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NONPARAMETRIC LIKELIHOOD RATIO TEST FOR SCALE

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ABSTRACT

In this study, a nonparametric two sample rank test is proposed for scale alternatives. The two samples are assumed to have continuous distribution functions with the difference in respective location parameters (medians) known. Mood and Klotz tests are considered and compared with the proposed likelihood ratio test for normal, uniform and exponential ditributions by using Monte Carlo emprical power simulation study. Monte Carlo emprical power study for these distributions are computed for equal sample sizes 10, 20, 30 and 45. Likelihood ratio score test is proposed for particular alternatives which is most powerfull for normal and symmetric distributions.

Key words: Nonparametric scale tests, nonparametric dispersion tests, score value, score function, Monte Carlo

ÖLÇEK İÇİN PARAMETRİK OLMAYAN OLABİLİRLİK ORAN TESTİ

ÖZET

Bu çalışmada, ölçek alternatifleri için parametrik olmayan iki örnek rank testi önerilmiştir. İki örneğin konumları (medyanları) bilinen ve sürekli dağılımlara sahip oldukları varsayılmıştır. Mood ve Klotz'un testleri önerilen olabilirlik oran skor testi ile normal, tek-düze ve üstel dağılımlar üzerinde Monte Carlo deneysel güç benzetim çalışmasıyla karşılaştırılmıştır. Bu dağılımlar üzerinde Monte Carlo deneysel güç karşılaştırması 10, 20, 30 ve 45 hacimli eşit sayıdaki örneklemler üzerinde gerçekleştirilmiştir. Olabilirlik oran skor testi özellikle normal ve simetrik dağılımları içeren bazı seçenekler için daha güçlü olduğundan önerilir.

Anahtar kelimeler: Parametrik olmayan ölçek testi, parametrik olmayan yayılım testi, skor değeri, skor fonksiyonu, Monte Carlo

1. INTRODUCTION

Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be two samples with continuous cumulative distribution functions F and G.Set $F(x) = \omega((x-\theta)/\sigma_X)$ and $G(x) = \omega((x-\theta)/\sigma_Y)$ with ω a distribution function and σ_X and σ_Y scale parameters. Assuming that the median difference is known, we take θ to be the common unknown median for F and G. Under this assumption, the null hypothesis

$$H_0 : \sigma_X = \sigma_Y \qquad [1.1]$$

gives F=G and statistics of the form $\sum_{i=1}^{N} s_i I_i$ have a distribution independent of ω under H_0. Here $N = n + m$, s_i are assigned score values and I_i is an indicator function so that $I_i = 1$ if the ith smallest order statistic in the pooled X and Y sample is an X and is zero otherwise. Test of the form (1.1) for scale alternatives have been given by Freund-Ansari-Bradley, David-Barton, Siegel-Tukey, Sukhatme, Moses, Mood and Klotz (2,3,5). The test of Mood uses scores

$$ms_i = \left(i - \frac{N+1}{2} \right)^2 \qquad [1.2]$$

and the test of Klotz uses scores

$$ks_i = \left[\Phi^{-1}\left(\frac{i}{N+1} \right) \right]^2 \qquad [1.3]$$

where Φ is the standard normal cumulative function. The test of Klotz, which has greater efficiency, gives more weight to the extreme ranks than does the Mood test. Mood gives the value $15/2\pi^2 = 0.76$ and Klotz gives the value near 1 for the limiting Pitman efficiency of their tests relative to the F-test under normality (2). Therefore, the question comes whether or not a new test can be obtained if even it is more powerfull. Thus it is proposed to use the test statistic

$$ss_i = \sum_{i=1}^{N} s_\sigma I_i = \sum_{i=1}^{N} \left\{ \frac{f_\sigma\left[\Phi^{-1}\left(\frac{i}{N+1} \right) \right]}{f\left[\Phi^{-1}\left(\frac{i}{N+1} \right) \right]} \right\} I_i \qquad [1.4]$$

which shall be called the likelihood ratio score test statistic (1). If we take two different probability functions as f_σ is the normal probability function with σ standard deviation and f is the standart normal probability function, it can be introduced many test statistics by using different functions depend upon different σ values. However, Monte Carlo emprical power study show that the likelihood ratio test statistic for σ=1.1 is more powerful than other alternative test statistics (1). Therefore, the proposed test statistic is

$$ss_i = \sum_{i=1}^{N} \frac{1}{1.1} e^{0.08677 z_i^2} I_i, \quad -\infty < z_i < \infty \qquad [1.5]$$

where

$$z_i = \Phi^{-1}\left(\frac{i}{N+1} \right).$$

2. EMPRICAL POWER STUDY

In emprical power study, the likelihood ratio score test is compared with Klotz and Mood tests since they have higher efficiency relative to the F-test under normality. In order to define the most powerfull likelihood ratio test statistic or score values s_σ, different f_σ probability functions were used. For this reason, different standard deviations were used as 1.1, 1.2, 1.3, 1.4, 1.5, 2 and 3. These tests were compared on normal, uniform and exponential distributions. Both sample has equal sizes (10, 20, 30, 45) and were taken from same type but different scaled distributions.

The hypothesis (1.1) were tested 10,000 times at $\alpha = 0.05$ level. The critical values of the tests were obtained exact critical value tables or calculated from the normal approximation theory (1,2,3,4,5).

Table 1. The power comparisons of tests on normal distributions ($F \sim N(0;1)$)

$n_X = n_Y = 10$	Critical value $\alpha = 0.05$	G normal distributions			
		N(0;1)	**N(0;1.5)**	**N(0;2)**	**N(0;3)**
Mood	444.500000	0.0483	0.2338	0.4699	0.7613
Klotz	10.864300	0.0462	0.2390	0.4794	0.7681
Ss(1.1)	10.017910	0.0526	0.2620	0.5087	0.7877
Ss(1.2)	9.938695	0.0530	0.2641	0.5106	0.7890
Ss(1.3)	9.788358	0.0527	0.2627	0.5082	0.7866
Ss(1.4)	9.588797	0.0533	0.2637	0.5091	0.7871
Ss(1.5)	9.357212	0.0536	0.2658	0.5104	0.7882
Ss(2.0)	8.064865	0.0544	0.2665	0.5120	0.7883
Ss(3.0)	5.978676	0.0541	0.2662	0.5102	0.7852
$n_X = n_Y = 20$					
Mood	3289.88300	0.0515	0.4221	0.7816	0.9759
Klotz	22.31768	0.0526	0.4616	0.8266	0.9858
Ss(1.1)	20.14127	0.0519	0.4636	0.8272	0.9859
Ss(1.2)	20.11672	0.0524	0.4648	0.8271	0.9867
Ss(1.3)	19.95484	0.0527	0.4668	0.8277	0.9864
Ss(1.4)	19.68604	0.0524	0.4661	0.8272	0.9863
Ss(1.5)	19.33827	0.0526	0.4670	0.8275	0.9861
Ss(2.0)	17.08996	0.0538	0.4662	0.8242	0.9847
Ss(3.0)	12.96832	0.0541	0.4629	0.8205	0.9836
$n_X = n_Y = 30$					
Mood	10715.03000	0.0527	0.5725	0.9206	0.9986
Klotz	33.64420	0.0523	0.6320	0.9536	0.9993
Ss(1.1)	30.25255	0.0523	0.6326	0.9549	0.9994
Ss(1.2)	30.28984	0.0522	0.6307	0.9544	0.9994
Ss(1.3)	30.13666	0.0524	0.6303	0.9544	0.9996
Ss(1.4)	29.82614	0.0525	0.6283	0.9530	0.9996
Ss(1.5)	29.39259	0.0524	0.6271	0.9517	0.9995
Ss(2.0)	26.31549	0.0519	0.6221	0.9487	0.9994
Ss(3.0)	20.23179	0.0528	0.6174	0.9442	0.9991
$n_X = n_Y = 45$					
Mood	35092.71000	0.0486	0.7240	0.9829	1.0
Klotz	50.35557	0.0486	0.7964	0.9941	1.0
Ss(1.1)	45.40125	0.0491	0.7995	0.9941	1.0
Ss(1.2)	45.53111	0.0501	0.7982	0.9938	1.0
Ss(1.3)	45.40488	0.0508	0.7966	0.9938	1.0
Ss(1.4)	45.05494	0.0507	0.7933	0.9936	1.0
Ss(1.5)	44.52084	0.0508	0.7913	0.9932	1.0
Ss(2.0)	40.33488	0.0514	0.7788	0.9914	1.0
Ss(3.0)	31.40405	0.0530	0.7682	0.9894	1.0

Table 2. The power comparisons of tests on uniform distributions (F ~ U(4;6)

$n_X = n_Y = 10$	Critical value $\alpha = 0.05$	G uniform distributions			
		U(4;6)	**U(2.5;7.5)**	**U(1.5;8.5)**	**U(0;10)**
Mood	444.500000	0.0483	0.6011	0.7480	0.8239
Klotz	10.864300	0.0462	0.5963	0.7370	0.8104
Ss(1.1)	10.017910	0.0526	0.6209	0.7536	0.8215
Ss(1.2)	9.938695	0.0530	0.6206	0.7541	0.8213
Ss(1.3)	9.788358	0.0527	0.6169	0.7493	0.8149
Ss(1.4)	9.588797	0.0533	0.6171	0.7494	0.8148
Ss(1.5)	9.357212	0.0536	0.6181	0.7498	0.8150
Ss(2.0)	8.064865	0.0544	0.6183	0.7500	0.8144
Ss(3.0)	5.978676	0.0541	0.6154	0.7465	0.8121
$n_X = n_Y = 20$					
Mood	3289.88300	0.0515	0.8990	0.9737	0.9864
Klotz	22.31768	0.0526	0.9152	0.9728	0.9847
Ss(1.1)	20.14127	0.0519	0.9154	0.9722	0.9838
Ss(1.2)	20.11672	0.0524	0.9146	0.9714	0.9835
Ss(1.3)	19.95484	0.0527	0.9123	0.9706	0.9832
Ss(1.4)	19.68604	0.0524	0.9108	0.9698	0.9829
Ss(1.5)	19.33827	0.0526	0.9099	0.9692	0.9824
Ss(2.0)	17.08996	0.0538	0.9037	0.9662	0.9821
Ss(3.0)	12.96832	0.0541	0.8995	0.9638	0.9812
$n_X = n_Y = 30$					
Mood	10715.03000	0.0527	0.9802	0.9975	0.9991
Klotz	33.64420	0.0523	0.9856	0.9970	0.9989
Ss(1.1)	30.25255	0.0523	0.9842	0.9971	0.9989
Ss(1.2)	30.28984	0.0522	0.9839	0.9968	0.9988
Ss(1.3)	30.13666	0.0524	0.9824	0.9966	0.9988
Ss(1.4)	29.82614	0.0525	0.9812	0.9965	0.9987
Ss(1.5)	29.39259	0.0524	0.9805	0.9964	0.9986
Ss(2.0)	26.31549	0.0519	0.9775	0.9955	0.9984
Ss(3.0)	20.23179	0.0528	0.9735	0.9949	0.9983
$n_X = n_Y = 45$					
Mood	35092.71000	0.0486	0.9980	1.0000	1.0
Klotz	50.35557	0.0486	0.9986	1.0000	1.0
Ss(1.1)	45.40125	0.0491	0.9986	1.0000	1.0
Ss(1.2)	45.53111	0.0501	0.9983	0.9999	1.0
Ss(1.3)	45.40488	0.0508	0.9982	0.9999	1.0
Ss(1.4)	45.05494	0.0507	0.9982	0.9999	1.0
Ss(1.5)	44.52084	0.0508	0.9981	0.9999	1.0
Ss(2.0)	40.33488	0.0514	0.9977	0.9999	1.0
Ss(3.0)	31.40405	0.0530	0.9963	0.9999	1.0

Normal and uniform distributions are symmetric distributions. However, exponential distribution are nonsymmetric distributions. The power comparison of likelihood ratio score test is also performed on different exponential distributions after normalization by their medians.

Table 3. The means and the medians of exponential distributions with parameters

	$\lambda = 1$	$\lambda = 2$	$\lambda = 3$	$\lambda = 4$
Mean	1	2	3	4
Median	0.693147	1.386290	2.079440	2.772590

Table 4. The power comparisons of tests on exponential distributions ($F \sim Exp(1)$

$n_X = n_Y = 10$	Critical value $\alpha = 0.05$	G exponential distributions			
		Exp(1)	**Exp(2)**	**Exp(3)**	**Exp(4)**
Mood	444.500000	0.0480	0.5431	0.7785	0.8684
Klotz	10.864300	0.0468	0.5436	0.7731	0.8674
Ss(1.1)	10.017910	0.0542	0.5678	0.7911	0.8767
Ss(1.2)	9.938695	0.0546	0.5691	0.7896	0.8756
Ss(1.3)	9.788358	0.0542	0.5663	0.7886	0.8727
Ss(1.4)	9.588797	0.0542	0.5674	0.7863	0.8722
Ss(1.5)	9.357212	0.0552	0.5689	0.7873	0.8723
Ss(2.0)	8.064865	0.0557	0.5703	0.7881	0.8727
Ss(3.0)	5.978676	0.0558	0.5689	0.7854	0.8711
$n_X = n_Y = 20$					
Mood	3289.88300	0.0545	0.8585	0.9827	0.9969
Klotz	22.31768	0.0528	0.8795	0.9851	0.9965
Ss(1.1)	20.14127	0.0522	0.8782	0.9826	0.9960
Ss(1.2)	20.11672	0.0515	0.8762	0.9814	0.9955
Ss(1.3)	19.95484	0.0517	0.8737	0.9798	0.9947
Ss(1.4)	19.68604	0.0523	0.8710	0.9786	0.9942
Ss(1.5)	19.33827	0.0522	0.8675	0.9770	0.9936
Ss(2.0)	17.08996	0.0529	0.8572	0.9731	0.9924
Ss(3.0)	12.96832	0.0531	0.8484	0.9696	0.9911
$n_X = n_Y = 30$					
Mood	10715.03000	0.0488	0.9570	0.9992	1.0000
Klotz	33.64420	0.0527	0.9724	0.9993	1.0000
Ss(1.1)	30.25255	0.0529	0.9704	0.9991	1.0000
Ss(1.2)	30.28984	0.0525	0.9675	0.9989	0.9999
Ss(1.3)	30.13666	0.0526	0.9630	0.9987	0.9998
Ss(1.4)	29.82614	0.0517	0.9595	0.9985	0.9998
Ss(1.5)	29.39259	0.0512	0.9576	0.9985	0.9997
Ss(2.0)	26.31549	0.0514	0.9466	0.9971	0.9996
Ss(3.0)	20.23179	0.0519	0.9365	0.9952	0.9992
$n_X = n_Y = 45$					
Mood	35092.71000	0.0509	0.9951	1.0000	1.0
Klotz	50.35557	0.0469	0.9979	1.0000	1.0
Ss(1.1)	45.40125	0.0465	0.9974	1.0000	1.0
Ss(1.2)	45.53111	0.0476	0.9969	1.0000	1.0
Ss(1.3)	45.40488	0.0471	0.9963	1.0000	1.0
Ss(1.4)	45.05494	0.0471	0.9957	1.0000	1.0
Ss(1.5)	44.52084	0.0477	0.9945	1.0000	1.0
Ss(2.0)	40.33488	0.0486	0.9902	0.9999	1.0
Ss(3.0)	31.40405	0.0487	0.9848	0.9998	1.0

3. CONCLUSIONS

The proposed likelihood ratio score test is especially well performed on symmetric distributions since the rejection probability is close to 0.05 for F=G. Also, the rejection probability of proposed test is higher than other tests for $F \neq G$.

On the other hand, it is not easy to say that the likelihood ratio test is well performed for scale test on nonsymmetric distributions. However, the power results are reasonable good.

The table of score values or of weights of likelihood ratio test for observations is shown in Table 5.

Table 5. The score values of likelihood ratio test for $4 \leq N \leq 30$

	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
N															
4	0.966	0.914													
5	0.985	0.923	0.909												
6	1.003	0.934	0.911												
7	1.019	0.945	0.917	0.909											
8	1.034	0.956	0.923	0.910											
9	1.048	0.966	0.930	0.914	0.909										
10	1.061	0.976	0.938	0.918	0.910										
11	1.073	0.985	0.945	0.923	0.912	0.909									
12	1.084	0.994	0.952	0.929	0.915	0.909									
13	1.095	1.003	0.959	0.934	0.919	0.911	0.909								
14	1.105	1.011	0.966	0.940	0.923	0.914	0.909								
15	1.115	1.019	0.973	0.945	0.927	0.917	0.911	0.909							
16	1.124	1.027	0.979	0.950	0.932	0.920	0.912	0.909							
17	1.133	1.034	0.985	0.956	0.936	0.923	0.915	0.910	0.909						
18	1.142	1.041	0.991	0.961	0.941	0.927	0.917	0.912	0.909						
19	1.150	1.048	0.997	0.966	0.945	0.930	0.920	0.914	0.910	0.909					
20	1.158	1.054	1.003	0.971	0.949	0.934	0.923	0.916	0.911	0.909					
21	1.165	1.061	1.008	0.976	0.954	0.938	0.926	0.918	0.913	0.910	0.909				
22	1.173	1.067	1.014	0.981	0.958	0.941	0.929	0.921	0.915	0.911	0.909				
23	1.180	1.073	1.019	0.985	0.962	0.945	0.932	0.923	0.917	0.912	0.909	0.909			
24	1.186	1.079	1.024	0.990	0.966	0.946	0.936	0.926	0.919	0.914	0.910	0.909			
25	1.193	1.084	1.029	0.994	0.970	0.952	0.939	0.929	0.921	0.915	0.912	0.909	0.909		
26	1.200	1.090	1.034	0.999	0.974	0.956	0.942	0.931	0.923	0.917	0.913	0.910	0.909		
27	1.206	1.095	1.039	1.003	0.978	0.959	0.945	0.934	0.926	0.919	0.914	0.911	0.909	0.909	
28	1.212	1.100	1.043	1.007	0.981	0.963	0.948	0.937	0.928	0.921	0.916	0.912	0.910	0.909	
29	1.218	1.105	1.048	1.011	0.985	0.966	0.951	0.940	0.930	0.923	0.918	0.914	0.911	0.909	0.909
30	1.224	1.110	1.052	1.015	0.989	0.969	0.954	0.942	0.933	0.925	0.919	0.915	0.912	0.910	0.909

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