\Phi(x,y)\phi(y)ds(y), \quad x \in D \tag{33}$$

The operators S_{ρ} and K_{ρ}^{ι} are defined from $C(D)$ to $C(\partial D)$, such that

$$(S_{\rho}\psi_1)(x) = 2\int_{D}\Phi(x,y)\rho(y)\psi_1(y)dy, \quad x \in \partial D \tag{34}$$

$$(K_p^{\iota}\psi_1)(x) = 2\frac{\partial}{\partial\nu(x)}\int_{D}\Phi(x,y)\rho(y)\psi_1(y)dy, \quad x \in \partial D \tag{35}$$

Finally, the operatör $S_{\rho}^{\wedge}$ be defined from $C(D)$ to $C(D)$, such that

$$(S_{p}^{\wedge}\psi_1)(x) = 2\int_{D}\Phi(x,y)\rho(y)\psi_1(y)dy, \quad x \in D. \tag{36}$$

Let K_0, S_0, T_0 and K_0^{ι} show the operators corresponding to K, S, T and K^{ι}, respectively, with Φ replaced by Φ_0.

Theorem 2.2. The functions u_0 and w defined by equations 26-27 are restricted to $IR^2 \setminus \overline{D}$ and D, respectively. Then the functions $\psi, \phi \in C(\partial D)$ and $\psi_1 \in C(D)$ satisfy the following integral equations

$$(K_0 - n_0 K)\psi + (1+n_0)\psi + (S_0 - \sqrt{n_0}S)\phi - \sqrt{n_0}S_{\rho}\psi_1 = 2f, \quad \text{on} \ \partial D \tag{37}$$

$$(T_0 - T)\psi + \left(K_0^{\iota} - \frac{1}{\sqrt{n_0}}K^{\iota}\right)\phi - \left(1 + \frac{1}{\sqrt{n_0}}\right)\phi - \frac{1}{\sqrt{n_0}}K_{\rho}^{\iota}\psi_1$$
$$+\lambda k\left[(K_0 - K)\psi + \left(S_0 - \frac{1}{\sqrt{n_0}}S\right)\phi + 2\psi - \frac{1}{\sqrt{n_0}}S_{\rho}\psi_1\right] = 2g, \quad \text{on} \ \partial D \tag{38}$$

$$\sqrt{n_0}K^{\wedge}\psi + S^{\wedge}\phi + S_{\rho}^{\wedge}\psi_1 - 2\psi_1 = 0, \quad \text{in} \ D. \tag{39}$$

Proof. Firstly, we will obtain the integral equation 37. When $x \in IR^2 \setminus \overline{D} \to x \in \partial D$, the limit value of u_0 in equation 26 is

$$u_0^+(x) = \int\limits_{\partial D} \frac{\partial \Phi_0(x,y)}{\partial \nu(y)} \psi(y) ds(y) + \frac{1}{2}\psi(x) + \int\limits_{\partial D} \Phi_0(x,y)\phi(y)ds(y).$$

When $x \in D \to x \in \partial D$, the limit value of w in equation 27 is

$$w^-(x) = \sqrt{n_0}\left[\int\limits_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \psi(y)ds(y) - \frac{1}{2}\psi(x)\right] + \int\limits_{\partial D} \Phi(x,y)\phi(y)ds(y)$$

$$+ \int\limits_D \Phi(x,y)\rho(y)\psi_1(y)dy.$$

From the condition 23 and the operators 28, 29,34, we obtain

$$2f(x) = (K_0\psi)(x) + \psi(x) + (S_0\phi)(x) - n_0(K\psi)(x) + n_0\psi(x) - \sqrt{n_0}(S\phi)(x) - \sqrt{n_0}(S_\rho\psi_1)(x).$$

Thus, for $\forall x \in \partial D$, the equation 37 is obtained.

Now, we will obtain the integral equation 38. We take the derivative of the function u_0 in the direction ν. When $x \in IR^2 \setminus \overline{D} \to x \in \partial D$, the limit value of $\dfrac{\partial u_0}{\partial \nu}$ is

$$\frac{\partial u_0^+}{\partial \nu}(x) = \frac{\partial}{\partial \nu(x)} \int\limits_{\partial D} \frac{\partial \Phi_0(x,y)}{\partial \nu(y)} \psi(y)ds(y) + \frac{\partial}{\partial \nu(x)} \int\limits_{\partial D} \Phi_0(x,y)\phi(y)ds(y) - \frac{1}{2}\phi(x).$$

We take the derivative of the function w in the direction ν. When $x \in D \to x \in \partial D$, the limit value of $\dfrac{\partial w}{\partial \nu}$ is

$$\frac{\partial w^-}{\partial \nu}(x) = \sqrt{n_0}\frac{\partial}{\partial \nu(x)} \int\limits_{\partial D} \frac{\partial \Phi(x,y)}{\partial \nu(y)} \psi(y)ds(y) + \frac{\partial}{\partial \nu(x)} \int\limits_{\partial D} \Phi(x,y)\phi(y)ds(y)$$

$$+ \frac{1}{2}\phi(x) + \frac{\partial}{\partial \nu(x)} \int\limits_D \Phi(x,y)\rho(y)\psi_1(y)dy.$$

From the condition 24 and the operators 28, 31, 34, 35 we have

$$2g(x) = (T_0\psi)(x) + (K_0'\phi)(x) - \phi(x) - (T\psi)(x) - \frac{(K'\phi)(x)}{\sqrt{n_0}} - \frac{\phi(x)}{\sqrt{n_0}} - \frac{(K_\rho'\psi_1)(x)}{\sqrt{n_0}}$$

$$+\lambda(x)k\left[(K_0\psi)(x)+\psi(x)+(S_0\phi)(x)-(K\psi)(x)+\psi(x)-\frac{(S\phi)(x)}{\sqrt{n_0}}-\frac{(S_\rho\psi_1)(x)}{\sqrt{n_0}}\right].$$

Thus, for $\forall x \in \partial D$, the equation 38 is obtained.

Finally, for the integral equation 39, if we write the operators 32, 33 and 36 in the function w defined by equation 27, then we obtain

$$2w(x)=\sqrt{n_0}\left(K^\wedge\psi\right)(x)+\left(S^\wedge\phi\right)(x)+\left(S_\rho^\wedge\psi_1\right)(x).$$

Since $w(x) \in C^2(D) \cap C^1(\overline{D})$ and $\psi_1(x) \in C(D)$, we can write $w(x) = \psi_1(x)$. Thus, we satisfy the equation 39 for $\forall x \in D$.

Equations 37 - 39 can be written in operator notation as

$$[A+B]\begin{bmatrix}\psi\\\phi\\\psi_1\end{bmatrix}=\begin{bmatrix}2f\\2g\\0\end{bmatrix},\tag{40}$$

where the matrixes A and B are described in the following forms

$$A=\begin{bmatrix}(1+n_0)I & 0 & 0\\ 2\lambda kI & -\left(1+\frac{1}{\sqrt{n_0}}\right)I & 0\\ 0 & 0 & -2I\end{bmatrix}$$

and

$$B=\begin{bmatrix}K_0-n_0K & S_0-\sqrt{n_0}S & -\sqrt{n_0}S_\rho\\ (T-T_0)+\lambda k(K_0-K) & \left(K_0^t-\frac{1}{\sqrt{n_0}}K^t\right)+\lambda k\left(S_0-\frac{1}{\sqrt{n_0}}S\right) & -\frac{1}{\sqrt{n_0}}\left(K_\rho^t+\lambda kS_\rho\right)\\ \sqrt{n_0}K^\wedge & S^\wedge & S_\rho^\wedge\end{bmatrix}.$$

The operator A clearly has a bounded inverse [2,4]. The operators in the matrix B are weakly singular operators. Thus, the operator B is compact in the space $C(\partial D) \times C(\partial D) \times C(D)$ [7].

In the following theorem, we will denote that the $A+B$ operator is injective.

Theorem 2.3. The boundary value problem 11-15 has a unique solution.

Proof. Let us consider the problem 21-24. From the uniqueness theorem 2.1, if $f = g = 0$ then $u_0 = 0$ in $IR^2 \setminus \overline{D}$ and $u = 0$ in D. Since $u = \sqrt{n}w$, then $w = 0$ in D, where the functions u_0 and w are defined by equations 26 and 27, respectively. From the equation 39, we have $w - \psi_1 = 0$ for $x \in D$ and so $\psi_1 = 0$. Thus, the equations 37 and 38 reduce to

$$(K_0 - n_0 K)\psi + (1 + n_0)\psi + (S_0 - \sqrt{n_0} S)\phi = 0,$$

$$(T - T_0)\psi + \left(K_0^t - \frac{1}{\sqrt{n_0}} K^t \right)\phi - \left(1 + \frac{1}{\sqrt{n_0}} \right)\phi + \lambda k \left[(K_0 - K)\psi + \left(S_0 - \frac{1}{\sqrt{n_0}} S \right)\phi + 2\psi \right] = 0.$$

Using the jump relations of potential theory [16], we obtain

$$u_0^+ - u_0^- = \psi \qquad \frac{\partial u_0^+}{\partial \nu} - \frac{\partial u_0^-}{\partial \nu} = -\phi, \qquad \text{on } \partial D$$

$$w^+ - w^- = \sqrt{n_0}\psi \qquad \frac{\partial w^+}{\partial \nu} - \frac{\partial w^-}{\partial \nu} = -\phi, \qquad \text{on } \partial D.$$

Since $u_0 = 0$ in $IR^2 \setminus \overline{D}$ and $w = 0$ in D, then $u_0^+ = \frac{\partial u_0^+}{\partial \nu} = w^- = \frac{\partial w^-}{\partial \nu} = 0$. Thus, we have

$$u_0^- + \frac{1}{\sqrt{n_0}} w^+ = 0 \qquad \frac{\partial u_0^-}{\partial \nu} + \frac{\partial w^+}{\partial \nu} = 0, \qquad \text{on } \partial D \tag{41}$$

Since n_0 is real, from the Divergence theorem and equation 41, we have

$$\text{Im} \int_{\partial D} w^+ \frac{\overline{\partial w^+}}{\partial \nu} ds = \text{Im} \sqrt{n_0} \int_{\partial D} u_0^- \frac{\overline{\partial u_0^-}}{\partial \nu} ds = 0.$$

Since the function w is radiating solution of the Helmholtz equation for $x \in IR^2 \setminus \overline{D}$, from the Rellich's lemma, $w = 0$ in $IR^2 \setminus \overline{D}$ and so $\frac{\partial w}{\partial \nu} = 0$. Since $w^+ = \frac{\partial w^+}{\partial \nu} = 0$ on ∂D and from equation 41,

$u_0^- = \frac{\partial u_0^-}{\partial \nu} = 0$. Then we obtain $\psi = \phi = 0$. Since $\psi = \phi = \psi_1 = 0$, the $A + B$ operatör is injective [18]. Since B is compact and $A + B$ is injective, the inhomogeneous system 40 has a unique solution ψ, ϕ, ψ_1 from the fundamental results of the Riesz's theory for compact operators (see: Theorem 1.16, Corollary 1.17 and Corollary 1.20 in [7]). Finally the boundary-value problem 21-24 has a unique solution.

To formulate the linear method, firstly, we consider the interior boundary-value problem.

3. The Interior Boundary Value Problem

The interior boundary value problem is to find the functions $u_0, u \in C^2(D) \cap C^1(\overline{D})$ to the differential equations

$$\Delta u_0 + k^2 u_0 = 0, \quad \text{in } D \tag{42}$$

$$\nabla \cdot \left(\frac{1}{n} \nabla u \right) + k^2 u = 0, \quad \text{in } D \tag{43}$$

and the boundary conditions

$$u_0 - u = f, \quad \text{on } \partial D \tag{44}$$

$$\frac{\partial u_0}{\partial \nu} - \frac{1}{n_0} \frac{\partial u}{\partial \nu} + \lambda k \left(u_0 - \frac{1}{n_0} u \right) = g, \quad \text{on } \partial D. \tag{45}$$

Theorem 3.1 Let $D_0 = \{ x \in D : \operatorname{Im} n(x) > 0 \}$ be different from empty set. The solution of the interior boundary value problem 42–45 is unique.

Proof. Let $u_0, u \in C^2(D) \cap C^1(\overline{D})$ be the solution of the homogeneous interior boundary value problem, that is, assume $f = g = 0$. Then, applying of the Divergence theorem to the function $\overline{u} \left(\frac{1}{n} \nabla u \right)$ and using the condition (i) of 5 and homogeneous boundary conditions, we obtain

$$\int_D \left[\frac{1}{n} |\nabla u|^2 - k^2 |u|^2 \right] dy = \int_{\partial D} \overline{u} \frac{1}{n} \frac{\partial u}{\partial \nu} ds(y) = \int_{\partial D} \overline{u}_0 \frac{\partial u_0}{\partial \nu} ds(\mathrm{y}) + \int_{\partial D} \lambda k \left(1 - \frac{1}{n_0} \right) |u_0|^2 ds(y)$$

$$= \int_D \left[\frac{1}{\overline{n}} |\nabla u|^2 - k^2 |u|^2 \right] dy + \int_{\partial D} \left(1 - \frac{1}{n_0} \right) \left(\lambda - \overline{\lambda} \right) k |u_0|^2 ds(y).$$

Taking imaginary parts of this equation, we have

$$\operatorname{Im} \int_D \frac{1}{n} |\nabla u|^2 dy = \operatorname{Im} \int_D \frac{1}{\overline{n}} |\nabla u|^2 dy + \operatorname{Im} \int_{\partial D} \left(1 - \frac{1}{n_0} \right) \left(\lambda - \overline{\lambda} \right) k |u_0|^2 ds(y). \tag{46}$$

Since $\text{Im}\dfrac{1}{n} \le 0$ in D, the left-hand of equation 46 is negative or zero. Since $\text{Im}\dfrac{1}{\overline{n}} \ge 0$ in D and $\text{Im}\left(\lambda - \overline{\lambda}\right) \ge 0$ on ∂D, due to the condition (i) of 5, the right side of equation 46 is pozitive or zero. Thus, we get

$$\text{Im}\int_D \frac{1}{\overline{n}}\left|\nabla u\right|^2 dy = 0.$$

Since $\text{Im}\dfrac{1}{\overline{n}} > 0$ in $D_0 \subset D$, then $\nabla u = 0$. Since u satisfies the equation 43, then $u = 0$ in D. From the unique continuation principle, we have $u = \dfrac{\partial u}{\partial v} = 0$ on ∂D. Also since u_0 satisfies the equation 42, from the homogeneous boundary conditions and the Helmholtz representation, $u_0 = 0$ in D.

We will show the existence of the solution of the interior boundary value problem 42-45. Again, using the change of variables $u(x) = \sqrt{n(x)}w(x)$, the interior boundary value problem 42-45 takes form

$$\Delta u_0 + k^2 u_0 = 0, \qquad \text{in } D \tag{47}$$

$$\Delta w + (k^2 n + p)w = 0, \quad \text{in } D \tag{48}$$

$$u_0 - \sqrt{n_0}\,w = f, \quad \text{on } \partial D \tag{49}$$

$$\frac{\partial u_0}{\partial v} - \frac{1}{\sqrt{n_0}}\frac{\partial w}{\partial v} + \lambda k \left(u_0 - \frac{1}{\sqrt{n_0}}w\right) = g, \quad \text{on } \partial D \tag{50}$$

where the function p is defined by equation 25. Now, for $\psi, \phi \in C(\partial D)$ ve $\psi_1 \in C(D)$, we use the function $u_0(x)$ for $x \in IR^2 \setminus \partial D$ defined by equation 26 and let's define the following function,

$$w(x) = \int_{\partial D} \left[\sqrt{n_0}\,\frac{\partial \Phi(x,y)}{\partial v(y)}\psi(y) - \Phi(x,y)\phi(y)\right]ds(y) + \int_D \Phi(x,y)\rho(y)\,\psi_1(y)dy,$$

$$x \in IR^2 \setminus \partial D \tag{51}$$

where the functions Φ_0, Φ ve ρ are defined as Section 2.

Theorem 3.2. Let the functions u_0 and w defined by equations 26 and 51, respectively, are restricted to D.

Then the functions $\psi, \phi \in C(\partial D)$ and $\psi_1 \in C(D)$ satisfy the integral equations

$$(K_0 - n_0 K)\psi - (1 - n_0)\psi + (S_0 + \sqrt{n_0}S)\varphi - \sqrt{n_0}S_\rho\psi_1 = 2f , \quad \text{on} \ \partial D \tag{52}$$

$$(T_0 - T)\psi + \left(K_0' + \frac{1}{\sqrt{n_0}}K' \right)\phi + \left(1 + \frac{1}{\sqrt{n_0}} \right)\phi - \frac{1}{\sqrt{n_0}}K_\rho'\psi_1$$

$$+ \lambda k \left[(K_0 - K)\psi + \left(S_0 + \frac{1}{\sqrt{n_0}}S \right)\phi - \frac{1}{\sqrt{n_0}}S_\rho\psi_1 \right] = 2g , \ \text{on} \ \partial D \tag{53}$$

$$\sqrt{n_0}K^\wedge \psi - S^\wedge \phi + S_\rho^\wedge \psi_1 - 2\psi_1 = 0 , \quad in \ D. \tag{54}$$

To prove, the similar way as Theorem 2.2 can be done .

Theorem 3.3 The interior boundary value problem 42-45 has a unique solution.

 Proof. For the proof, we will examine the interior boundary value problem 47-50. From the uniqueness theorem 3.1, if $f = g = 0$ then $u_0 = w = 0$ in D. Since $\sqrt{n_0}K^\wedge \psi - S^\wedge \phi + S_\rho^\wedge \psi_1 = 2w$, from the equation 54, $\psi_1 = 0$. Thus, the equations 52 and 53 reduce to

$$(K_0 - n_0 K)\psi - (1 - n_0)\psi + (S_0 + \sqrt{n_0}S)\phi = 0$$

and

$$(T_0 - T)\psi + \left(K_0' + \frac{1}{\sqrt{n_0}}K' \right)\phi + \left(1 + \frac{1}{\sqrt{n_0}} \right)\phi + \lambda k \left[(K_0 - K)\psi + \left(S_0 + \frac{1}{\sqrt{n_0}}S\phi \right) \right] = 0 .$$

From the jump relations, we obtain

$$u_0^+ - u_0^- = \psi \qquad \frac{\partial u_0^+}{\partial v} - \frac{\partial u_0^-}{\partial v} = -\phi , \qquad \text{on} \ \partial D$$

$$w^+ - w^- = \sqrt{n_0}\psi \qquad \frac{\partial w^+}{\partial v} - \frac{\partial w^-}{\partial v} = \phi , \qquad \text{on} \ \partial D.$$

Since $u_0 = w = 0$ in D, then $u_0^- = \dfrac{\partial u_0^-}{\partial v} = w^- = \dfrac{\partial w^-}{\partial v} = 0$. Thus, we have

$$u_0^+ - \frac{1}{\sqrt{n_0}}w^+ = 0 \qquad \frac{\partial u_0^+}{\partial v} + \frac{\partial w^+}{\partial v} = 0, \qquad \text{on} \ \partial D.$$

Thanks to n_0 positive real constant in condition (i) of 5, we obtain

$$\operatorname{Im} \int_{\partial D} u_0^+ \frac{\partial \overline{u_0^+}}{\partial \nu} ds = -\frac{1}{\sqrt{n_0}} \operatorname{Im} \int_{\partial D} w^+ \frac{\partial \overline{w^+}}{\partial \nu} ds.$$ (55)

The two integrals in equation 55 is pozitive or zero. Since u_0 and w are radiating solution of the Helmholtz equation for $x \in IR^2 \setminus \overline{D}$, from the Rellich's lemma, we have either $u_0 = 0$ or $w = 0$. Thus we have either $u_0^+ = \frac{\partial u_0^+}{\partial \nu} = 0$ or $w^+ = \frac{\partial w^+}{\partial \nu} = 0$ on ∂D. Then, we obtain $\psi = \phi = 0$. Thus, the existence of the solution of the interior boundary value problem 47-50 is obtained from the fundamental results of the Riesz's theory.

4. The Linear Method for The Inverse Scattering Problem

We will formulate the linear method for the solution of the inverse scattering problem defined by the boundary value problem 6-10. This problem is associated with the determine the support $\overline{D}$ of $n(x) - n_0$ from the information about the far-field pattern $u_\infty(\hat{x}, d)$ in the section 1. For $\forall \varepsilon > 0$, there exists a solution $g_y \in L^2(\Omega)$ such that

$$\left\| \int_\Omega u_\infty(\hat{x}, \mathrm{d}) g_y(\mathrm{d}) ds(\mathrm{d}) - \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}.y} \right\|_{L^2(\Omega)} < \varepsilon \quad \text{for } y \in D.$$

When $y \rightarrow \partial D$, both $\|g_y\|_{L^2(\Omega)}$ and $\|v_g\|_{L^2(D)}$ become unbounded [10,11].

First of all, we shall form the integral equation for the linear method. We will come up with a basic solution that provides equation 48. Let be $\Omega_\varepsilon = \{ y : |x - y| \le \varepsilon \} \subset D$. We take the integral

$$I(x, z) = \int_D \Phi(x, y) m(y) \Gamma(y, z) dy, \qquad z \in IR^2$$ (56)

where $\Gamma \in C^2(D) \cap C^1(\overline{D})$. Let $I(x, z)$ be the solution of equation 48 and

$$m(y) = -n_0 + n(y) + \frac{p(y)}{k^2}$$ (57)

Let be

$$I(x,z) = \int_{\Omega_\varepsilon} \Phi(x,y)m(y)\Gamma(y,z)dy + \int_{D\backslash\Omega_\varepsilon} \Phi(x,y)m(y)\Gamma(y,z)dy \quad = I_1(x,z) + I_2(x,z).$$

Since $\Delta\Phi(x,y) + k^2 n_0 \Phi(x,y) = 0$ for $x \neq y$, then we get $(\Delta + k^2 n_0) I_2(x,z) = 0$. Hence

$$(\Delta + k^2 n_0) I(x,z) = (\Delta + k^2 n_0) I_1(x,z)$$

$$= \Delta \int_{\Omega_\varepsilon} \Phi(x,y)m(y)\Gamma(y,z)dy + k^2 n_0 \int_{\Omega_\varepsilon} \Phi(x,y)m(y)\Gamma(y,z)dy. \quad (58)$$

Applying the divergence theorem to the first integral on the right-hand of equation 58, we get

$$\int_{\Omega_\varepsilon} \Delta\Phi(x,y)m(y)\Gamma(y,z)dy = -\int_{\partial\Omega_\varepsilon} \nabla_x\Phi(x,y)v(y)m(y)\Gamma(y,z)ds(y), \qquad \nabla_x = -\nabla_y$$

$$= \int_{\partial\Omega_\varepsilon} \nabla_y\left[\frac{i}{4}H_0^{(1)}\left(k\sqrt{n_0}\,|x-y|\right)\right]v(y)m(y)\Gamma(y,z)ds(y)$$

$$= \int_0^{2\pi} -\frac{\partial}{\partial\varepsilon}\frac{i}{4}H_0^{(1)}\left(k\sqrt{n_0}\,\varepsilon\right)m(x+\varepsilon\theta)\Gamma(x+\varepsilon\theta,z)\varepsilon d\theta$$

$$= \int_0^{2\pi} \frac{ik\sqrt{n_0}}{4}H_1^{(1)}\left(k\sqrt{n_0}\,\varepsilon\right)m(x+\varepsilon\theta)\Gamma(x+\varepsilon\theta,z)\varepsilon d\theta.$$

Thus, from $\lim_{\varepsilon\to 0}\sqrt{n_0}\,\varepsilon H_1^{(1)}\left(k\sqrt{n_0}\,\varepsilon\right) = -\frac{i}{\pi}\left(\frac{2}{k}\right)$ given in [20], we have

$$\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \Delta\Phi(x,y)m(y)\Gamma(y,z)dy = m(x)\Gamma(x,z). \qquad (59)$$

Applying the mean value theorem [1] to the second integral on the right-hand of equation 58, we get

$$\int_{\Omega_\varepsilon} \Phi(x,y)m(y)\Gamma(y,z)dy = \Phi(x,a)m(a)\Gamma(a,z)\left|\int_{\Omega_\varepsilon} dy\right|, \qquad 0 < x-a < \varepsilon$$

$$= \frac{i\pi\varepsilon^2}{4}H_0^{(1)}\left(k\sqrt{n_0}\,|x-a|\right)m(a)\Gamma(a,z).$$

Therefore

$$\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} \Phi(x,y) m(y) \Gamma(y,z) dy = 0 \tag{60}$$

From equations 59, 60 and function 57, the equation 58 takes the form

$$\left(\Delta + k^2 \left[n(x) + \frac{p(x)}{k^2} - m(x) \right] \right) I(x,z) = m(x) \Gamma(x,z) .$$

Since $I(x,z)$ satisfies the equation 48 and $\Phi(x,y)$ is a solution of the Helmholtz equation,

$$\Gamma(x,z) = \Phi(x,z) - k^2 I(x,z)$$

satisfies the equation 48. If we write the integral 56 in the last equation, then we obtain the Lippmann Schwinger integral equation [16]

$$\Gamma(x,z) = \Phi(x,z) + \int_D \Phi(x,y) \left[k^2 n_0 - \left(k^2 n(y) + p(y) \right) \right] \Gamma(y,z) dy .$$

Thus $\Gamma(x,z)$ is a basic solution for the equation 48. From the Theorem 8.3 given in [16], the solution of $\Gamma(x,z)$ is a solution of the following problem

$$\Delta w + \left(k^2 n + p \right) w = 0, \qquad x \in IR^2 \tag{61}$$

$$w(x) = \Phi(x,z) + w^s(x) \tag{62}$$

$$\lim_{r \to \infty} \sqrt{r} \left(\frac{\partial w^s}{\partial r} - ikw^s \right) = 0. \tag{63}$$

With the change of variables $u(x) = \sqrt{n(x)} w(x)$, the problem 61-63 is takes form

$$\nabla \cdot \left(\frac{1}{n} \nabla u \right) + k^2 u = 0, \qquad x \in IR^2 \tag{64}$$

$$u(x) = \sqrt{n(x)} \Phi(x,z) + u^s(x), \tag{65}$$

where $u^s(x)$ satisfies the Sommerfeld radiation condition 10.

From the Theorem 8.7 given in [16] and the condition (ii) of 5, the problem 64-65 has at most one solution. Thus, the orijinal problem 61-63 has at most one solution and the Fredholm alternative [18] guarantee the existence of a fundamental solution for the equation 48.

Secondly, we will give the following lemma.

Lemma 4.1. Let D be a bounded domain with $C^2(\partial D)$, $x^* \in \partial D$ and $B_R = \left\{ x \in IR^2 : |x - x^*| \le R \right\}$ If the function $u \in C^2(D) \cap C^1(\overline{D})$ is the solution of the following equation

$$\nabla . \left(\frac{1}{n} \nabla u \right) + k^2 u = 0 \qquad \text{in } D, \tag{66}$$

then there exists a constant $C > 0$ such that

$$\|u\|_{C(\partial D)} \le C \left(\left\| \frac{\partial u}{\partial v} \right\|_{C(\partial D)} + \|u\|_{C(\partial D \setminus B_R)} \right). \tag{67}$$

Proof. The proof can be done in the similar way to proof of Lemma 4.4 given in [19]. Let $\lambda \in C^{0,\alpha}(\partial D)$ be positive function with support $\partial D \setminus B_R$. Now, we will show that any solution of equation (66) satisfying the boundary condition

$$\frac{\partial u}{\partial v} - \lambda k u = g \tag{68}$$

must vanish identically in D. We suppose that the solution of the problem 66 and 68 is not unique i.e. let $u = u_1 - u_2$. Thus the function u satisfy the homogeneous boundary condition

$$\frac{\partial u}{\partial v} - \lambda k u = 0. \tag{69}$$

We take the homogeneous problem 66 and 69. By applying the divergence theorem to the function $\overline{u} \left(\frac{1}{n} \nabla u \right)$ and then taking imaginary parts, we have

$$\text{Im} \int_D \left[-k^2 |u|^2 + \frac{1}{n} |\nabla u|^2 \right] dy = \text{Im} \frac{1}{n_0} \int_{\partial D} \lambda k |u|^2 \, ds(y). \tag{70}$$

From $\operatorname{Im} \dfrac{1}{n} \leq 0$, the left-hand of equation 70 is negatif or zero. From $\operatorname{Im}\lambda \geq 0$ and $n_0 \in IR$ on ∂D, the right-hand of equation 70 is positive or zero. Moreover, since $\lambda \neq 0$, $u = 0$ on $\partial D \setminus B_R$, the boundary condition 69 implies that $\dfrac{\partial u}{\partial \nu} = 0$. From the unique continuation principle, we obtain that $u = 0$ in D. Thus, the problem 66 and 68 has at most one solution.

To show existence of the solution of the boundary value problem 66 and 68, we use the inverse operator's existence theorem [18]. Firstly, we define the function

$$\Phi_1(x, y) = \sqrt{\dfrac{n(x)}{n_0}}\, \Gamma(x, y)$$

and let this function be the fundamental solution to equation 66. With the function Φ in the operators S and K^t which were defined in the operators 29 and 31 replaced by Φ_1. Therefore, for $\phi \in C(\partial D)$, we define the function

$$u(x) = \int_{\partial D} \Phi_1(x, y)\phi(y)\, ds(y), \qquad x \in IR^2 \setminus \partial D.$$

The function u restricted to D solves the problem 66 and 68. The function ϕ satisfies the integral equation

$$K^t \phi + \phi - \lambda k S \phi = 2g \quad \text{on } \partial D. \tag{71}$$

This integral equation is obtained from the jump relations and the boundary condition 68. If $g = 0$, since $u = 0$ in D, from the unique continuation principle, then $u^- = \dfrac{\partial u^-}{\partial \nu} = 0$. From the continuity of the single-layer potential and the uniqueness of the solution of the exterior Dirichlet problem given in [16], we have that $u^- = u^+ = \dfrac{\partial u^+}{\partial \nu} = 0$. From the jump relations, we obtain that $\dfrac{\partial u^+}{\partial \nu} - \dfrac{\partial u^-}{\partial \nu} = -\phi$. Thus, $\phi = 0$. This ensures the existence of the solution. That is, since the homogeneous equation $\left(I + K^t - \lambda k S\right)\phi = 0$ has to the solution $\phi = 0$, the operatör $I + K^t - \lambda k S$ is injective. Thus, from the inverse operator's existence theorem, the inhomogeneous equation 71 for all $g \in C(\partial D)$ has a unique solution and the solution depends continuously on the function g. Since the inverse operator $\left(I + K^t - \lambda k S\right)^{-1} : C(\partial D) \to C(\partial D)$ exists and bounded, then the constant $C_1 > 0$ exists such that

$$\|u\|_{C(\partial D)} \leq C_1 \|g\|_{C(\partial D)}. \tag{72}$$

From the boundary condition 68 and since the function λ is support $\partial D \setminus B_R$, then

$$\|g\|_{C(\partial D)} = \left\|\frac{\partial u}{\partial \nu} - \lambda k u\right\|_{C(\partial D)} \leq \left\|\frac{\partial u}{\partial \nu}\right\|_{C(\partial D)} + c\|u\|_{C(\partial D \setminus B_R)}, \qquad c > 0.$$

Writing the last inequality in the inequality 72, we get the inequality 67.

Teorem 4.2. If the sequences $u_{0,j}$ and u_j are solutions of the interior boundary value problem

$$\Delta u_{0,j} + k^2 u_{0,j} = 0, \qquad \text{in } D \tag{73}$$

$$\nabla.\left(\frac{1}{n}\nabla u_j\right) + k^2 u_j = 0, \qquad \text{in } D \tag{74}$$

$$u_{0,j} - u_j = -\Phi_0\left(.,y_j\right), \qquad \text{on } \partial D \tag{75}$$

$$\frac{\partial u_{0,j}}{\partial \nu} - \frac{1}{n_0}\frac{\partial u_j}{\partial \nu} + \lambda k\left(u_{0,j} - \frac{1}{n_0}u_j\right) = -\frac{\partial \Phi_0\left(.,y_j\right)}{\partial \nu} - \lambda k \Phi_0\left(.,y_j\right), \qquad \text{on } \partial D \tag{76}$$

Then

$$\lim_{j\to\infty}\|u_{0,j}\|_{C^1(\partial D)} = \infty, \tag{77}$$

where the sequences y_j are defined by

$$y_j = y^* - \frac{R}{j}v\left(y^*\right) \tag{78}$$

for $R > 0$ is sufficently small and y^* is a point on ∂D.

Proof. We assume that there exists a positive constant c_1 such that

$$\|u_{0,j}\|_{C^1(\partial D)} \leq c_1, \qquad j \to \infty \tag{79}$$

For $R > 0$ sufficently small and $y^* \in \partial D$, we take the set of points in $IR^2 \setminus \overline{D}$ defined with the sequences

$$z_j = y^* + \frac{R}{j}v\left(y^*\right).$$

Let's define the sequence

$$u_j = u_j + \sqrt{n}\Gamma(.,z_j) \quad \text{in } D. \tag{80}$$

From the boundary condition 75 and the sequence 80, we obtain

$$u_{0,j} - u_j = -\left[\Phi_0(.,y_j) + \sqrt{n_0}\Gamma(.,z_j)\right] \quad \text{on } \partial D. \tag{81}$$

Again from the boundary condition 76 and the derivative of the sequence 80 in the direction ν, we obtain

$$\frac{\partial u_{0,j}}{\partial \nu} - \frac{1}{n_0}\frac{\partial u_j}{\partial \nu} + \lambda k\left(u_{0,j} - \frac{1}{n_0}u_j\right) = -\left[\frac{\partial \Phi_0(.,y_j)}{\partial \nu} + \frac{1}{\sqrt{n_0}}\frac{\partial \Gamma(.,z_j)}{\partial \nu}\right]$$

$$-\lambda k\left[\Phi_0(.,y_j) + \frac{1}{\sqrt{n_0}}\Gamma(.,z_j)\right] \quad \text{on } \partial D. \tag{82}$$

The right-hand of equations 81 and 82 are defined, respectively, by the sequences

$$f_j = \Phi_0(.,y_j) + \sqrt{n_0}\Gamma(.,z_j) \quad \text{on } \partial D,$$

$$g_j = \frac{\partial \Phi_0(.,y_j)}{\partial \nu} + \frac{1}{\sqrt{n_0}}\frac{\partial \Gamma(.,z_j)}{\partial \nu} + \lambda k\left[\Phi_0(.,y_j) + \frac{1}{\sqrt{n_0}}\Gamma(.,z_j)\right] \quad \text{on } \partial D.$$

Let the disk B_r and λ be as defined as the Lemma 4.1. Then there exists a constant $c_2 > 0$ such that

$$\left\|f_j\right\|_{C(\partial D \setminus B_R)} \le \sup_{.\in \partial D \setminus B_R}\left|\Phi_0(.,y_j)\right| + \sup_{.\in \partial D \setminus B_R}\left|\sqrt{n_0}\Gamma(.,z_j)\right| \le c_2. \tag{83}$$

The norm of sequence g_j is given by the following inequality

$$\left\|g_j\right\|_{C(\partial D)} \le \left\|\frac{\partial \Phi_0(.,y_j)}{\partial \nu} + \frac{1}{\sqrt{n_0}}\frac{\partial \Gamma(.,z_j)}{\partial \nu}\right\|_{C(\partial D)} + \left\|\lambda k\left[\Phi_0(.,y_j) + \frac{1}{\sqrt{n_0}}\Gamma(.,z_j)\right]\right\|_{C(\partial D)}.$$

Taking the first norm on the right-hand of the above inequation and using as in the proof of Lemma 4.2 [8], there exists a constant $c_3 > 0$ such that

$$\left\|\frac{\partial \Phi_0(.,y_j)}{\partial \nu} + \frac{1}{\sqrt{n_0}}\frac{\partial \Gamma(.,z_j)}{\partial \nu}\right\|_{C(\partial D)} \le c_3.$$

Thus, the λ with support $\partial D \setminus B_R$, there exists a constant $c_4 > 0$ such that

$$\left\| g_j \right\|_{C(\partial D)} \leq c_3 + \sup_{.\in \partial D \setminus B_R} \left| \lambda k \left[\Phi_0(.,y_j) + \frac{1}{\sqrt{n_0}} \Gamma(.,z_j) \right] \right| \leq c_4 . \tag{84}$$

From the Lemma 4.1, there exists a constant $c_5 > 0$ such that

$$\left\| u_j \right\|_{C(\partial D)} \leq c_5 \left(\left\| u_j \right\|_{C(\partial D \setminus B_R)} + \left\| \frac{\partial u_j}{\partial v} \right\|_{C(\partial D)} \right) . \tag{85}$$

From the boundary condition 75, the assumption 79 and the inequality 83, we obtain

$$\left\| u_j \right\|_{C(\partial D \setminus B_R)} \leq \left\| u_{0,j} \right\|_{C(\partial D \setminus B_R)} + \left\| f_j \right\|_{C(\partial D \setminus B_R)} \leq c_1 + c_2 . \tag{86}$$

From the boundary conditions 75, 76, the equation 81, the assumption 79, the inequalities 84 and 86, there exists a constant $c_6 > 0$ such that

$$\left\| \frac{\partial u_j}{\partial v} \right\|_{C(\partial D)} \leq \left| n_0 \right| \left\| \frac{\partial u_{0,j}}{\partial v} + \lambda k u_{0,j} \right\|_{C(\partial D)} + \left\| -\lambda k u_j \right\|_{C(\partial D)} + \left| n_0 \right| \left\| g_j \right\|_{C(\partial D)}$$

$$\leq \left| n_0 \right| \left[\left\| \frac{\partial u_{0,j}}{\partial v} \right\|_{C(\partial D)} + \left\| \lambda k u_{0,j} \right\|_{C(\partial D \setminus B_R)} \right] + \left\| -\lambda k u_j \right\|_{C(\partial D \setminus B_R)} + \left| n_0 \right| c_4 \leq c_6 . \tag{87}$$

When we write the inequalities 86 and 87 in the inequality 85, we obtain the following inequality

$$\left\| u_j \right\|_{C(\partial D)} \leq c_7 \quad , \quad c_7 > 0 . \tag{88}$$

For the sequence f_j, we have

$$\left\| f_j \right\|_{C(\partial D)} = \left\| \Phi_0(.,y_j) + \sqrt{n_0} \Gamma(.,z_j) \right\|_{C(\partial D)} = \left\| u_j - u_{0,j} \right\|_{C(\partial D)} \leq \left\| u_j \right\|_{C(\partial D)} + \left| -1 \right| \left\| u_{0,j} \right\|_{C(\partial D)} .$$

From the assumption 79 and the inequality 88, $\left\| f_j \right\|_{C(\partial D)}$ is bounded which is a contradiction. Because f_j is nondefined in $\partial D \cap B_R$ and bounded according to the norm on $C(\partial D \setminus B_R)$. Therefore, $\left\| u_{0,j} \right\|_{C^1(\partial D)}$ is unbounded as $j \to \infty$.

To formulate the linear method for the solution of the inverse scattering problem, we will benefit from the information about the far-field model $u_\infty(\hat{x}, d)$, where $\hat{x} = \dfrac{x}{|x|}$ and d are unit vector on the unit circle Ω. Recall that for this end, the Herglotz wave function in the form

$$v_g(x) = \int_\Omega e^{ikx.d} g(d) ds(d) \tag{89}$$

is a solution of the Helmholtz equation , where $g \in L^2(\Omega)$ is the kernel of v_g. Our aim is to show that there exists a function $g = g(., y_j) \in L_2(\Omega)$ such that

$$\left\| \int_\Omega u_\infty(\hat{x}, d) g(d) ds(d) - \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}.y_j} \right\|_{L^2(\Omega)} < \varepsilon \qquad \text{for } \forall \varepsilon > 0,$$

where $y_j \in D$ is defined by sequence 78. We will also show that it is $\lim\limits_{j\to\infty} \left\| g(., y_j) \right\|_{L^2(\Omega)} = \infty$. Thus, the boundary of D is characterized by points where the norm $\left\| g(., y_j) \right\|_{L^2(\Omega)}$ is unlimited.

Theorem 4.3 There exists $g = g(., y_j) \in L^2(\Omega)$ such that

$$\left\| \int_\Omega u_\infty(\hat{x}, d) g(d) ds(d) - \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}.y_j} \right\|_{L^2(\Omega)} < \varepsilon, \qquad \text{for } \forall \varepsilon > 0 \tag{90}$$

and $\lim\limits_{j\to\infty} \left\| g(., y_j) \right\|_{L^2(\Omega)} = \infty$. Moreover, if v_g is the Herglotz wave function defined by function 89, then $\lim\limits_{j\to\infty} \left\| v_g(., y_j) \right\|_{L^2(D)} = \infty$.

Proof. From Theorem 3.3, the interior boundary value problem 73-76 has a solution which is not generally a Herglotz wave function. However, a Herglotz wave function $U_{0,j}$ with kernel g is shown to exist and this function approaches $u_{0,j}$ in $C^1(\overline{D})$ given in [8,11]. Let u_0 be show the total field, solving the original exterior boundary value problem 6-10, and the functions u_0^* and u^* be defined $u_0^*(y) = u_0(y, -\hat{x})$ and $u^*(y) = u(y, -\hat{x})$, respectively. From the reciprocity relation [16] and the far-field pattern $u_\infty(\hat{x}, d)$, we obtain

$$\int_\Omega u_\infty(\hat{x},\mathrm{d})g(\mathrm{d})ds(\mathrm{d}) = \int_\Omega u_\infty(-\mathrm{d},-\hat{x})g(\mathrm{d})ds(\mathrm{d})$$

$$= \int_\Omega \frac{e^{i\pi/4}}{\sqrt{8\pi k}}\left\{\int_{\partial D}\left[u_0^*(y)\frac{\partial e^{-ik\mathrm{d}.y}}{\partial v} - e^{-ik\mathrm{d}.y}\frac{\partial u_0^*}{\partial v}(y)\right]ds(y)\right\}g(\mathrm{d})ds(\mathrm{d})$$

$$= \frac{e^{i\pi/4}}{\sqrt{8\pi k}}\int_{\partial D}\left[u_0^*(y)\frac{\partial U_{0,j}}{\partial v}(y) - \frac{\partial u_0^*}{\partial v}(y)U_{0,j}(y)\right]ds(y). \qquad (91)$$

Since $U_{0,j} \approx u_{0,j}$ in $C^1\left(\overline{D}\right)$, the integral on the right-hand of equation 91 become

$$\int_{\partial D}\left[u_0^*(y)\frac{\partial U_{0,j}}{\partial v}(y) - \frac{\partial u_0^*}{\partial v}(y)U_{0,j}(y)\right]ds(y) \approx \int_{\partial D}\left[u_0^*(y)\frac{\partial u_{0,j}}{\partial v}(y) - \frac{\partial u_0^*}{\partial v}(y)u_{0,j}(y)\right]ds(y).$$

Applying the conditions 75-76 and then the conditions 8-9 , respectively, the last equation is in the form below

$$\int_{\partial D}\left(u_0^*(y)\frac{\partial U_{0,j}}{\partial v}(y) - \frac{\partial u_0^*}{\partial v}(y)U_{0,j}(y)\right)ds(y) \approx \frac{1}{n_0}\int_{\partial D}\left[u^*(y)\frac{\partial u_j}{\partial v}(y) - \frac{\partial u^*}{\partial v}(y)u_j(y)\right]ds(y)$$

$$- \int_{\partial D}\left[u_0^*(y)\frac{\partial \Phi_0}{\partial v}(y,y_j) - \frac{\partial u_0^*}{\partial v}(y)\Phi_0(y,y_j)\right]ds(y). \qquad (92)$$

Let's apply the Divergens theorem to the first integral on the right hand of equation 92. We get

$$\frac{1}{n_0}\int_{\partial D}\left[u^*(y)\frac{\partial u_j}{\partial v}(y) - \frac{\partial u^*}{\partial v}(y)u_j(y)\right]ds(y)$$

$$= \int_{\partial D}\left[u(y,-\hat{x})\frac{1}{n_0}\nabla u_j(y)v(y) - u_j(y)\frac{1}{n_0}\nabla u(y,-\hat{x})v(y)\right]ds(y)$$

$$= \int_D\left[div\left[u(y,-\hat{x})\frac{1}{n(y)}\nabla u_j(y)\right] - div\left[u_j(y)\frac{1}{n(y)}\nabla u(y,-\hat{x})\right]\right]dy$$

$$= \int_D\left[u(y,-\hat{x})\left(-k^2 u_j(y)\right) - u_j(y)\left(-k^2 u(y,-\hat{x})\right)\right]dy = 0. \qquad (93)$$

From the Helmholtz representation and the Green's formula, the last integral in the right-hand of equation 92 is

$$\int_{\partial D}\left[u_0(y,-\hat{x})\frac{\partial \Phi_0}{\partial v}(y,y_j) - \frac{\partial u_0}{\partial v}(y,-\hat{x})\Phi_0(y,y_j)\right]ds(y) = -u_0(y_j,-\hat{x}) = e^{-ik\hat{x}.y_j}. \qquad (94)$$

From the equations 93 and 94, the equation 92 is in the form below

$$\int_{\partial D} \left[u_0^*(y) \frac{\partial U_{0,j}}{\partial \nu}(y) - \frac{\partial u_0^*}{\partial \nu}(y) U_{0,j}(y) \right] ds(y) \approx e^{-ik\hat{x}.y_j} \tag{95}$$

When we write the equation 95 in the equation 91, we get

$$\int_{\Omega} u_\infty(\hat{x},\mathrm{d}) g(\mathrm{d}) ds(\mathrm{d}) \approx \frac{e^{i\pi/4}}{\sqrt{8\pi k}} e^{-ik\hat{x}.y_j} .$$

Hence there is a function $g \in L^2(\Omega)$ that satisfies the equation 90. We assume that $\left\| g(.,y_j) \right\|_{L^2(\Omega)}$ is bounded as $j \to \infty$. Hence $\left\| U_{0,j} \right\|_{C^1(D)}$ is bounded. This implies that $\left\| u_{0,j} \right\|_{C^1(D)}$ is bounded as $j \to \infty$. This result is contradict with the Theorem 4.2. Thus, the theorem is proved.

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