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ON THE SENSITIVITY OF TWO-POINT BOUNDARY VALUE PROBLEM FOR THE SYSTEM OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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ABSTRACT

In this study, a system of homogeneous, linear, ordinary differential equations with constant coefficients related with the sensitivity of two-point boundary value problem is investigated in the interval of $[t_0, t_1]$. The problem is given as follows:

$$\frac{d}{dt}x(t) = Ax(t)$$

$$Lx(t_0) = \varphi, \quad Rx(t_1) = \psi$$

Key Words: Linear system, condition number, sensitivity.

SABİT KATSAYILI LİNEER ADİ DİFERENSİYEL DENKLEM SİSTEMİ İÇİN İKİ NOKTA SINIR DEĞER PROBLEMİNİN HASSASİYETİ ÜZERİNE

ÖZET

Bu çalışmada sabit katsayılı homogen lineer adi diferensiyel denklem sistemi için $[t_0, t_1]$ aralığında tanımlı

$$\frac{d}{dt}x(t) = Ax(t)$$

$$Lx(t_0) = \varphi, \quad Rx(t_1) = \psi$$

şeklinde verilen iki nokta sınır değer probleminin hassasiyeti incelenmiştir.

Anahtar Kelimeler: Lineer sistem, şart sayısı, hassasiyet.

1. INTRODUCTION

There have been huge developments in the calculation techniques in the last years. The Main subject among these developments is to examine the closeness of the numerical solutions to the exact solution [1,2,3,4]. In our work, the set of numbers used by a computer and wether the problem is well conditioned or ill conditioned is very important. Here, the dependence between the solution, for two-point boundary value problem (TPBVP) which has many applications, and the datum is examined.

TPBVP related with systems of differentials equations has been examined in terms of existence and uniqueness theorem [5], the synthesis of optimal filtering algorithms [6,7], wave propagation [8] etc. There are many computational methods [9,10,11] that have been developed for solving TPBVP.

2. THE TWO POINT BOUNDARY VALUE PROBLEM FOR AN ORDINARY DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

Definition 1. To find the solution of problem in the interval of [t_0,t_1] described

$$\frac{d}{dt}x(t) = Ax(t) \qquad [1]$$

$$Lx(t_0) = \varphi, \quad Rx(t_1) = \psi$$

is called “two-point boundary value problem for a homogeneous, linear, ordinary differential equation with constant coefficients”.

Where A is N×N; L, k×N; R, (N-k)×N type real matrices, φ is a vector with k components and ψ is a vector with (N-k) components and x(t) is unknown N dimensional vector function on the given interval to be found.

Definition 2 (Fundamental Matrix). Let

$$\{\psi_1(t), \psi_2(t), \ldots, \psi_N(t)\}$$

N dimensional vector is the base of vector space of the functions of solutions of the system given in (1).

Then, the matrix

$$\psi(t) = [\psi_1(t), \psi_2(t), \ldots, \psi_N(t)]$$

is called a fundamental matrix of the system given in (1).

Namely, if ψ (t) is fundamental matrix, then

$$\frac{d}{dt}\psi(t) = A\psi(t); \quad \det \psi(t) \neq 0$$

is satisfied (1).

The only solution to the Cauchy problem

$$\frac{d}{dt}X(t) = A\,X(t)$$
$$X(0) = I$$

is X(t) = e^{tA} which is an exponential matrix function. So, e^{tA} is a fundamental matrix of the given system (12).

Theorem 1. Let us consider problem [1]. If Φ (t) is a fundamental matrix of the system

$$\frac{d}{dt}x(t) = Ax(t)$$

and

$$H = \begin{pmatrix} L \\ R\Phi(t_1 - t_0)\Phi^{-1}(0) \end{pmatrix} \qquad [2]$$

is a non-singular matrix ($\det H \neq 0$), then there exists a unique solution to the problem [1] and it is given below

$$x(t) = e^{(t-t_0)A} H^{-1} \begin{bmatrix} \varphi \\ \psi \end{bmatrix} \qquad [3]$$

(13).

Note: Let T, t be any numbers. Then for any $\Phi(t)$ fundamental matrix

$$\Phi(T-t)\,\Phi^{-1}(0) = e^{(T-t)A}$$

is satisfied. Therefore, the condition (2) is independent of the choice of fundamental matrix.

3. SENSITIVITY OF THE SOLUTION

In this part, the effect of the input datum's errors to the solution of problem (1) will be given. In the following lemmas and theorem, these are given as sensitivity of the solution of these problems. In these lemmas and theorem, we determine an upper bound for the absolute errors that have been made.

Lemma 1. Let A be a non singular matrix and the following two point boundary value problem be given:

$$\frac{d}{dt} X(t) = AX(t) \qquad [4]$$

$$LX(t_0) = \varphi, \quad RX(t_1) = \psi$$

and

$$\frac{d}{dt} Y(t) = A\,Y(t) \qquad [5]$$

$$L\,Y(t_0) = \tilde{\varphi}, \quad R\,Y(t_1) = \tilde{\psi}.$$

Then the following inequality, between the solutions, will be satisfied

$$\frac{\|X(t) - Y(t)\|}{\|X(t)\|} \leq \mu\left(e^{(t-t_0)A} H^{-1}\right) \frac{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right\|}{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\|} \qquad [6]$$

Where the interval $[t_0, t_1]$ is small enough.

Proof. The solutions to the given problems can be the following, respectively:

$$X(t) = e^{(t-t_0)A} H^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$$

and

$$Y(t) = e^{(t-t_0)A} H^{-1} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix}.$$

By using them, we can obtain that

$$\|X(t)-Y(t)\| = \left\| e^{(t-t_0)A} H^{-1} \left\{ \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right\} \right\|$$

$$\leq \mu\left(e^{(t-t_0)A} H^{-1}\right) \frac{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right\|}{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\|} \left\| e^{(t-t_0)A} H^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\|$$

$$\leq \mu\left(e^{(t-t_0)A} H^{-1}\right) \frac{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right\|}{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \right\|} \|X(t)\|$$

are satisfied. Which is required to prove the inequality [6].

With this lemma an upper bound has been determined by the relative error of the problems [4] and [5]. It is that the condition number $\mu\left(e^{(t-t_0)A} H^{-1}\right)$ has effective role on the upper bound.

Lemma 2. Let A be a nonsingular matrix. Consider the following two-point boundary value problems on the interval of $[t_0,t_1]$:

$$\frac{d}{dt} X(t) = AX(t)$$
$$LX(t_0) = \varphi, \quad RX(t_1) = \psi \qquad [7]$$

and

$$\frac{d}{dt} Y(t) = A\, Y(t)$$
$$\tilde{L}\, Y(t_0) = \varphi, \quad \tilde{R}\, Y(t_1) = \psi. \qquad [8]$$

The relation given below can be held between the solutions of these problems:

$$\frac{\|X(t)-Y(t)\|}{\|X(t)\|} \leq \mu\left(e^{(t-t_0)A} H^{-1}\right) \mu(H) \frac{\|H-\tilde{H}\|}{\|H\|} \frac{1}{1-\mu(H)\dfrac{\|H-\tilde{H}\|}{\|H\|}} \qquad [9]$$

Proof. Let the solutions to the given problems be respectively:

$$X(t) = e^{(t-t_0)A} H^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$$

and

$$Y(t) = e^{(t-t_0)A} \tilde{H}^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}.$$

Where H and $\tilde{H}$ are given as follows:

$$H = \begin{pmatrix} L \\ Re^{(t_1-t_0)A} \end{pmatrix}, \quad \tilde{H} = \begin{pmatrix} \tilde{L} \\ \tilde{R}e^{(t_1-t_0)A} \end{pmatrix}$$

Then,

$$X(t)-Y(t)=e^{(t-t_0)A}H^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}-e^{(t-t_0)A}\tilde{H}^{-1}\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$$
$$=-e^{(t-t_0)A}(\tilde{H}^{-1}-H^{-1})\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$$
$$=-e^{(t-t_0)A}(\tilde{H}^{-1}-H^{-1})(e^{(t-t_0)A}H^{-1})^{-1}(e^{(t-t_0)A}H^{-1})\begin{pmatrix}\varphi\\ \psi\end{pmatrix}$$

is satisfied. Therefore the inequality of

$$X(t)-Y(t)=-e^{(t-t_0)A}(\tilde{H}^{-1}-H^{-1})(e^{(t-t_0)A}H^{-1})^{-1}X(t) \qquad [10]$$

can be written. For the relative error

$$\frac{\|X(t)-Y(t)\|}{\|X(t)\|}\le\|e^{(t-t_0)A}(\tilde{H}^{-1}-H^{-1})\|.\|(e^{(t-t_0)A}H^{-1})^{-1}\|$$
$$=\|e^{(t-t_0)A}H^{-1}H(\tilde{H}^{-1}-H^{-1})\|.\|(e^{(t-t_0)A}H^{-1})^{-1}\|$$
$$\le\|e^{(t-t_0)A}H^{-1}\|.\|H(\tilde{H}^{-1}-H^{-1})\|.\|(e^{(t-t_0)A}H^{-1})^{-1}\|$$

is obtained. Therefore

$$\frac{\|X(t)-Y(t)\|}{\|X(t)\|}\le\mu\left(e^{(t-t_0)A}H^{-1}\right)\|H(\tilde{H}^{-1}-H^{-1})\| \qquad [11]$$

is satisfied. Furthermore, by taking the norm of both sider of

$$H(\tilde{H}^{-1}-H^{-1})=(\tilde{H}-H)\tilde{H}^{-1}$$
$$\|H(\tilde{H}^{-1}-H^{-1})\|\le\|\tilde{H}-H\|.\|\tilde{H}^{-1}\|$$
$$\le\|\tilde{H}-H\|\frac{1}{\frac{1}{\|\tilde{H}^{-1}\|}-\|H-\tilde{H}\|}$$
$$=\frac{1}{\|H\|}\|\tilde{H}-H\|\frac{1}{\frac{1}{\|\tilde{H}^{-1}\|\|H\|}-\frac{\|H-\tilde{H}\|}{\|H\|}}$$
$$=\frac{1}{\|H\|}\|\tilde{H}-H\|\frac{1}{\frac{1}{\mu(H)}-\frac{\|H-\tilde{H}\|}{\|H\|}}$$
$$=\frac{\|H-\tilde{H}\|}{\|H\|}\frac{1}{\frac{1}{\mu(H)}\left(1-\mu(H)\frac{\|H-\tilde{H}\|}{\|H\|}\right)}$$
$$=\mu(H)\frac{\|H-\tilde{H}\|}{\|H\|}\frac{1}{1-\mu(H)\frac{\|H-\tilde{H}\|}{\|H\|}}$$

is obtained. If we substitute this result in (1 1) the required result (9) will be obtained. This completes the proof of the lemma.

Theorem 2. Let us consider the following two problems:

$$\frac{d}{dt}X(t) = AX(t)$$
$$LX(t_0) = \varphi, \quad RX(t_1) = \psi \qquad [12]$$

and

$$\frac{d}{dt}Y(t) = \tilde{A}\,Y(t)$$
$$\tilde{L}Y(t_0) = \tilde{\varphi}, \quad \tilde{R}Y(t_1) = \tilde{\psi} \qquad [13]$$

where $\tilde{A}, \tilde{L}, \tilde{R}, \tilde{\varphi}, \tilde{\psi}$ are matrices and vectors of A, L, R, φ and ψ which are very close to each other in the sense of norm, respectively. Finally $t \in [t_0, t_1]$. Since

$$H = \begin{pmatrix} L \\ Re^{TA} \end{pmatrix} \qquad [14]$$

so, the inequality of

$$\| X(t) - Y(t) \| \le \| X(t) \| \ \mu\left(e^{(t-t_0)A}H^{-1}\right) \mu(H) \, \|H - \tilde{H}\| \frac{1}{\|H\| - \mu(H)\|H - \tilde{H}\|}$$
$$+ \| X(t) \| \, \mu\left(e^{(t-t_0)A}\tilde{H}^{-1}\right) \frac{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right\|}{\left\| \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right\|} + \|A - \tilde{A}\| \, e^{T\|A\|} \, e^{T\|\tilde{A}\|} \, \|\tilde{H}^{-1}\| \left\| \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right\|$$

is valid.

Proof. The solutions to these problems, respectively, are

$$X(t) = e^{(t-t_0)A} H^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$$

and

$$Y(t) = e^{(t-t_0)\tilde{A}} \tilde{H}^{-1} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix}$$

$$X(t) - Y(t) = e^{(t-t_0)A} H^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - e^{(t-t_0)\tilde{A}} \tilde{H}^{-1} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix}$$
$$= e^{(t-t_0)A} H^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} e^{(t-t_0)A} \tilde{H}^{-1} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} + e^{(t-t_0)A} \tilde{H}^{-1} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} - e^{(t-t_0)\tilde{A}} \tilde{H}^{-1} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix}$$
$$= e^{(t-t_0)A} \left(H^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \tilde{H}^{-1} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix}\right) + \left(e^{(t-t_0)A} - e^{(t-t_0)\tilde{A}}\right) \tilde{H}^{-1} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix}$$
$$= e^{(t-t_0)A} \left(H^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \tilde{H}^{-1} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} + \tilde{H}^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \tilde{H}^{-1} \begin{pmatrix} \varphi \\ \psi \end{pmatrix}\right) + \left(e^{(t-t_0)A} - e^{(t-t_0)\tilde{A}}\right) \tilde{H}^{-1} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix}$$
$$= e^{(t-t_0)A} [H^{-1} - \tilde{H}^{-1}] \begin{pmatrix} \varphi \\ \psi \end{pmatrix} + e^{(t-t_0)A} \tilde{H}^{-1} \left[\begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right] + \left[e^{(t-t_0)A} - e^{(t-t_0)\tilde{A}} \right] \tilde{H}^{-1} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix}$$

Let $C_1 = e^{(t-t_0)A} (H^{-1} - \tilde{H}^{-1}) \begin{pmatrix} \varphi \\ \psi \end{pmatrix}$

$C_2 = e^{(t-t_0)A} \tilde{H}^{-1} \left(\begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix} \right)$

$C_3 = \left(e^{(t-t_0)A} - e^{(t-t_0)\tilde{A}} \right) \tilde{H}^{-1} \begin{pmatrix} \tilde{\varphi} \\ \tilde{\psi} \end{pmatrix}$

Therefore,

$$\| X(t) - Y(t) \| \leq \| C_1 \| + \| C_2 \| + \| C_3 \|$$

will be satisfied. For the C_1 and C_2 we can write the following by getting help from the Lemma 1 and 2:

$$\|C_1\| \leq \| X(t) \| . \mu(e^{(t-t_0)A} H^1)\, \mu(H)\, \|H - \widetilde{H}\| \frac{1}{\| H \| - \mu(H) \| H - \widetilde{H} \|}$$

and

$$\| C_2 \| \leq \| X(t) \| \mu\left(e^{(t-t_0)A} \widetilde{H}^{-1}\right) \frac{\left\| \begin{pmatrix} \varphi \\ \psi \end{pmatrix} - \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{\psi} \end{pmatrix} \right\|}{\left\| \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{\psi} \end{pmatrix} \right\|}.$$

For C_3 we can also write the following inequality.

$$\| C_3 \| \leq \| e^{(t-t_0)A} - e^{(t-t_0)\widetilde{A}} \| \| \widetilde{H}^{-1} \| \left\| \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{\psi} \end{pmatrix} \right\|$$

$$\leq \|A - \widetilde{A}\| \; e^{T\|A\|} \; e^{T\|\widetilde{A}\|} \| \widetilde{H}^{-1} \| . \left\| \begin{pmatrix} \widetilde{\varphi} \\ \widetilde{\psi} \end{pmatrix} \right\|$$

This completes the proof of the theorem.

Here closeness of the solutions depends on the condition number, $T = t_1 - t_0$ and $(H) = \|H\|.\|H^{-1}\|$ is seen.

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