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ON THE SENSITIVITY OF THE TWO-POINT BOUNDARY VALUE PROBLEM FOR A LINEAR DIFFERENCE EQUATIONS SYSTEM

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ABSTRACT

This paper presents the sensitivity notion of the solution of the homogeneous two-point boundary value problem for the systems of the linear difference equations of the form

$$\begin{cases} x(n+1) = Ax(n) \\ Lx(n_0) = \varphi ; Rx(n_1) = \psi ; \{n : n, n_0, n_1 \in Z, n_0 \le n \le n_1\}, \end{cases}$$

where A NxN matrix, L kxN matrix and R (N-k)xN matrix are real matrices, φ and ψ are real column vectors of N and N-k orderly.

Key Words: Difference equations, two-point boundary value problem, sensitivity

LİNEER FARK DENKLEMLER SİSTEMİ İÇİN İKİ-NOKTA SINIR DEĞER PROBLEMİNİN HASSASİYETİ ÜZERİNE

ÖZET

Bu çalışma lineer fark denklemler sistemi için

$$\begin{cases} x(n+1) = Ax(n) \\ Lx(n_0) = \varphi ; Rx(n_1) = \psi ; \{n : n, n_0, n_1 \in Z, n_0 \le n \le n_1\}, \end{cases}$$

biçimindeki homogen iki-nokta sınır değer probleminin çözümünün hassasiyeti kavramını vermektedir. Burada A, L ve R matrisleri sırasıyla Nx N, kxN ve (N-k)xN tipinde reel matrisler, φ ve ψ sırasıyla N ve N-k bileşenli reel sütun vektörleridir.

Anahtar Kelimeler: Fark denklemleri, iki-nokta sınır değer problemi, hassasiyet AMS subject classification 39 A 99

1. INTRODUCTION

The theory of difference equations is a lot richer than the corresponding theory of differential equations. Many authors have been studied on difference equations and some problems related with them, such as stability theory (1), existence and uniqueness theorem (2,3), transmission of information (4), signal processing, oscillation (5), control and dynamic systems (6,7), etc. (see also 8). Knowledge about the accuracy in the computation of the solution components is important (9). First-order sensitivity analysis involves examination of the effects of differential variations in the fixed coefficients or boundary conditions of a mathematical model. Sensitivity calculations may be required for gradient evaluation in optimizations, in experimental design and analysis, and in many phases of chemical process design (10). Our aim in this article is to present sensitivity notion about the TPBVP. The system of difference equations x(n+1) = Ax(n) has N-linearly independent solutions. Here A is NxN nonsingular matrix and $x(n)=(x_1(n),x_2(n),...x_N(n))^T$ is N column vector. The matrix obtained by using these independent solutions as its columns is called fundamental matrix of the system and denoted by $\psi(A, n) = (\psi_1, \psi_2, ..., \psi_n)$ (1,2,6).

2. TWO-POINT BOUNDARY VALUE PROBLEMS FOR THE SYSTEM OF LINEAR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

We consider the system of difference equations

$$x(n+1) = Ax(n).$$

If for some $n_0 \geq 0$, $x(n_0) = x_0$ is specified, then the considered system with this condition is called as an initial value problem. If the matrix A is a N×N real nonsingular matrix then this problem has

$$x(n) = A^{n-n_0} x(n_0) \quad [2.1]$$

the unique solution. Now we consider the initial value problem about the system of the nonhomogeneous difference equations

$$x(n+1) = Ax(n) + f(n),\ x(n_0)=x_0.$$

Therefore the solution of this system is

$$x(n) = A^{n-n_0} x(n_0) + \sum_{k=n_0}^{n-1} A^{n-k-1} f(k) \quad [2.2]$$

(3,11). We give the following two definitions and a lemma for the sake of use in the sequel:

Definition 2.1. Let A NxN matrix, L kxN matrix and R (N-k)xN matrix are real matrices, φ, ψ and f(n) are real column vectors with N, (N-k) and N elements orderly. Then the following problem is called as a two-point boundary value problem for the system of the linear difference equations with constant coefficients (2):

$$\begin{cases} x(n+1) = Ax(n) + f(n) \\ Lx(n_0) = \varphi;\ Rx(n_1) = \psi;\ \{n : n, n_0, n_1 \in Z, n_0 \leq n \leq n_1\} \end{cases} \quad [2.3]$$

Definition 2.2. For an NxN nonsingular matrix A the number $\mu(A) = \|A\|\|A^{-1}\|$ is called as the condition number of A, where $\|A\|$ denotes matrix norm of A (9,12).

Lemma 2.1. Let A and B are real NxN matrices and let $\|B\| < \|A\|$ then the following inequality is satisfied:

$$\left\|(A+B)^n - A^n\right\| \le \frac{\|B\|.\|A\|^{(n-1)}}{1-e^{\frac{\|B\|}{\|A\|}}} \qquad [2.4]$$

Proof. To show this inequality we need to consider the following initial-value problem :

$$\begin{aligned} X(n+1) &= AX(n)+BX(n) \\ X(0) &= I \end{aligned} \qquad [2.5]$$

With the help of [2.1] the unique solution of this problem is

$$X(n) = (A+B)^n \qquad [2.6]$$

If we consider this problem as a nonhomogeneous system problem then with the help of [2.2] the unique solution of this problem is

$$X(n) = A^n + \sum_{k=0}^{n-1} A^{n-k-1}BX(k) \qquad [2.7]$$

Since the [2.5] problem has only one solution we can write the following equality of

$$(A+B)^n - A^n = \sum_{k=0}^{n-1} A^{n-k-1}B(A+B)^k$$

By using this equality we continue as follows:

$$\begin{aligned} \left\|(A+B)^n - A^n\right\| &\le \|B\| \sum_{k=0}^{n-1}\left\|A^{n-k-1}\right\| \left\|(A+B)^k\right\| \\ &\le \|B\| \sum_{k=0}^{n-1}\|A\|^{n-k-1}(\|A\|+\|B\|)^k \\ &\le \|B\| \sum_{k=0}^{n-1}\|A\|^{n-1}\left(1+\frac{\|B\|}{\|A\|}\right)^k \\ &\le \|B\|\,\|A\|^{n-1}\sum_{k=0}^{n-1} e^{k\frac{\|B\|}{\|A\|}} \\ &\le \|B\|\,\|A\|^{n-1}\sum_{k=0}^{\infty} e^{k\frac{\|B\|}{\|A\|}} \\ &= \|B\|\,\|A\|^{n-1}\frac{1}{1-e^{\frac{\|B\|}{\|A\|}}} \end{aligned}$$

$$\left\|(A+B)^n - A^n\right\| \le \frac{\|B\|.\|A\|^{(n-1)}}{1-e^{\frac{\|B\|}{\|A\|}}}$$

This completes the proof.

2.1. Existing and Uniqueness Theorems For The TPBVP

Theorem 2.1. We consider the following two-point boundary value problem;

$$\begin{cases} x(n+1) = Ax(n) \\ Lx(n_0) = \varphi\,;\, Rx(n_1) = \psi;\, \{n : n, n_0, n_1 \in Z, n_0 \le n \le n_1\} \end{cases}$$

Then the following two cases can be met:

Case 1. If A is a nonsingular matrix and ψ (A,n) is the fundamental matrix of ,

$x(n+1) = Ax(n)$, $n = 0, \pm1, \pm2, \cdots$.

Furthermore let W be given as $W = \Psi(A, n_1 - n_0)\Psi^{-1}(0)$.

Case 2. If A is a singular matrix then there exists an orthogonal matrix U such that

$$U^* AU = \begin{pmatrix} A_1 & B \\ 0 & A_0 \end{pmatrix}$$

is satisfied. Where A_0 is $N_0 \times N_0$ real matrix having all eigenvalues as 0 and det $A_1 \neq 0$. $\Psi(A_1, n)$ is the fundamental matrix of $x(n+1) = A_1 x(n)$. Let W be given as

$$W = \Psi(A, n_1 - n_0)\left(\Psi^{-1}(A_1, 0), 0\right)U^* .$$

For these two cases we can define the matrix H as

$$H = \begin{pmatrix} L \\ RW \end{pmatrix}.$$

If the matrix H is a nonsingular matrix, then the problem has a unique solution (2).

Proof.

Case 1. By the hypothesis, A is a nonsingular matrix. Since ψ (A,n) is the fundamental matrix then the general solution of the given $x(n+1) = Ax(n)$ difference system can be written as

$$x(n) = \Psi(A, n - m)\Psi^{-1}(A, 0)x(m).$$

Therefore

$$x(n_1) = \Psi(A, n_1 - n_0)\Psi^{-1}(A, 0)x(n_0);$$

is satisfied. For the given two boundary conditions in the theorem

$$Lx(n_0) = \varphi, R\Psi(A, n_1 - n_0)\Psi^{-1}(A, 0)x(n_0) = \psi,$$

should be verified. So by using $W = \Psi(A, n_1 - n_0)\Psi^{-1}(A, 0)$ we can see that $\det H \neq 0$ is true. Therefore the problem has a unique solution in this case.

Case 2. By the hypothesis, A is a singular matrix. Therefore the general solution of the differerence system can be written as

$$x(n) = \Psi(A, n - m)\left(\Psi^{-1}(A_1, 0), 0\right)U^* x(m)$$

So

$$x(n_1) = \Psi(A, n_1 - n_0)(\Psi^{-1}(A_1, 0), 0)U^* x(n_0)$$

is satisfied. We also know that

$$Lx(n_0) = \varphi, R\Psi(A, n_1 - n_0)(\Psi^{-1}(A_1, 0), 0)U^* x(n_0) = \psi$$

must hold. By using

$$W = \Psi(A, n_1 - n_0)(\Psi^{-1}(A_1, 0), 0)U^*$$

we can see that $\det H \neq 0$ is satisfied. So the problem has a unique solution for this case.

It is clear that the solution of the TPBVP is $x(n) = A^{n-n_0} H^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, n_0 \leq n \leq n_1$.

3. ON THE SENSITIVITY OF TPBVP

In this part, we examine the behavior of the solution of given any TPBVP according to the small changes of initial values. Therefore, here we examine the sensitivity property of a TPBVP.

Theorem 3.1. We consider the following two TPBVP's :

$$\begin{cases} x(n+1) = Ax(n) \\ Lx(n_0) = \varphi ; Rx(n_1) = \psi ; \{n : n, n_0, n_1 \in Z, n_0 \leq n \leq n_1\} \end{cases} \quad [3.1]$$

Suppose the following problem is given

$$\begin{cases} y(n+1) = Ay(n) \\ Ly(n_0) = \tilde{\varphi} ; Ry(n_1) = \tilde{\psi} ; \{n : n, n_0, n_1 \in Z, n_0 \leq n \leq n_1\} \end{cases} \quad [3.2]$$

Then, for the problems [3.1] and [3.2]

$$\frac{\|x(n) - y(n)\|}{\|x(n)\|} \leq \mu\left(A^{n-n_0} H^{-1}\right) \frac{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right\|}{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\|}, n_0 \leq n \leq n_1 \quad [3.3]$$

is satisfied. Where A is a nonsingular matrix.

Proof. Since A is a nonsingular matrix then the solutions of [3.1] and [3.2] can be written orderly as

$$x(n) = A^{n-n_0} H^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, n_0 \leq n \leq n_1,$$
$$y(n) = A^{n-n_0} H^{-1} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix}, n_0 \leq n \leq n_1.$$

Here the matrix H has the following form

$$H = \begin{pmatrix} L \\ RW \end{pmatrix} = \begin{pmatrix} L \\ RA^{n_1-n_0} \end{pmatrix}.$$

Therefore, by the hypothesis we can see that the following is satisfied:

$$x(n)-y(n)=A^{n-n_0}H^{-1}\left[\begin{pmatrix}\varphi\\ \psi\end{pmatrix}-\begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}\right], n_0 \le n \le n_1 \qquad [3.4]$$

$$\|x(n)-y(n)\| \le \left\|A^{n-n_0}H^{-1}\right\|\left\|\left(A^{n-n_0}H^{-1}\right)^{-1}\right\| \frac{\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}-\begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}\right\|}{\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|}\left\|A^{n-n_0}H^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|$$

$$\le \mu\left(A^{n-n_0}H^{-1}\right)\frac{\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}-\begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}\right\|}{\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|}\|x(n)\|$$

$$\|x(n)-y(n)\| \le \mu\left(A^{n-n_0}H^{-1}\right)\frac{\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}-\begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}\right\|}{\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|}\|x(n)\|, n_0 \le n \le n_1$$

$$\frac{\|x(n)-y(n)\|}{\|x(n)\|} \le \mu\left(A^{n-n_0}H^{-1}\right)\frac{\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}-\begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}\right\|}{\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix}\right\|}, n_0 \le n \le n_1 .$$

If we examine the upper bound of this relative error, we get the result that the condition number $\mu(A^{n-n_0}H^{-1})$ is effective. Therefore as the condition number μ gets bigger, the relative error also gets bigger. By looking at the above upper bound we see that this holds locally. In the above inequality if the upper bound does not depend on the number n then it will be the global upper bound for the relative error.

Now we give the following theorem, which deals with the sensitivity of the problem, only when the elements of L and R have small changes.

Theorem 3.2. Let the following problems are given:

$$\begin{cases} x(n+1)=Ax(n) \\ Lx(n_0)=\varphi\,; Rx(n_1)=\psi; \{n : n, n_0, n_1 \in Z, n_0 \le n \le n_1\} \end{cases} \qquad [3.5]$$

$$\begin{cases} y(n+1)=Ay(n) \\ \tilde{L}y(n_0)=\varphi\,; \tilde{R}y(n_1)=\psi; \{n : n, n_0, n_1 \in Z, n_0 \le n \le n_1\}. \end{cases} \qquad [3.6]$$

Then for these two problems the inequality of

$$\frac{\|x(n)-y(n)\|}{\|x(n)\|} \le \mu\left(A^{n-n_0}H^{-1}\right)\left(\mu(H)\frac{\|L-\tilde{L}\|}{\|L\|}+\frac{\|R-\tilde{R}\|}{\|R\|}\mu\left(A^{n_1-n_0}\right)\right)\frac{1}{1-\mu(H)\frac{\|L-\tilde{L}\|}{\|L\|}+\frac{\|R-\tilde{R}\|}{\|R\|}\mu\left(A^{n_1-n_0}\right)} \qquad [3.7]$$

is satisfied.

Proof. As it can be obtained easily the solutions of these problems are

$$x(n) = A^{n-n_0} H^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, n_0 \le n \le n_1$$

$$y(n) = A^{n-n_0} \tilde{H}^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}, n_0 \le n \le n_1$$

orderly. By using the hypotheses and these solutions we can see that the below equalities and inequalities are satisfied:

$$x(n) - y(n) = -A^{n-n_0} \left(\tilde{H}^{-1} - H^{-1}\right) \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \qquad [3.8]$$

$$= -A^{n-n_0} \left(\tilde{H}^{-1} - H^{-1}\right) \left(A^{n-n_0} H^{-1}\right)^{-1} x(n)$$

$$\frac{\|x(n) - y(n)\|}{\|x(n)\|} \le \left\| A^{n-n_0} \left(\tilde{H}^{-1} - H^{-1}\right) \right\| \left\| \left(A^{n-n_0} H^{-1}\right)^{-1} \right\|$$

$$\le \left\| A^{n-n_0} H^{-1} \right\| \left\| H\left(\tilde{H}^{-1} - H^{-1}\right) \right\| \left\| \left(A^{n-n_0} H^{-1}\right)^{-1} \right\|$$

$$= \mu\left(A^{n-n_0} H^{-1}\right) \left\| H\left(\tilde{H}^{-1} - H^{-1}\right) \right\|$$

$$H\left(\tilde{H}^{-1} - H^{-1}\right) = -\left(\tilde{H} - H\right)\tilde{H}^{-1}$$

$$\left\| H\left(\tilde{H}^{-1} - H^{-1}\right) \right\| \le \left\| \tilde{H} - H \right\| \left\| \tilde{H}^{-1} \right\|$$

$$= \left\| \tilde{H} - H \right\| \frac{1}{\sigma_1(\tilde{H})}$$

Here $\sigma_1(\tilde{H})$ is the least singular values of the matrix $\tilde{H}$.

$$\left\| H\left(\tilde{H}^{-1} - H^{-1}\right) \right\| \le \left\| \tilde{H} - H \right\| \frac{1}{\sigma_1(H) - \left\| H - \tilde{H} \right\|}$$

$$= \frac{1}{\|H\|} \left\| \tilde{H} - H \right\| \frac{1}{\frac{\sigma_1(H)}{\|H\|} - \frac{\left\| H - \tilde{H} \right\|}{\|H\|}}$$

$$= \mu(H) \frac{\left\| \tilde{H} - H \right\|}{\|H\|} \frac{1}{1 - \mu(H) \frac{\left\| H - \tilde{H} \right\|}{\|H\|}}$$

$$\frac{\|x(n) - y(n)\|}{\|x(n)\|} \le \mu\left(A^{n-n_0} H^{-1}\right) \mu(H) \frac{\left\| \tilde{H} - H \right\|}{\|H\|} \frac{1}{1 - \mu(H) \frac{\left\| H - \tilde{H} \right\|}{\|H\|}} \qquad [3.9]$$

$$H-\tilde{H}=\begin{pmatrix} L-\tilde{L} \\ (R-\tilde{R})\ A^{n_1-n_0} \end{pmatrix}$$

$$\left\|H-\tilde{H}\right\| \le \left\|\begin{pmatrix} L-\tilde{L} \\ 0 \end{pmatrix}\right\| + \left\|\begin{pmatrix} 0 \\ (R-\tilde{R})\ A^{n_1-n_0} \end{pmatrix}\right\|$$

$$\le \left\|L-\tilde{L}\right\| + \left\|(R-\tilde{R})A^{n_1-n_0}\right\|$$

$$\frac{\left\|H-\tilde{H}\right\|}{\|H\|} \le \frac{\left\|L-\tilde{L}\right\|}{\|H\|} + \frac{\left\|R-\tilde{R}\right\|}{\|H\|}\left\|A^{n_1-n_0}\right\|$$

$$\le \frac{\left\|L-\tilde{L}\right\|}{\|L\|} + \frac{\left\|R-\tilde{R}\right\|}{\left\|RA^{n_1-n_0}\right\|}\left\|A^{n_1-n_0}\right\|$$

$$\le \frac{\left\|L-\tilde{L}\right\|}{\|L\|} + \frac{\left\|R-\tilde{R}\right\|}{\|R\|}\frac{\left\|A^{n_1-n_0}\right\|}{\sigma_1\left(A^{n_1-n_0}\right)}$$

$$\le \frac{\left\|L-\tilde{L}\right\|}{\|L\|} + \frac{\left\|R-\tilde{R}\right\|}{\|R\|}\mu\left(A^{n_1-n_0}\right)$$

$$\frac{\|x(n)-y(n)\|}{\|x(n)\|} \le \mu\left(A^{n-n_0}H^{-1}\right)\left(\mu(H)\frac{\left\|L-\tilde{L}\right\|}{\|L\|} + \frac{\left\|R-\tilde{R}\right\|}{\|R\|}\mu\left(A^{n_1-n_0}\right)\right)\frac{1}{1-\mu(H)\frac{\left\|L-\tilde{L}\right\|}{\|L\|} + \frac{\left\|R-\tilde{R}\right\|}{\|R\|}\mu\left(A^{n_1-n_0}\right)}$$

Thus, the proof of the theorem is completed.

Now, we have arrived at the point that when the elements of A, L, R, φ, ψ have too small changes and the behavior of the solutions of some TPBVPs regarding with these changes. To do these, we need the following theorem:

Theorem 3.3.

$$\begin{cases} x(n+1)=Ax(n) \\ Lx(n_0)=\varphi;\ Rx(n_1)=\psi;\ \{n: n, n_0, n_1 \in Z, n_0 \le n \le n_1\}, \end{cases} \quad [3.10]$$

$$\begin{cases} y(n+1)=\tilde{A}y(n) \\ \tilde{L}y(n_0)=\tilde{\varphi}\ ;\ \tilde{R}y(n_1)=\tilde{\psi}\ ;\ \{n: n, n_0, n_1 \in Z, n_0 \le n \le n_1\}. \end{cases} \quad [3.11]$$

Suppose the problems [3.10] and [3.11] be considered. Where as we mentioned above, $\tilde{A}, \tilde{L}, \tilde{R}, \tilde{\varphi}, \tilde{\psi}$ and A, L, R, φ, ψ are too close to each other respectively in the sense of norm respectively. In addition to these the inequality of $\left\|R-\tilde{R}\right\| \le (0.1)\|R\|$ be at hand. The closeness of these solutions depends on the condition number $\mu(H)=\|H\|\left\|H^{-1}\right\|$. Let

$$S_1 = \|x(n)\|\mu\left(A^{n-n_0}H^{-1}\right)\mu(H)\left[\|L-\tilde{L}\| + \|R-\tilde{R}\|\|A^{n_1-n_0}\| + (1.1)\|R\|\frac{\|A-\tilde{A}\|\|A\|^{n_1-n_0-1}}{1-e^{\frac{\|A-\tilde{A}\|}{\|A\|}}}\right]$$

$$\cdot\frac{1}{\|H\| - \mu(H)\left[\|L-\tilde{L}\| + \|R-\tilde{R}\|\|A^{n_1-n_0}\| + (1.1)\|R\|\frac{\|A-\tilde{A}\|\|A\|^{n_1-n_0-1}}{1-e^{\frac{\|A-\tilde{A}\|}{\|A\|}}}\right]}$$

$$S_2 = \mu\left(A^{n-n_0}\tilde{H}^{-1}\right)\frac{\left\|\begin{pmatrix}\varphi\\ \psi\end{pmatrix} - \begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}\right\|}{\left\|\begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}\right\|}\|x(n)\|, \quad S_3 = \frac{\|A-\tilde{A}\|\|A\|^{n-n_0-1}}{1-e^{\frac{\|A-\tilde{A}\|}{\|A\|}}}\|\tilde{H}^{-1}\|\left\|\begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}\right\|.$$

Then the inequality of

$$\|y(n) - x(n)\| \le S_1 + S_2 + S_3 \qquad [3.12]$$

holds.

Proof. As it can be obtained easily, the solutions of the problems [3.10] and [3.11] are

$$x(n) = A^{n-n_0}H^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix},\ n_0 \le n \le n_1,\ H = \begin{pmatrix}L\\ RA^{n_1-n_0}\end{pmatrix},$$

$$y(n) = \tilde{A}^{n-n_0}\tilde{H}^{-1}\begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix},\ n_0 \le n \le n_1,\ \tilde{H} = \begin{pmatrix}\tilde{L}\\ \tilde{R}\ \ \tilde{A}^{n_1-n_0}\end{pmatrix},$$

orderly. By using them we can see that the following equalities are satisfied:

$$x(n) - y(n) = A^{n-n_0}\left[H^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix} - \tilde{H}^{-1}\begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}\right] + \left[A^{n-n_0} - \tilde{A}^{n-n_0}\right]\tilde{H}^{-1}\begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}$$

$$= A^{n-n_0}\left[H^{-1} - \tilde{H}^{-1}\right]\begin{pmatrix}\varphi\\ \psi\end{pmatrix} + A^{n-n_0}\tilde{H}^{-1}\left[\begin{pmatrix}\varphi\\ \psi\end{pmatrix} - \begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}\right] + \left[A^{n-n_0} - \tilde{A}^{n-n_0}\right]\tilde{H}^{-1}\begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}.$$

For the sake of shortness let us write the following.

$$C_1 = A^{n-n_0}\left[H^{-1} - \tilde{H}^{-1}\right]\begin{pmatrix}\varphi\\ \psi\end{pmatrix},\ C_2 = A^{n-n_0}\tilde{H}^{-1}\left[\begin{pmatrix}\varphi\\ \psi\end{pmatrix} - \begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}\right],\ C_3 = \left[A^{n-n_0} - \tilde{A}^{n-n_0}\right]\tilde{H}^{-1}\begin{pmatrix}\tilde{\varphi}\\ \tilde{\psi}\end{pmatrix}$$

So, the following inequality can be written:

$$\|x(n) - y(n)\| \le \|C_1\| + \|C_2\| + \|C_3\|.$$

With the help of the Lemma 2.1, the Theorem 3.1 and finally the Theorem 3.2 the followings are satisfied, orderly.

$$\|C_3\| \le \left\|A^{n-n_0} - \tilde{A}^{n-n_0}\right\| \left\|\tilde{H}^{-1}\right\| \left\|\begin{pmatrix}\tilde{\varphi} \\ \tilde{\psi}\end{pmatrix}\right\|$$

$$\|C_3\| \le \frac{\left\|A-\tilde{A}\right\| \, \|A\|^{n-n_0-1}}{1-e^{\frac{\|A-\tilde{A}\|}{\|A\|}}} \left\|\tilde{H}^{-1}\right\| \left\|\begin{pmatrix}\tilde{\varphi} \\ \tilde{\psi}\end{pmatrix}\right\| \qquad [3.13]$$

$$\|C_2\| \le \mu\left(A^{n-n_0}\tilde{H}^{-1}\right) \frac{\left\|\begin{pmatrix}\varphi \\ \psi\end{pmatrix} - \begin{pmatrix}\tilde{\varphi} \\ \tilde{\psi}\end{pmatrix}\right\|}{\left\|\begin{pmatrix}\tilde{\varphi} \\ \tilde{\psi}\end{pmatrix}\right\|} \|x(n)\| \qquad [3.14]$$

$$\|C_1\| \le \|x(n)\| \mu\left(A^{n-n_0}H^{-1}\right)\mu(H)\left\|\tilde{H} - H\right\| \frac{1}{\|H\| - \mu(H)\left\|H - \tilde{H}\right\|} \qquad [3.15]$$

By the hypotheses on H and $\tilde{H}$ we can write the following inequality

$$\begin{aligned}
\left\|H-\tilde{H}\right\| &\le \left\|\begin{pmatrix}L-\tilde{L} \\ 0\end{pmatrix}\right\| + \left\|\begin{pmatrix}0 \\ RA^{n_1-n_0} - \tilde{R}\tilde{A}^{n_1-n_0}\end{pmatrix}\right\| \\
&\le \left\|L-\tilde{L}\right\| + \left\|RA^{n_1-n_0} - \tilde{R}\tilde{A}^{n_1-n_0}\right\| \\
\left\|H-\tilde{H}\right\| &\le \left\|L-\tilde{L}\right\| + \left\|(R-\tilde{R})A^{n_1-n_0} + \tilde{R}\left(A^{n_1-n_0} - \tilde{A}^{n_1-n_0}\right)\right\| \\
&\le \left\|L-\tilde{L}\right\| + \left\|R-\tilde{R}\right\| \left\|A^{n_1-n_0}\right\| + \left\|\tilde{R}\right\| \left\|A^{n_1-n_0} - \tilde{A}^{n_1-n_0}\right\|
\end{aligned}$$

is obtained easily. By the Lemma 2.1 that the last inequality takes the following form:

$$\left\|H-\tilde{H}\right\| \le \left\|L-\tilde{L}\right\| + \left\|R-\tilde{R}\right\| \left\|A^{n_1-n_0}\right\| + \left[\|R\| + \left\|R-\tilde{R}\right\|\right] \frac{\left\|A-\tilde{A}\right\| \|A\|^{n_1-n_0-1}}{1-e^{\frac{\|A-\tilde{A}\|}{\|A\|}}}$$

Thus, by using the results obtained above and adding the hypotheses of $\left\|R-\tilde{R}\right\| \le (0.1)\|R\|$ to these we can see that the inequality (3.15) takes the following form

$$C_1 \leq \|x(n)\| \mu\left(A^{n-n_0} H^{-1}\right) \mu(H) \left[\|L-\tilde{L}\| + \|R-\tilde{R}\| \left\| A^{n_1-n_0} \right\| + (1.1)\|R\| \frac{\|A-\tilde{A}\| \|A\|^{n_1-n_0-1}}{1-e^{\frac{\|A-\tilde{A}\|}{\|A\|}}} \right]$$

$$\cdot \frac{1}{\|H\| - \mu(H) \left[\|L-\tilde{L}\| + \|R-\tilde{R}\| \left\| A^{n_1-n_0} \right\| + (1.1)\|R\| \frac{\|A-\tilde{A}\| \|A\|^{n_1-n_0-1}}{1-e^{\frac{\|A-\tilde{A}\|}{\|A\|}}} \right]}$$

So, this completes the proof.

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